AN ALTERNATIVE PROOF AND APPLICATIONS OF A THEOREM OF E. G. EFFROS

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1. Introduction. We present an alternative proof of a theorem of E. G. Effros. Our Theorem 1 is a version of Theorem 2.1 of [8]. Our proof of Theorem 1 is of a topological nature, whereas the proof of Theorem 2.1 in [8] has a more analytic character.

Theorem 2.1 of [8] has become a well-known and valuable tool in the study of topological homogeneity and properties of continua. It has been exploited by topologists in many ways beginning with the papers [15], [10], and [13]. Part of the justification for presenting our alternative proof of Effros' theorem is that our proof may be more accessible to the topologists who use the result.

Section 2 of this paper presents the statement of Theorem 1 preceded by the basic definitions needed to understand it. In particular, the concept of *micro-transitivity*, which is essential to Theorem 1, is introduced.

The proof of Theorem 1 appears in Section 3. The proof of Theorem 2.1 in [8] relies on the concept of a *Borel section* and on theorems concerning its existence. Instead of using these ideas, our proof employs an ingenious technique which was introduced by Homma in [12], and which played an important role in the study of tame and wild embeddings in manifolds in the 1960's and 1970's.

The proof in Section 3 was independently discovered by Torunczyk (unpublished). He refers to this result as the *Open Mapping Principle.* This name is inspired by comparison with the fundamental result of functional analysis known as the Open Mapping Theorem. Indeed, there is a strong resemblance between the proof in Section 3 and the proofs of the Open Mapping Theorem found in standard functional analysis texts. (For instance, see [7, Theorem 1, p. 55].) We reinforce this comparison by showing, in Section 4, that the Open Mapping Theorem is an easy corollary of our Theorem 1.

Section 4 also contains a deduction of Theorem 2.1 of [8] from Theorem 1. This deduction requires some argument because Theorem 2.1 of [8] involves issues which at first glimpse don't appear to be covered by Theorem 1.

In Section 5, the notion of *micro-homogeneity* is introduced. Theorem 1 has implications that relate the homogeneity and micro-homogeneity of a topological space. Theorem 2 sets forth conditions under which these implications are valid. Limitations to the applicability of Theorem 1 in this context are illustrated by two examples. Some questions concerning the relation between homogeneity and micro-homogeneity are posed.

At one point, the proof of Theorem 1 relies essentially on a result of Hausdorff. Hausdorff's theorem, which appeared in [11], says that a metrizable open

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image of a complete metric space has a complete metric. Though this result is of a fundamental nature, it has not been reproduced in any of the standard topology texts. In order to provide the reader with easy access to a modern proof of this classical result, we have appended such a proof as Section 6.

The appearance of Homma's technique in the proof of Theorem 1 suggests that Theorem 1 may have applications to tame and wild embeddings in manifolds. This is indeed the case. In [3] (now in preparation), the notion of a *micro-unknotted* closed embedding is introduced and is related to Theorem 1. It is shown that a *tame* closed embedding of a k-manifold in an n-manifold is micro-unknotted if and only if $k \neq n-2$. The micro-unknottedness of wild embeddings is also explored. It is shown that some wild embeddings of S^2 into S^3 (such as the Alexander Horned Sphere [1]) are micro-unknotted, while others (e.g., a certain Fox–Artin wild sphere [9]) are micro-knotted.

I would like to express my gratitude to Charles Hagopian for first telling me the intriguing statement of Effros' theorem and for not telling me a proof, to David Bellamy and Henryk Torunczyk for helpful and stimulating conversations on topics related to this paper, and lastly to Judy Kennedy for encouraging me to write the paper. (See [14].)

2. Basic definitions and the statement of Theorem 1. A *topological group* is a group G endowed with a topology which makes the following two functions continuous:

$$(g,h) \mapsto gh : G \times G \to G \quad \text{and} \quad g \mapsto g^{-1} : G \to G.$$

An *action* of a topological group G on a topological space X is a continuous function

$$(g,x) \mapsto gx : G \times X \to X$$

such that $(\mathrm{id}_G)x = x$ for every $x \in X$ and $g(hx) = (gh)x$ for $g, h \in G$ and $x \in X$. Observe that for each $g \in G$, the map $x \mapsto gx : X \to X$ is a homeomorphism whose inverse is the map $x \mapsto g^{-1}x : X \to X$.

We introduce some useful terminology. Suppose a topological group G acts on a topological space X. For $H, K \subset G$ and $Y \subset X$, define

$$HK = \{hk : h \in H \text{ and } k \in K\},$$
$$H^{-1} = \{h^{-1} : h \in H\}, \quad \text{and}$$
$$HY = \{hy : h \in H \text{ and } y \in Y\}.$$

In addition, for $g \in G$ and $x \in X$ define

$$gH = \{g\}H, \quad Hg = H\{g\}, \quad gY = \{g\}Y, \quad \text{and} \quad Hx = H\{x\}.$$

Suppose a topological group G acts on a topological space X. The action of G on X is *transitive* if $Gx = X$ for each $x \in X$. The action of G on X is *micro-transitive* if for every $x \in X$ and every neighborhood U of id_G in G, Ux is a neighborhood of x in X. (In this paper, a *neighborhood* of a point in a topological space means a subset of the space which contains the point in its interior; a neighborhood need not be an open set.)

A subset of a topological space is *nowhere dense* if its closure has empty interior. A topological space is of the *first category* if it can be represented as the union of a countable collection of nowhere dense subsets. A topological space which is not of the first category is of the *second category*. Recall that a metric on a topological space is *complete* if every Cauchy sequence with respect to this metric converges. The *Baire Category Theorem* asserts that every complete metric space is of the second category. (See [6, Theorem 4.1, p. 299].)

Recall that a topological space is *separable* if it has a countable dense subset.

By a *(complete) metric group* we shall mean a topological group whose topology is induced by a (complete) metric.

Suppose ρ is a metric on a topological space X and $S \subset X$. The *ρ-diameter* of S is the number $\sup\{\rho(x,y): x, y \in S\}$ and is denoted $\rho\text{-diam}(S)$.

THEOREM 1. *Suppose that a separable complete metric group G acts transitively on a metric space X. Then the following are equivalent.*

(A) *G acts micro-transitively on X.*

(B) *X has a complete metric.*

(C) *X is of the second category.*

3. The proof of Theorem 1. Throughout this section, we suppose that G is a complete metric group which acts transitively on a metric space X. For each $x \in X$, define the map $\gamma_x: G \to X$ by $\gamma_x(g) = gx$ for $g \in G$.

LEMMA 1. *The following are equivalent.*

(a) *G acts micro-transitively on X.*

(b) *$\gamma_x: G \to X$ is an open map for every $x \in X$.*

(c) *$\gamma_x: G \to X$ is an open map for some $x \in X$.*

Proof. First assume (a). We shall prove (b). Let $x \in X$. To see that $\gamma_x: G \to X$ is an open map, let U be an open subset of G, and take $g \in U$. It suffices to show that $\gamma_x(U)$ is a neighborhood of $\gamma_x(g)$ in X. Since $g^{-1}U$ is a neighborhood of id_G in G, and since G acts micro-transitively on X, then $g^{-1}Ux$ is a neighborhood of x in X. It follows that $Ux = \gamma_x(U)$ is a neighborhood of $gx = \gamma_x(g)$ in X.

