ALGEBRAIC APPROXIMATION OF MAPPINGS INTO SPHERES

J. Bochnak and Wojciech Kucharz

1. Introduction. Let $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^p$ be real algebraic sets. A map $f: X \to Y$ is said to be a polynomial map if it is the restriction to X of a polynomial map from $\mathbf{R}^n$ to $\mathbf{R}^p$. We say that a map $f: X \to Y$ is entire rational if there exist polynomials f_i and g_i in $\mathbf{R}[x_1, \dots, x_n]$, $g_i^{-1}(0) \cap X = \varnothing$, $i = 1, \dots, p$, such that $f(x) = (f_1(x)/g_1(x), \dots, f_p(x)/g_p(x))$ for x in X. Very little seems to be known about polynomial and entire rational maps between real algebraic sets, their classification, the relationship with other classes of maps, etc. In this paper we address some of these questions for entire rational maps, mostly in the case $Y = S^k = \{y \in \mathbf{R}^{k+1} \mid y_1^2 + \cdots + y_{k+1}^2 = 1\}$, the unit sphere. A different behavior of polynomial and entire rational maps is often a characteristic feature.

We denote by $\mathcal{R}(X, Y)$ the set of entire rational maps from X to Y. If X and Y are compact and nonsingular, we denote by $\mathcal{E}(X, Y)$ the set of smooth (i.e., C^∞) maps from X to Y equipped with the C^∞ topology.

THEOREM 1.1. *For each positive integer n, the set $\mathcal{R}(S^n, S^k)$ is dense in $\mathcal{E}(S^n, S^k)$, provided that $k = 1$, 2 or 4.*

This theorem contrasts with a result of Wood [18] saying that every polynomial map from S^n to S^k is a constant map if $n = 2^m > k$.

We note that in Theorem 1.1, S^n cannot be replaced by an arbitrary compact nonsingular real algebraic set.

EXAMPLE 1.2. It is shown in [13] (cf. also [5]) that for each pair (n, k) of positive integers there exist a nonsingular real algebraic set X, a smooth diffeomorphism $\varphi_{n,k}: X \to S^n \times S^k$, and a point s_0 in S^k such that $Y = \varphi_{n,k}^{-1}(S^n \times \{s_0\})$ is a smooth submanifold of X which cannot be isotoped in X to a nonsingular algebraic subset of X. It follows, by using Thom's isotopy lemma [1], that $\pi \circ \varphi_{n,k}: X \to S^k$, where $\pi: S^n \times S^k \to S^k$ is the natural projection, cannot be approximated by entire rational maps from X to S^k. If $k = 1$, 2, or 4, then $\pi \circ \varphi_{n,k}$ is not homotopic to an entire rational map. The last observation follows from the next theorem.

THEOREM 1.3. *Let X be a compact nonsingular real algebraic set and let $f: X \to S^k$ be a smooth map. If $k = 1$, 2, or 4, then the following conditions are equivalent:*

(i) *f can be approximated in the C^∞ topology by entire rational maps from X to S^k.*

(ii) *f is homotopic to an entire rational map from X to S^k.*

Received March 7, 1986.
Michigan Math. J. 34 (1987).

We do not know whether the assumption "$k=1$, 2 or 4" in Theorems 1.1 and 1.3 is necessary.

We shall more closely examine maps into S^1 and S^2. Given a compact nonsingular n-dimensional real algebraic set X, we denote by $H_k^{\mathrm{alg}}(X, \mathbf{Z}_2)$ the subgroup of $H_k(X, \mathbf{Z}_2)$ of homology classes represented by k-dimensional algebraic subsets of X ([2], [3], [5], [6]) and by $H_{\mathrm{alg}}^{n-k}(X, \mathbf{Z}_2)$ the subgroup of $H^{n-k}(X, \mathbf{Z}_2)$ of cohomology classes corresponding, via Poincaré duality, to homology classes in $H_k^{\mathrm{alg}}(X, \mathbf{Z}_2)$.

THEOREM 1.4. *Given a compact nonsingular n-dimensional real algebraic set X and a smooth map $f: X \to S^1$, the following conditions are equivalent:*

(i) *f can be approximated in the C^∞ topology by entire rational maps from X to S^1.*

(ii) *For any regular value y of f the homology class represented by $f^{-1}(y)$ in $H_{n-1}(X, \mathbf{Z}_2)$ belongs to $H_{n-1}^{\mathrm{alg}}(X, \mathbf{Z}_2)$.*

(iii) *$f^*(u)$ belongs to $H_{\mathrm{alg}}^1(X, \mathbf{Z}_2)$, where $f^*: H^1(S^1, \mathbf{Z}_2) \to H^1(X, \mathbf{Z}_2)$ is the homomorphism induced by f and u is a generator of $H^1(S^1, \mathbf{Z}_2)$.*

COROLLARY 1.5. *Let X be a compact nonsingular one-dimensional real algebraic set. Then $\mathcal{R}(X, S^1)$ is dense in $\mathcal{E}(X, S^1)$.*

Proof. By definition, $H_{\mathrm{alg}}^1(X, \mathbf{Z}_2) = H^1(X, \mathbf{Z}_2)$. The conclusion follows from Theorem 1.4. □

There are no such neat results for maps into S^2. However, we have the following.