Obviously (b) implies (c). To see that (c) implies (b), assume that $\gamma_x: G \to X$ is an open map for some particular $x \in X$. Let $z \in X$. We shall show that $\gamma_z: G \to X$ is also an open map. Since G acts transitively on X, there is an $h \in G$ such that $hx = z$. A homeomorphism $\tau: G \to G$ is defined by the formula $\tau(g) = gh$ for $g \in G$. It is easily verified that $\gamma_z = \gamma_x \circ \tau$. We conclude that γ_z is an open map.

Finally we prove that (b) implies (a). Assume (b). Let $x \in X$ and let U be a neighborhood of id_G in G. Then $\gamma_x(\mathrm{int}(U)) = (\mathrm{int}(U))x$ must be an open subset of X. Since $x \in (\mathrm{int}(U))x$, then Ux is a neighborhood of x in X. This establishes (a). □

Proof that (A) *implies* (B). Suppose that G acts micro-transitively on X. Fix $x \in X$, and consider the map $\gamma_x: G \to X$. The transitivity of the action of G on X implies that γ_x is surjective, and Lemma 1 implies that γ_x is open. Thus, X is the image of G under an open map. Now the existence of a complete metric on X follows directly from a theorem of Hausdorff [11] which states that a metrizable open image of a complete metric space has a complete metric. Because the proof of Hausdorff's result is non-trivial, and because it can't be found in standard topology texts, we have given a sketch of it in the Appendix. □

Proof that (B) *implies* (C). Here we simply invoke the Baire Category Theorem. □

Proof that (C) *implies* (A). We begin with two definitions. X is *G-countably covered* if for every $x \in X$ and every neighborhood U of id_G in G, there is a sequence $\{h_i\}$ of homeomorphisms of X such that $\{h_i(Ux): i \geq 1\}$ covers X. The action of G on X is *weakly micro-transitive* if for every $x \in X$ and every neighborhood U of id_G in G, $\mathrm{cl}(Ux)$ is a neighborhood of x in X.

The following three lemmas obviously entail that (C) implies (A).

LEMMA 2. *If G is separable, then X is G-countably covered.*

LEMMA 3. *If X is of the second category and is G-countably covered, then G acts weakly micro-transitively on X.*

LEMMA 4. *If G acts weakly micro-transitively on X, then G acts micro-transitively on X.*

Proof of Lemma 2. Suppose G is separable. Let $x \in X$ and let U be a neighborhood of id_G in G. Then the collection $\{gU: g \in G\}$ covers G. Since G is a separable metric space, there is a sequence $\{g_i\}$ in G such that $\{g_i U: i \geq 1\}$ covers G. Then $\{g_i Ux: i \geq 1\}$ covers X, because G acts transitively on X. As multiplication on the left by g_i defines a homeomorphism of X, we conclude that X is G-countably covered. □

Proof of Lemma 3. Assume that X is of the second category and is G-countably covered. Let $x \in X$ and let U be a neighborhood of id_G in G. We must show that $\mathrm{cl}(Ux)$ is a neighborhood of x in X.

There is a neighborhood V of id_G in G such that $V^{-1}V \subset U$. By hypothesis, there is a sequence $\{h_i\}$ of homeomorphisms of X such that $\{h_i(Vx): i \geq 1\}$ covers X. Since X is of the second category, then for some $i \geq 1$, $\mathrm{cl}(h_i(Vx))$ must have non-empty interior. Consequently, $\mathrm{cl}(Vx)$ has non-empty interior. As any non-empty open subset of $\mathrm{cl}(Vx)$ must intersect Vx, it follows that there is a $g \in V$ such that $gx \in \mathrm{int}(\mathrm{cl}(Vx))$. Consequently, $x \in g^{-1}(\mathrm{int}(\mathrm{cl}(Vx))) = \mathrm{int}(\mathrm{cl}(g^{-1}Vx)) \subset \mathrm{int}(\mathrm{cl}(V^{-1}Vx)) \subset \mathrm{int}(\mathrm{cl}(Ux))$. We conclude that $\mathrm{cl}(Ux)$ is a neighborhood of x in X. □

Proof of Lemma 4. Suppose that G acts weakly micro-transitively on X. Let ρ be a metric on X, and let σ be a *complete* metric on G.

Let $x_0 \in X$ and let U be a neighborhood of id_G in G. We must show that Ux is a neighborhood of x in X. To this end, let U_0 be a neighborhood of id_G in G such that $(\mathrm{cl}(U_0))^{-1}(\mathrm{cl}(U_0)) \subset U$. Since G acts weakly micro-transitively on X, there is an open subset M_0 of X such that $x_0 \in M_0 \subset \mathrm{cl}(U_0 x_0)$. We shall prove that $M_0 \subset U x_0$.

Take $y_0 \in M_0$. We must produce a $g \in U$ such that $g x_0 = y_0$.

In the special case that G is locally compact and U_0 is chosen to have compact closure, we could proceed as follows. Since $M_0 \subset \mathrm{cl}(U_0 x_0)$, there is a sequence $\{g_i\}$ in U_0 such that $\{g_i x_0\}$ converges to y_0. Then some subsequence of $\{g_i\}$ would converge to a $g \in \mathrm{cl}(U_0)$ such that $gx_0 = y_0$. Unfortunately, in the general case, there is no guarantee that $\{g_i\}$ or any of its subsequences converges in G. Instead, a more complicated strategy must be adopted.

It is at this point that we exploit the technique introduced by Homma in [12]. Roughly speaking, we shall alternate between moving x_0 toward y_0 and y_0 toward x_0, planning ahead at each move so that the subsequent move is possible and is close to id_G in G.

We now give the details. First, set $V_0 = U_0$. Invoke the weak micro-transitivity of the action of G on X to obtain an open subset N_0 of X such that $y_0 \in N_0 \subset \mathrm{cl}(V_0 y_0)$.

We shall construct eight sequences:

sequences $\{g_i\}$ and $\{h_i\}$ in G,
sequences $\{x_i\}$ and $\{y_i\}$ in X,
sequences $\{U_i\}$ and $\{V_i\}$ of neighborhoods of id_G in G, and
sequences $\{M_i\}$ and $\{N_i\}$ of open subsets of X.

These eight sequences are constructed to satisfy the following thirteen properties.

(1_i) $g_i \in U_{i-1}$.
(2_i) $h_i \in V_{i-1}$.
(3_i) $x_i = g_i x_{i-1}$.
(4_i) $y_i = h_i y_{i-1}$.
(5_i) $x_i \in N_{i-1}$.
(6_i) $y_i \in M_i$.
(7_i) $U_i g_i \cdots g_1 \subset U_0$.
(8_i) $V_i h_i \cdots h_1 \subset V_0$.
(9_i) σ-diam$(U_i g_i \cdots g_1) < 2^{-i}$.
(10_i) σ-diam$(V_i h_i \cdots h_1) < 2^{-i}$.
(11_i) $x_i \in M_i \subset \mathrm{cl}(U_i x_i)$.
(12_i) $y_i \in N_i \subset \mathrm{cl}(V_i y_i)$.
(13_i) ρ-diam$(M_i) < 1/i$.

The construction of the eight sequences proceeds by induction. We already have U_0, V_0, x_0, y_0, M_0, and N_0. Let $i \geq 1$ and inductively assume that for $1 \leq k \leq i-1$, we have g_k, h_k, x_k, y_k, U_k, V_k, M_k, and N_k satisfying (1_k) through (13_k).

(6_{i-1}), (12_{i-1}) and (11_{i-1}) imply that

$$y_{i-1} \in M_{i-1} \cap N_{i-1} \subset \mathrm{cl}(U_{i-1} x_{i-1}).$$

Hence, there is a $g_i \in U_{i-1}$ such that $g_i x_{i-1} \in N_{i-1}$; so (1_i) holds. Set $x_i = g_i x_{i-1}$; then (3_i) and (5_i) hold. (7_{i-1}) and (1_i) imply that $g_i g_{i-1} \cdots g_1 \in U_0$. It follows that we can choose a neighborhood U_i of id_G in G so that (7_i) and (9_i) hold. Since G acts weakly micro-transitively on X, we can find an open subset M_i of X satisfying (11_i) and (13_i). So far, we have g_i, x_i, U_i, and M_i satisfying (1_i), (3_i), (5_i), (7_i), (9_i), (11_i), and (13_i).

(5_i), (11_i), and (12_{i-1}) imply that

$$x_i \in N_{i-1} \cap M_i \subset \mathrm{cl}(V_{i-1} y_{i-1}).$$

Hence, there is an $h_i \in V_{i-1}$ such that $h_i y_{i-1} \in M_i$; so (2_i) holds. Set $y_i = h_i y_{i-1}$; then (4_i) and (6_i) hold. (8_{i-1}) and (2_i) imply that $h_i h_{i-1} \cdots h_1 \in V_0$. It follows that we can choose a neighborhood V_i of id_G in G so that (8_i) and (10_i) hold.

Since G acts weakly micro-transitively on X, we can find an open subset N_i of X satisfying (12_i). We have now completed the verification that the eight sequences can be constructed as desired.

For each $i \geq 1$, set $\tilde{g}_i = g_i \cdots g_1$ and set $\tilde{h}_i = h_i \cdots h_1$. From (1_i), (2_i), (9_i), and (10_i), we deduce that $\{\tilde{g}_i\}$ and $\{\tilde{h}_i\}$ are Cauchy sequences with respect to the metric σ on G. Since σ is a complete metric, it follows that $\{\tilde{g}_i\}$ and $\{\tilde{h}_i\}$ converge to elements g and h (respectively) of G. (7_i) and (8_i) imply that $\{\tilde{g}_i\} \subset U_0$ and $\{\tilde{h}_i\} \subset V_0$. Therefore, $g \in \mathrm{cl}(U_0)$ and $h \in \mathrm{cl}(V_0) = \mathrm{cl}(U_0)$. Now the choice of U_0 insures that $h^{-1}g \in U$.

It follows from (3_i) and (4_i) that $x_i = \tilde{g}_i x_0$ and $y_i = \tilde{h}_i y_0$ for each $i \geq 1$. Hence, $\{x_i\}$ converges to gx_0 and $\{y_i\}$ converges to hy_0. For each $i \geq 1$, since (6_i) and (11_i) imply that x_i and $y_i \in M_i$, then (13_i) implies that $\rho(x_i, y_i) < 1/i$. We conclude that $gx_0 = hy_0$. Thus, $h^{-1}g \in U$ and $(h^{-1}g)x_0 = y_0$. □

4. The Open Mapping Theorem and Effros' theorem. In this section, we illustrate the strength of Theorem 1 by deriving two known results from it. The first is the fundamental proposition of functional analysis known as the Open Mapping Theorem. (See [7, Theorem 1, p. 55].) (We thereby provide further justification for Torunczyk's name for Theorem 1: the *Open Mapping Principle.*) The second is Effros' theorem (Theorem 2.1 of [8]).

Recall that a *Frechet space* is a topological vector space whose topology is induced by a complete metric.

THE OPEN MAPPING THEOREM. *Suppose $\Lambda: E \to F$ is a continuous linear map between Frechet spaces. If Λ is surjective, then it is an open map.*

Proof. We regard E as a topological group with respect to vector addition, and we define an action of E on F by

$$(x, y) \mapsto \Lambda(x) + y : E \times F \to F.$$

Assume that Λ is surjective. Then E acts transitively on F.

It suffices to prove that E acts micro-transitively on F. For then, according to Lemma 1, the map $x \mapsto \Lambda(x) + 0 : E \to F$ is open.

In the special case that E is separable, micro-transitivity follows directly from Theorem 1. In the general case, without the separability hypothesis, we can't use Lemma 2. However, we shall still follow the outline of the proof that (C) implies (A) in Section 3. We shall give a direct proof that F is E-countably covered. We shall then invoke Lemmas 3 and 4 to complete the proof.

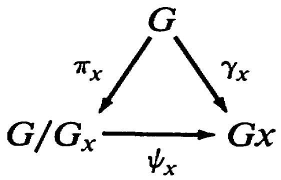
To prove that F is E-countably covered, let $y \in F$ and let V be an open neighborhood of 0 in E. Since the sequence $\{iV : i \geq 1\}$ covers E, and since Λ is surjective, then the sequence $\{\Lambda(iV) : i \geq 1\}$ covers F. For each $i \geq 1$, define the homeomorphism h_i of F by $h_i(z) = i(z - y)$ for $z \in F$. Then $h_i(\Lambda(V) + y) = \Lambda(iV)$ for each $i \geq 1$. Hence, $\{h_i(\Lambda(V) + y) : i \geq 1\}$ covers F. This proves F is E-countably covered. We now invoke Lemmas 3 and 4 to conclude that E acts micro-transitively on F. □

We now introduce definitions needed for the statement of Effros' theorem (Theorem 2.1 of [8]). For the remainder of this section suppose that a topological group G acts on a topological space X.

For each $x \in X$, the set $Gx = \{gx : g \in G\}$ is called the *orbit* of x under the action of G. Notice that distinct orbits are disjoint. The set $\{Gx : x \in X\}$ of all orbits is called the *orbit space* determined by the action of G on X, and is denoted X/G. The *natural projection* $\pi : X \to X/G$ is defined by the formula $\pi(x) = Gx$ for $x \in X$. X/G is given the quotient topology: a subset U of X/G is an open subset of X/G if and only if $\pi^{-1}(U)$ is an open subset of X. The natural projection $\pi : X \to X/G$ is an open map; indeed, if V is an open subset of X, then so is $\pi^{-1}(\pi(V)) = GV$.

Let $x \in X$. Define the map $\gamma_x : G \to Gx$ by $\gamma_x(g) = gx$ for $g \in G$. According to Lemma 1, G acts micro-transitively on the orbit Gx if and only if $\gamma_x : G \to Gx$ is an open map.

Let $x \in X$. Set $G_x = \{g \in G : gx = x\}$. G_x is a subgroup of G called the *stabilizer subgroup* of x. Let G_x act on G by multiplication on the right. The orbit space of this action, G/G_x, is the set $\{gG_x : g \in G\}$ of left cosets of G_x in G. Let $\pi_x : G \to G/G_x$ denote the natural projection; thus, $\pi_x(g) = gG_x$ for $g \in G$. As noted above, π_x is an open map.



Let $x \in X$. For $g, h \in G$, $gG_x = hG_x$ if and only if $gx = hx$. Hence, for $g, h \in G$, $\pi_x(g) = \pi_x(h)$ if and only if $\gamma_x(g) = \gamma_x(h)$. It follows that a *bijection*

$$\psi_x : G/G_x \to Gx$$

is defined by the formula $\psi_x(gG_x) = gx$ for $g \in G$. So $\psi_x \circ \pi_x = \gamma_x$. ψ_x is continuous, because γ_x is continuous and π_x is an open map. Since π_x is continuous, then ψ_x is an open map if and only if γ_x is an open map. These observations entail the next lemma.

LEMMA 5. *Let $x \in X$. G acts micro-transitively on the orbit Gx if and only if $\psi_x : G/G_x \to Gx$ is a homeomorphism.*

Recall that a topological space X is T_0 if for each pair of distinct points $x, y \in X$, either $x \notin \mathrm{cl}\{y\}$ or $y \notin \mathrm{cl}\{x\}$.

EFFROS' THEOREM (Theorem 2.1 of [8]). *Suppose a separable complete metric group G acts on a separable complete metric space X. Then the following are equivalent.*

(A) *For each $x \in X$, $\psi_x : G/G_x \to Gx$ is a homeomorphism.*
(B) *Each orbit is of the second category (in itself).*
(C) *Each orbit is a G_δ subset of X.*
(D) *X/G is T_0.*

Proof. It is convenient to add a fifth condition:

(∗) G acts micro-transitively on each orbit.

Lemma 5 establishes the equivalence of (A) and (∗). The equivalence of (B) and (∗) follows from Theorem 1.

Since X has a complete metric, then according to a theorem of Mazurkiewicz [6, Theorem 8.3, p. 308], a subset of X is G_δ if and only if it has a complete metric. Hence, (C) is equivalent to the statement that every orbit has a complete metric. The latter statement is equivalent to (∗) by Theorem 1.

The equivalence of (C) and (D) is the content of the following lemma. □

LEMMA 6. *Suppose a topological group G acts on a separable complete metric space X. Then each orbit is a G_δ subset of X if and only if X/G is a T_0 space.*

Proof. First assume that each orbit is a G_δ subset of X. To prove that X/G is T_0, suppose that $x, y \in X$ and $\pi(x) \in \mathrm{cl}_{X/G}\{\pi(y)\}$ and $\pi(y) \in \mathrm{cl}_{X/G}\{\pi(x)\}$. We shall show that $\pi(x) = \pi(y)$.

We assert that $Gx \subset \mathrm{cl}_X(Gy)$. Indeed, suppose there is a $g \in G$ such that $gx \notin \mathrm{cl}_X(Gy)$. Then gx has an open neighborhood U in X which is disjoint from Gy. Since $\pi: X \to X/G$ is an open map, it follows that $\pi(U)$ is an open neighborhood of $\pi(gx) = \pi(x)$ in X/G which is disjoint from $\pi(Gy) = \pi(y)$. This contradicts the hypothesis that $\pi(x) \in \mathrm{cl}_{X/G}\{\pi(y)\}$. Our assertion is proved. A similar argument proves that $Gy \subset \mathrm{cl}_X(Gx)$. We conclude that $Gx \cup Gy \subset \mathrm{cl}_X(Gx) \cap \mathrm{cl}_X(Gy)$.

Gx and Gy are both dense subsets of $\mathrm{cl}_X(Gx) \cap \mathrm{cl}_X(Gy)$, because Gx is dense in $\mathrm{cl}_X(Gx)$ and Gy is dense in $\mathrm{cl}_X(Gy)$. Our hypothesis implies that Gx and Gy are both G_δ subsets of $\mathrm{cl}_X(Gx) \cap \mathrm{cl}_X(Gy)$. $\mathrm{cl}_X(Gx) \cap \mathrm{cl}_X(Gy)$ has a complete metric, because it is a closed subset of the complete metric space X. One version of the Baire Category Theorem [6, Theorem 4.1, p. 299] states that in a complete metric space, the intersection of two dense G_δ's is dense. Consequently, $Gx \cap Gy$ must be dense in $\mathrm{cl}_X(Gx) \cap \mathrm{cl}_X(Gy)$. In particular, $Gx \cap Gy \neq \varnothing$. It follows that $Gx = Gy$. So $\pi(x) = \pi(y)$.

In [8] there is a short and elementary argument establishing the opposite direction of this proof. (See the proof of (4)⇒(3) on p. 41 of [8].) For completeness, we recall the idea.

The separable metric space X has a countable basis of open sets $\{U_i\}$. We assert that $\{\pi(U_i)\}$ is a countable basis of open sets for X/G. Each $\pi(U_i)$ is an open set because $\pi: X \to X/G$ is an open map. Furthermore, if $x \in X$ and V is a neighborhood of $\pi(x)$ in X/G, then the continuity of π implies that $\pi(x) \in \pi(U_i) \subset V$ for some $i \geq 1$.

Now assume that X/G is T_0. Let $x \in X$. Set

$$V_i = \begin{cases} \pi(U_i) & \text{if } \pi(x) \in \pi(U_i), \\ X/G - \pi(U_i) & \text{if } \pi(x) \notin \pi(U_i). \end{cases}$$

The fact that X/G is T_0 implies that $\bigcap\{V_i : i \geq 1\} = \{\pi(x)\}$. Hence,

$$\bigcap\{\pi^{-1}(V_i) : i \geq 1\} = \pi^{-1}(\pi(x)) = Gx.$$

Since each V_i is either open or closed in X/G, then each $\pi^{-1}(V_i)$ is either open or closed in X. Thus, each $\pi^{-1}(V_i)$ is a G_δ subset of X. We conclude that Gx is a G_δ subset of X. □

5. Homogeneity implies micro-homogeneity. In this section, we introduce the concept of *micro-homogeneity*. Theorem 1 can be interpreted as a relationship between the homogeneity and micro-homogeneity of a topological space. We establish conditions under which this relationship is valid in Theorem 2. We present two examples which illustrate limits to the validity of this relationship, and pose several questions concerning this relationship.

Suppose X is a topological space. Let $\mathcal{H}(X)$ denote the homeomorphism group of X. The *natural action* of $\mathcal{H}(X)$ on X is defined by the formula $(h,x)\mapsto h(x)$: $\mathcal{H}(X)\times X\to X$. A topology on $\mathcal{H}(X)$ is *admissible* if it makes $\mathcal{H}(X)$ a topological group and makes the natural action of $\mathcal{H}(X)$ on X continuous.

Suppose X is a topological space. For $U\subset\mathcal{H}(X)$, a subset Z of X is *U–homogeneous* if for all $y,z\in Z$, there is an $h\in U$ such that $h(y)=z$. X is *homogeneous* if it is $\mathcal{H}(X)$-homogeneous. Suppose $\mathcal{H}(X)$ is endowed with an admissible topology. X is *micro-homogeneous* (with respect to the topology on $\mathcal{H}(X)$) if for every neighborhood U of id_X in $\mathcal{H}(X)$, each point of X has a U-homogeneous neighborhood. Thus X is micro-homogeneous if and only if, for every neighborhood U of id_X in $\mathcal{H}(X)$, X is covered by U-homogeneous open sets. In the case that X is a compact metric space with metric ρ, we observe that X is micro-homogeneous if and only if, for every neighborhood U of id_X in $\mathcal{H}(X)$, there is an $\epsilon>0$ such that if $x,z\in X$ and $\rho(x,z)<\epsilon$ then $h(x)=z$ for some $h\in U$. This is proved by taking ϵ to be a *Lebesgue number* of a cover of X by U-homogeneous open sets [7, Theorem 4.5, p. 234].

The next lemma connects these homogeneity notions with the transitivity concepts of previous sections.

LEMMA 7. *Suppose X is a topological space and $\mathcal{H}(X)$ is endowed with an admissible topology. X is homogeneous if and only if $\mathcal{H}(X)$ acts transitively on X. X is micro-homogeneous if and only if $\mathcal{H}(X)$ acts micro-transitively on X.*

Proof. The first assertion is immediate. We prove the second.

Suppose X is micro-homogeneous. Let U be a neighborhood of id_X in $\mathcal{H}(X)$ and let $x\in X$. Then x has a neighborhood N in X which is U-homogeneous. Hence, $Ux\supset N$. This proves that $\mathcal{H}(X)$ acts micro-transitively on X.

Now suppose that $\mathcal{H}(X)$ acts micro-transitively on X. Let U be a neighborhood of id_X in $\mathcal{H}(X)$ and let $x\in X$. There is a neighborhood V of id_X in $\mathcal{H}(X)$ such that $VV^{-1}\subset U$. Then Vx is a neighborhood of x in X. If $y,z\in Vx$, then there are $g,h\in V$ such that $g(x)=y$ and $h(x)=z$. Hence, $h\circ g^{-1}\in VV^{-1}\subset U$ and $h\circ g^{-1}(y)=z$. Consequently, Vx is U-homogeneous. This proves X is micro-homogeneous. □

Lemma 7 makes it clear that Theorem 1 applies to the notions of homogeneity and micro-homogeneity. Indeed, suppose X is a metric space of the second category, and suppose $\mathcal{H}(X)$ is endowed with an admissible topology which makes it a separable complete metric group. In this situation, Theorem 1 implies that if X is homogeneous then X is micro-homogeneous. To take advantage of this observation, we must find admissible topologies which make homeomorphism groups into separable complete metric groups. In the next paragraph we introduce a homeomorphism group topology which has these qualities when the underlying space is a locally compact separable metric space.

Suppose X and Y are Hausdorff spaces, and suppose $\mathcal{M}$ is a set of maps from X to Y. For $K \subset X$ and $U \subset Y$, set $\langle K, U\rangle = \{f \in \mathcal{M} : f(K) \subset U\}$. Recall that the *compact-open topology* on $\mathcal{M}$ is the topology which has a subbasis consisting of all sets of the form $\langle K, U\rangle$ where K is a compact subset of X and U is an open subset of Y. The *complemented compact-open topology* on $\mathcal{M}$ is the topology which has a subbasis consisting of all sets of the form $\langle K, U\rangle$ where K is a compact subset of X and U is an open subset of Y, as well as all sets of the form $\langle X-V, Y-L\rangle$ where V is an open subset of X and L is a compact subset of Y. Observe that if X is compact then the compact-open topology on $\mathcal{M}$ and the complemented compact-open topology on $\mathcal{M}$ coincide.

The admissibility of the complemented compact-open topology is studied in [4]. In particular, Theorem 3 of [4] tells us that if X is a locally compact Hausdorff space, then the complemented compact-open topology on $\mathcal{H}(X)$ is admissible. (*Warning:* In [4], the term *admissible* has a weaker meaning than it does here. In [4], *admissible* means only that the natural action of $\mathcal{H}(X)$ on X is continuous; it does not entail the continuity of the group operations on $\mathcal{H}(X)$.) When X is not locally compact, the complemented compact-open topology on $\mathcal{H}(X)$ may not be admissible. However, for any Hausdorff space X, the complemented compact-open topology on $\mathcal{H}(X)$ is a lower bound of all the admissible topologies on $\mathcal{H}(X)$. In other words, when X is Hausdorff, every admissible topology on $\mathcal{H}(X)$ contains the complemented compact-open topology. This is established by the proof of Theorem 3 of [4], although the statement of Theorem 3 asserts it only for locally compact X.

By introducing one-point compactifications, the next lemma establishes a basic connection between the compact-open topology and the complemented compact-open topology.

LEMMA 8. *Suppose X and Y are locally compact Hausdorff spaces. Let $X^* = X \cup \{\infty_X\}$ and $Y^* = Y \cup \{\infty_Y\}$ denote the one-point compactifications of X and Y, respectively. Suppose $\mathcal{M}$ is a set of maps from X to Y, and $\mathcal{N}$ is a set of maps from X^* to Y^* with the following two properties.*

(1) *For each $f \in \mathcal{M}$, there is an $f^* \in \mathcal{N}$ such that $f^* | X = f$ and $f^*(\infty_X) = \infty_Y$.*

(2) *If $g \in \mathcal{N}$ such that $g(X) \subset Y$ and $g(\infty_X) = \infty_Y$, then $g = f^*$ for some $f \in \mathcal{M}$. Endow $\mathcal{M}$ with the complemented compact-open topology, endow $\mathcal{N}$ with the compact-open topology, and set $\mathcal{M}^* = \{f^* : f \in \mathcal{M}\}$. Then $f \mapsto f^* : \mathcal{M} \to \mathcal{N}$ is an embedding. Also, if ρ is a metric on Y which extends to a metric on Y^*, then a metric σ on $\mathcal{M}$, inducing the complemented compact-open topology on $\mathcal{M}$, is defined by the formula*

$$\sigma(f, g) = \sup\{\rho(f(x), g(x)) : x \in X\}$$

for $f, g \in \mathfrak{M}$. *Furthermore, if* X *and* Y *are* σ*-compact, then* $\mathfrak{M}^*$ *is a* G_δ *subset of* $\mathfrak{N}$.

Proof. For each $F \subset \mathfrak{M}$, set $F^* = \{f^* : f \in F\}$.

To prove that $f \mapsto f^* : \mathfrak{M} \to \mathfrak{N}$ is continuous, let $f \in \mathfrak{M}$, let K be a compact subset of X^*, and let U be an open subset of Y^* such that $f^* \in \langle K, U\rangle$. We assert that $\langle K \cap X, U \cap Y\rangle$ is open in the complemented compact-open topology on $\mathfrak{M}$, and that $\langle K \cap X, U \cap Y\rangle^* \subset \langle K, U\rangle$. Both assertions are obvious in the case that $K \subset X$; so assume $\infty_X \in K$. Then $\infty_Y = f(\infty_X) \in U$. So $Y - U$ is compact. As $U \cap Y = Y - (Y - U)$, the first assertion follows. If $g \in \langle K \cap X, U \cap Y\rangle$, then $g^*(\infty_X) = \infty_Y \in U$; so $g^* \in \langle K, U\rangle$. This proves the second assertion.

To prove that $f \mapsto f^* : \mathfrak{M} \to \mathfrak{M}^*$ is an open map, we must consider two cases. First, if K is a compact subset of X, and U is an open subset of Y, then it is clear that $\langle K, U\rangle^* = \langle K, U\rangle \cap \mathfrak{M}^*$. Second, if V is an open subset of X, and L is a compact subset of Y, then it is clear that $X^* - V$ is compact, $Y^* - L$ is open, and $\langle X - V, Y - L\rangle^* = \langle X^* - V, Y^* - L\rangle \cap \mathfrak{M}^*$.

Suppose that ρ is a metric on Y that extends to a metric ρ^* on Y^*. The above formula for σ defines a metric on $\mathfrak{M}$; the issue is whether σ induces the complemented compact-open topology on $\mathfrak{M}$. According to [6, Theorem 8.2(3), p. 270], a metric σ^* on $\mathfrak{N}$, inducing the compact-open topology on $\mathfrak{N}$, is defined by the formula

$$\sigma^*(f, g) = \sup\{\rho^*(f(x), g(x)) : x \in X^*\}$$

for $f, g \in \mathfrak{N}$. For $f, g \in \mathfrak{M}$, it is clear that $\sigma^*(f^*, g^*) = \sigma(f, g)$. Thus, $f \mapsto f^* : (\mathfrak{M}, \sigma) \to (\mathfrak{N}, \sigma^*)$ is an isometry. Since $f \mapsto f^* : \mathfrak{M} \to \mathfrak{N}$ is an embedding, it follows that σ induces the complemented compact-open topology on $\mathfrak{M}$.

We now assume that X and Y are σ-compact. Then X and Y are covered by sequences of compacta $\{K_n\}$ and $\{L_n\}$, respectively. Since

$$\mathfrak{M}^* = \{g \in \mathfrak{N} : g(X) \subset Y \text{ and } g(\infty_X) = \infty_Y\},$$

then

$$\mathfrak{M}^* = \left(\bigcap \{\langle K_n, Y\rangle : n \geq 1\}\right) \cap \left(\bigcap \{\langle \{\infty_X\}, Y^* - L_n\rangle : n \geq 1\}\right).$$

We conclude that $\mathfrak{M}^*$ is a G_δ subset of $\mathfrak{N}$. □

We now use Lemma 8 to establish that, under the complemented compact-open topology, the homeomorphism group of a locally compact separable metric space is tractable.

LEMMA 9. *Suppose* X *is a locally compact separable metric space, and* $\mathfrak{H}(X)$ *is endowed with the complemented compact-open topology. Then both* X *and* $\mathfrak{H}(X)$ *are separable complete metric spaces.*

Proof. Let $X^* = X \cup \{\infty\}$ denote the one-point compactification of X, and endow $\mathfrak{H}(X^*)$ with the compact-open topology. Then X^* is a compact metric space, and X is an open subset of X^*. We appeal to the theorem of Mazurkiewicz [6, Theorem 8.3, p. 308] to conclude that X is a separable complete metric space.

It is well known that $\mathcal{H}(X^*)$ is a separable complete metric space. Indeed, if ρ is a metric on X^*, then a complete metric σ on $\mathcal{H}(X^*)$ is defined by the formula

$$\sigma(g,h)=\sup\{\rho(g(x),h(x)):x\in X^*\}+\sup\{\rho(g^{-1}(x),h^{-1}(x)):x\in X^*\}$$

for $g,h\in\mathcal{H}(X^*)$. $\mathcal{H}(X^*)$ is separable by Theorem 5.2 [6, p. 265].

Lemma 8 applies here with $\mathcal{H}(X)$ and $\mathcal{H}(X^*)$ substituted for $\mathfrak{M}$ and $\mathfrak{N}$, respectively. As X is σ-compact, we conclude that $\mathcal{H}(X)$ is homeomorphic to a G_δ subset of $\mathcal{H}(X^*)$. Now the theorem of Mazurkiewicz [6, Theorem 8.3, p. 308] implies that $\mathcal{H}(X)$ is a separable complete metric space. □

Lemmas 7 and 9 easily transform Theorem 1 into the following generalization of Lemma 4 of [10].

THEOREM 2. *Suppose X is a locally compact separable metric space, and $\mathcal{H}(X)$ is endowed with the complemented compact-open topology. If X is homogeneous, then it is micro-homogeneous.*

The problem of generalizing Theorem 2 to the non-metric setting provokes the following questions.

QUESTION 1. Suppose X is a compact Hausdorff space, and $\mathcal{H}(X)$ is endowed with the compact-open topology. Assume X is homogeneous. Must X be micro-homogeneous?

QUESTION 2. Suppose X is a locally compact Hausdorff space, and $\mathcal{H}(X)$ is endowed with the complemented compact-open topology. Assume X is homogeneous. Must X be micro-homogeneous?

One might hope to generalize Theorem 2 in a different direction, by enlarging the homeomorphism group topology to achieve more control at infinity. The complemented compact-open topology exerts very little control at infinity. Indeed, as Lemma 8 shows, the complemented compact-open topology is consistent with adding a single point at infinity. We now introduce two other homeomorphism group topologies which are potentially more useful because they are finer at infinity. Both topologies are admissible for a wide class of spaces. Unfortunately, as we shall illustrate, these homeomorphism group topologies don't guarantee that homogeneity implies micro-homogeneity. This situation provokes a question.

Suppose X and Y are Hausdorff spaces, and suppose $\mathfrak{M}$ is a set of maps from X to Y. The *closed-open topology* on $\mathfrak{M}$ is the topology which has a subbasis consisting of all sets of the form $\langle K,U\rangle$, where K is a closed subset of X and U is an open subset of Y. The *fine topology* on $\mathfrak{M}$ is the topology which has a basis consisting of all sets of the form $\{h\in\mathfrak{M}: h\subset W\}$, where W is an open subset of $X\times Y$. Now consider the following four topologies on $\mathfrak{M}$: the compact-open topology, the complemented compact-open topology, the closed-open topology, and the fine topology. We have listed them in order of increasing size; each is contained in the one that follows it. When X is compact, they all coincide. The list ends with the fine topology because, in the author's experience, this is the largest topology on $\mathfrak{M}$ which carries useful information about X and Y in a natural way.

If X is a normal Hausdorff space, then the closed-open topology on $\mathcal{H}(X)$ is admissible. (This is easy to prove and is left to the reader.) If X is a paracompact Hausdorff space, then the fine topology on $\mathcal{H}(X)$ is admissible. (The only difficulty to be faced in proving this statement is in establishing the continuity of the group operation $(h, g) \mapsto h \circ g: \mathcal{H}(X) \times \mathcal{H}(X) \to \mathcal{H}(X)$ with respect to the fine topology. The Composition Lemma A.10 [2, p. 33] is of aid here.)

There is a proposition, similar to Lemma 8, which relates a set of maps between two spaces with the closed-open topology to a set of maps between the Stone–Cech compactifications of the spaces with the compact-open topology. This proposition has the following corollary. Suppose X is a normal Hausdorff space, and βX is its Stone–Cech compactification. Each $h \in \mathcal{H}(X)$ has a unique extension $\beta h \in \mathcal{H}(\beta X)$. Endow $\mathcal{H}(X)$ with the closed-open topology, and endow $\mathcal{H}(\beta X)$ with the compact-open topology. Then $h \mapsto \beta h: \mathcal{H}(X) \to \mathcal{H}(\beta X)$ is an embedding. Furthermore, if X is σ-compact, then $h \mapsto \beta h$ carries $\mathcal{H}(X)$ onto a G_δ subset of $\mathcal{H}(\beta X)$. Unfortunately, as Example 1 illustrates, this information is of no value in proving that homogeneity implies micro-homogeneity with respect to the closed-open topology.

EXAMPLE 1. The underlying space of this example is $C \times \mathbf{R}$ where C is the deleted middle-thirds Cantor set in the unit interval $[0, 1]$ and $\mathbf{R}$ is the real line. $C \times \mathbf{R}$ is homogeneous because both C and $\mathbf{R}$ are homogeneous. Since $C \times \mathbf{R}$ is a locally compact separable metric space, then according to Theorem 2, $C \times \mathbf{R}$ is micro-homogeneous with respect to the complemented compact-open topology on $\mathcal{H}(C \times \mathbf{R})$. We shall argue that $C \times \mathbf{R}$ is *not* micro-homogeneous with respect to the closed-open topology on $\mathcal{H}(C \times \mathbf{R})$. Since the closed-open topology is contained in the fine topology, it follows that $C \times \mathbf{R}$ is not micro-homogeneous with respect to the fine topology on $\mathcal{H}(C \times \mathbf{R})$.

Endow $\mathcal{H}(C \times \mathbf{R})$ with the closed-open topology. Let $K = \{0\} \times \mathbf{R}$ and let $U = \{(x, y) \in C \times \mathbf{R}: |xy| < 1\}$. U is an open neighborhood of K in $C \times \mathbf{R}$ which *tapers* as $y \to \pm\infty$. Thus, $\langle K, U \rangle$ is an open neighborhood $\mathrm{id}_{C \times \mathbf{R}}$ in $\mathcal{H}(C \times \mathbf{R})$. We shall prove that $C \times \mathbf{R}$ is not micro-homogeneous by showing that $\langle K, U \rangle \cdot (0, 0) = \{h(0, 0): h \in \langle K, U \rangle\}$ does not contain an open neighborhood of $(0, 0)$ in $C \times \mathbf{R}$. In fact, we shall show that $\langle K, U \rangle \cdot (0, 0) \subset K$. This will suffice, because every open neighborhood of $(0, 0)$ in $C \times \mathbf{R}$ contains points (x, y) with $x > 0$. Let $h \in \langle K, U \rangle$. As K is a component of $C \times \mathbf{R}$, so is $h(K)$. Since $h(K) \subset U$, and since K is the only component of $C \times \mathbf{R}$ which is contained in U, then necessarily $h(K) = K$. As $(0, 0) \in K$, we conclude that $h(0, 0) \in K$.

For local compacta, the complemented compact-open topology is admissible. However, outside the class of local compacta, the complemented compact-open topology may fail to be admissible, and it may be difficult to find an admissible homeomorphism group topology for which homogeneity implies micro-homogeneity. Example 2 illustrates these difficulties.

EXAMPLE 2. The underlying space of this example is $J\times\mathbf{R}$ where J is the space of irrational numbers in the real line $\mathbf{R}$. J has a complete metric because it is a G_δ subset of $\mathbf{R}$; so $J\times\mathbf{R}$ is a separable complete metric space. We shall give a proof that the complemented compact-open topology on $\mathcal{H}(J\times\mathbf{R})$ is *not* admissible. $J\times\mathbf{R}$ is homogeneous because both J and $\mathbf{R}$ are homogeneous. However, $J\times\mathbf{R}$ is *not* micro-homogeneous with respect to either the closed-open topology or the fine topology on $\mathcal{H}(J\times\mathbf{R})$. A proof of this fact can be obtained from the proof given in Example 1 simply by changing all C's to J's. It is possible to describe an admissible topology $\mathcal{K}$ on $\mathcal{H}(J\times\mathbf{R})$ which makes $J\times\mathbf{R}$ microhomogeneous. $\mathcal{K}$ contains the complemented compact-open topology and is contained in the closed-open topology.

To prove that the complemented compact-open topology on $\mathcal{H}(J\times\mathbf{R})$ is not admissible, endow $\mathcal{H}(J\times\mathbf{R})$ with the complemented compact-open topology, let $x\in J$, and set $U=(x-1,x+1)\cap J$. We shall indicate why one can't find neighborhoods V of $\mathrm{id}_{J\times\mathbf{R}}$ in $\mathcal{H}(J\times\mathbf{R})$ and W of $(x,0)$ in $J\times\mathbf{R}$ so that $VW\subset U\times\mathbf{R}$. Suppose such V and W exist. We can assume that

$$V=\left(\bigcap\{\langle K_i,M_i\rangle:1\le i\le m\}\right)\cap\left(\bigcap\{\langle (J\times\mathbf{R})-N_k,(J\times\mathbf{R})-L_k\rangle:1\le k\le n\}\right),$$

where K_i and L_k are compact subsets of $J\times\mathbf{R}$ and M_i and N_k are open subsets of $J\times\mathbf{R}$. Also we can assume that $W=((x-\delta,x+\delta)\cap J)\times(-\delta,\delta)$ for some $\delta>0$. Since $\mathrm{id}_{J\times\mathbf{R}}\in V$, then $K_i\subset M_i$ for $1\le i\le m$, and $L_k\subset N_k$ for $1\le k\le n$. Let

$$C=\left(\bigcup\{K_i:1\le i\le m\}\right)\cup\left(\bigcup\{L_k:1\le k\le n\}\right).$$

C is compact. Since no compact subset of J has non-empty interior, then we can find rational numbers $p<q<r<s$ such that $(p,q)\subset(x-\delta,x+\delta)$, $(r,s)\subset(x+1,\infty)$, and $\pi(C)$ is disjoint from $(p,q)\cup(r,s)$, where $\pi:J\times\mathbf{R}\to J$ denotes projection. There is a homeomorphism $g:J\to J$ such that $g((p,q)\cap J)=(r,s)\cap J$, $g((r,s)\cap J)=(p,q)\cap J$, and $g=\mathrm{id}$ on $J-((p,q)\cup(r,s))$. Set

$$h=g\times\mathrm{id}_{\mathbf{R}}\in\mathcal{H}(J\times\mathbf{R}).$$

It is easy to verify that $h\in V$. Let $z\in(p,q)\cap J$. Then $(z,0)\in W$. So $h(z,0)\in VW$. Now $h(z,0)=(g(z),0)\in(r,s)\times\mathbf{R}\subset(x+1,\infty)\times\mathbf{R}$. Since $U\subset(x-1,x+1)$, then $h(z,0)\notin U\times\mathbf{R}$.

We shall now describe an admissible topology $\mathcal{K}$ on $\mathcal{H}(J\times\mathbf{R})$ which makes $J\times\mathbf{R}$ micro-homogeneous. Let $K\subset J\times\mathbf{R}$. For $x\in J$, let $K\,|\,x=K\cap(\{x\}\times\mathbf{R})$; and for $V\subset J$, let $K\,|\,V=K\cap(V\times\mathbf{R})$. K is *continuous* if it has the following two properties: (1) $K\,|\,x$ is compact for each $x\in J$, and (2) for every $x\in J$ and every neighborhood U of $K\,|\,x$ in $J\times\mathbf{R}$, there is a neighborhood V of x in J such that $K\,|\,V\subset U$. Appendix A of [2] is a source of information about such continuous sets. Let $\mathcal{K}$ denote the topology on $\mathcal{H}(J\times\mathbf{R})$ which has a subbasis consisting of all sets of the form $\langle K,U\rangle$ as well as all sets of the form $\langle X-U,X-K\rangle$, where K is a continuous subset of $J\times\mathbf{R}$ and U is an open subset of $J\times\mathbf{R}$. (An appropriate name for $\mathcal{K}$ is the *complemented continuous-open topology* on $\mathcal{H}(J\times\mathbf{R})$.) Each compact subset of $J\times\mathbf{R}$ is continuous, and each continuous subset of $J\times\mathbf{R}$

is closed. It follows that $\mathcal{K}$ contains the complemented compact-open topology and is contained in the closed-open topology. The proofs that $\mathcal{K}$ is admissible and that $\mathcal{K}$ makes $J\times\mathbf{R}$ micro-homogeneous are left as exercises for the reader. $\square$

The existence of the admissible topology $\mathcal{K}$ making $J\times\mathbf{R}$ micro-homogeneous prompts the following question.

QUESTION 3. Suppose X is a homogeneous complete separable metric space. Is there necessarily an admissible topology on $\mathcal{K}(X)$ which makes X micro-homogeneous?

The local compactness of $\mathbf{R}$ appears to be a crucial factor in the proof that the topology $\mathcal{K}$ on $\mathcal{K}(J\times\mathbf{R})$ has the desired properties. Thus, in studying Question 3, one should consider a space such as $J\times\ell_2$ (where J is the space of irrational numbers in the real line and ℓ_2 is the Hilbert space of square-summable sequences), because ℓ_2 is not locally compact. The author has been unable to answer Question 3 for the space $J\times\ell_2$.

6. Appendix. We sketch a proof of the following result.

A THEOREM OF F. HAUSDORFF. *If $f: X\to Y$ is an open map from a complete metric space X onto a metric space Y, then Y has a complete metric.*

This theorem originally appeared in [11]. A modern generalization of it can be found in [5].

Proof. By completing a metric on Y, we can regard Y as a dense subset of a complete metric space Z [6, Theorem 6.1, p. 304]. According to a theorem of Mazurkiewicz [6, Theorem 8.3, p. 308], it suffices to show that Y is a G_δ subset of Z.

Let ρ and σ denote complete metrics on X and Z, respectively. For each $n\geq 1$, let $\mathfrak{Z}_n$ denote the collection of all open subsets of X of ρ-diameter $<1/n$, and let $\mathcal{U}_n$ denote the collection of all open subsets of Y of σ-diameter $<1/n$.

It is easy to construct, for each $n\geq 1$, a set A_n and functions $\alpha_n: A_n\to\mathfrak{Z}_n$ and $\lambda_n: A_{n+1}\to A_n$ such that

(1) $\{\alpha_n(a): a\in A_n\}$ covers X, and

(2) $\bigcup\{\alpha_{n+1}(b): b\in\lambda_n^{-1}(a)\}=\alpha_n(a)$ for each $a\in A_n$.

We leave this construction to the reader. Some authors call the sequence $\{(\alpha_n, A_n, \lambda_n): n\geq 1\}$ a *sieve* in X.

Nor is it difficult to construct, for each $n\geq 1$, a subcollection $\mathcal{V}_n$ of $\mathcal{U}_n$ and functions $\beta_n: \mathcal{V}_n\to A_n$ and $\mu_n: \mathcal{V}_{n+1}\to\mathcal{V}_n$ such that

(3) $\mathcal{V}_n$ is a locally finite cover of Y,

(4) each element of $\mathcal{V}_{n+1}$ intersects only finitely many distinct elements of $\mathcal{V}_n$,

(5) $V\subset\mu_n(V)$ for each $V\in\mathcal{V}_{n+1}$,

(6) $V\subset f(\alpha_n(\beta_n(V)))$ for each $V\in\mathcal{V}_n$, and

(7) $\lambda_n\circ\beta_{n+1}=\beta_n\circ\mu_n$.

$$X \xrightarrow{\quad f \quad} Y$$

$$\begin{array}{ccccc}
\mathfrak{U}_{n+1} & \xleftarrow{\alpha_{n+1}} & A_{n+1} & \xleftarrow{\beta_{n+1}} & \mathcal{V}_{n+1} \\
 & & \downarrow \lambda_n & & \downarrow \mu_n \\
\mathfrak{U}_n & \xleftarrow{\alpha_n} & A_n & \xleftarrow{\beta_n} & \mathcal{V}_n
\end{array}$$

Again, we leave this construction to the reader, with the suggestion that he exploit the paracompactness of Y.

For each open subset V of Y, let V^* denote the union of all the open subsets W of Z such that $W \cap Y \subset V$. Then V^* is an open subset of Z and $V^* \cap Y = V$. Furthermore, since Y is dense in Z, $\sigma\text{-diam}(V^*) = \sigma\text{-diam}(V)$.

For each $n \geq 1$, set $G_n = \bigcup \{V^* : V \in \mathcal{V}_n\}$. Then G_n is an open subset of Z which contains Y. Thus, $Y \subset \bigcap \{G_n : n \geq 1\}$. We shall prove that Y is a G_δ subset of Z by showing that $Y = \bigcap \{G_n : n \geq 1\}$.

Let $z \in \bigcap \{G_n : n \geq 1\}$. We must show $z \in Y$. For each $n \geq 1$, there is a $V_n \in \mathcal{V}_n$ such that $z \in V_n^*$. Thus, $\bigcap \{V_i^* : 1 \leq i \leq n\}$ is a non-empty open subset of Z for each $n \geq 1$. Since Y is dense in Z, it follows that $\varnothing \neq Y \cap (\bigcap \{V_i^* : 1 \leq i \leq n\}) = \bigcap \{V_i : 1 \leq i \leq n\}$, for each $n \geq 1$.

We shall now produce a sequence $\{W_n : n \geq 1\}$ so that for each $n \geq 1$, $W_n \in \mathcal{V}_n$, $\mu_n(W_{n+1}) = W_n$, and $W_n \cap V_n \neq \varnothing$. For each $n \geq 1$, set

$$J_n = \{W \in \mathcal{V}_n : W \cap (\bigcap \{V_i : 1 \leq i \leq n+1\}) \neq \varnothing\}.$$

J_n is non-empty and finite because $V_n \in J_n$ and V_{n+1} intersects only finitely many elements of $\mathcal{V}_n$. Set $J_\infty = J_1 \times J_2 \times \cdots$. For each $n \geq 1$, we make J_n a compact Hausdorff space by endowing it with the discrete topology. We endow J_∞ the associated product topology. Then, according to the Tychonoff theorem [6, Theorem 1.4(4), p. 224], J_∞ is also a compact Hausdorff space. Next, for each $n \geq 1$, set $K_n = \{(W_1, \ldots, W_n) \in J_1 \times \cdots \times J_n : \mu(W_{i+1}) = W_i \text{ for } 1 \leq i \leq n\}$. Then, for each $n \geq 1$, it is easy to see that $K_n \neq \varnothing$, and that if $(W_1, \ldots, W_n, W_{n+1}) \in K_{n+1}$, then $(W_1, \ldots, W_n) \in K_n$. It follows that if we set $L_n = K_n \times J_{n+1} \times J_{n+2} \times \cdots$ for each $n \geq 1$, then $\{L_n : n \geq 1\}$ is a sequence of non-empty closed subsets of J_∞ which is decreasing: $L_1 \supset L_2 \supset \cdots$. Since J_∞ is compact, we conclude that $\bigcap \{L_n : n \geq 1\} \neq \varnothing$. Select an element $(W_1, W_2, \ldots) \in \bigcap \{L_n : n \geq 1\}$. Then $(W_1, \ldots, W_n) \in K_n$ for each $n \geq 1$. It follows that $W_n \in \mathcal{V}_n$, $\mu(W_{n+1}) = W_n$, and $W_n \cap V_n \neq \varnothing$ for each $n \geq 1$.

For each $n \geq 1$, let $a_n = \beta_n(W_n)$. Then $W_n \subset f(\alpha_n(a_n))$ and $\lambda_n(a_{n+1}) = a_n$. It follows that $f(\alpha_n(a_n)) \cap V_n \neq \varnothing$ for each $n \geq 1$. Hence, we can choose an $x_n \in \alpha_n(a_n)$ so that $f(x_n) \in V_n$.

For each $n \geq 1$, $x_n \in \alpha_n(a_n)$, $\rho\text{-diam}(\alpha_n(a_n)) < 1/n$, and $\alpha_{n+1}(a_{n+1}) \subset \alpha_n(a_n)$, because $\lambda_n(a_{n+1}) = a_n$. Consequently $\{x_n\}$ is a Cauchy sequence in X. Since the metric ρ is complete, it follows that $\{x_n\}$ converges to a point $x \in X$. Hence, $\{f(x_n)\}$ converges to $f(x)$.

Since $f(x_n)$ and $z \in V_n^*$ and $\sigma\text{-diam}(V_n^*) < 1/n$ for each $n \geq 1$, then $\{f(x_n)\}$ must converge to z. We conclude that $z = f(x)$. Therefore $z \in Y$. □

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