THEOREM 1.6. *Let X be a compact nonsingular real algebraic set and let $f: X \to S^2$ be a smooth map. If $f^*: H^2(S^2, \mathbf{Z}) \to H^2(X, \mathbf{Z})$ is the zero homomorphism, then f can be approximated in the C^∞ topology by entire rational maps from X to S^2.*

The case of maps from surfaces to S^2 deserves special attention. Given a closed smooth n-dimensional submanifold Y of a compact smooth $2n$-dimensional manifold X, we denote by $\#_2(Y, Y; X)$ the modulo 2 self-intersection number of Y in X [10, pp. 132–133].

THEOREM 1.7. *Let X be a nonsingular two-dimensional real algebraic set. Assume that X is compact, connected, and nonorientable as a smooth manifold. Then $\mathcal{R}(X, S^2)$ is dense in $\mathcal{E}(X, S^2)$ in each of the following cases:*

(i) *There exists an algebraic nonsingular curve C in X such that*

$$\#_2(C, C; X) = 1.$$

(ii) *$H_1^{\mathrm{alg}}(X, \mathbf{Z}_2) = H_1(X, \mathbf{Z}_2)$.*

(iii) *The genus of X (as a smooth surface) is odd.*

We should mention that the assumption "X is nonorientable" in Theorem 1.7 is essential. Indeed, in [8] we generalize Loday's result [14], concerning polynomial maps, showing that every rational map from $S^1 \times S^1$ to S^2 is null homotopic. Thus $\mathcal{R}(S^1 \times S^1, S^2)$ is not dense in $\mathcal{E}(S^1 \times S^1, S^2)$. On the other hand there exists a nonsingular real algebraic set X diffeomorphic to $S^1 \times S^1$ such that

$\mathcal{R}(X, S^2)$ is dense in $\mathcal{E}(X, S^2)$ [8]. We do not know if there exists a nonsingular two-dimensional real algebraic set X such that X is compact, connected, and nonorientable as a smooth manifold and $\mathcal{R}(X, S^2)$ is not dense in $\mathcal{E}(X, S^2)$. We note that, in general, condition (ii) of Theorem 1.7 is not satisfied [13].

All theorems stated in this section are proved in Section 2 by making use of some results on Grassmannians and algebraic vector bundles.

2. Entire rational maps into Grassmannians. Let us start with a description of affine real algebraic models of the Grassmannians over **F**, where $\mathbf{F} = \mathbf{R}$, **C**, or **H** (the quaternions). Since the case $\mathbf{F} = \mathbf{R}$ has been considered in [2] and the other cases are similar, we shall give only a short, uniform treatment of all three cases.

Let $d(\mathbf{F}) = \dim_{\mathbf{R}} \mathbf{F}$, that is, $d(\mathbf{R}) = 1$, $d(\mathbf{C}) = 2$, and $d(\mathbf{H}) = 4$. On the (right) **F**-vector space $\mathbf{F}^n$ we define the inner product by

$$\langle x, y \rangle = \sum_{i=1}^{n} \bar{y}_i x_i,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. For z in **F**, $\bar{z}$ denotes the conjugate of z. Let $M(n, \mathbf{F})$ be the space of all $n \times n$ matrices with coefficients in **F**. We identify, using the canonical basis in $\mathbf{F}^n$, elements of $M(n, \mathbf{F})$ with (right) **F**-linear endomorphisms of $\mathbf{F}^n$. An **F**-linear endomorphism $A: \mathbf{F}^n \to \mathbf{F}^n$ is said to be self-adjoint if $A = A^*$, where $A^*: \mathbf{F}^n \to \mathbf{F}^n$ is the **F**-linear endomorphism defined by the condition $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for all x and y in $\mathbf{F}^n$. If A is identified with the matrix (a_{ij}), then A^* corresponds to $(\bar{a}_{ji})$. Recall that each **F**-vector subspace V of $\mathbf{F}^n$ of dimension p determines the **F**-linear orthogonal projection $A: \mathbf{F}^n \to \mathbf{F}^n$ onto V. This map satisfies the conditions: $A^2 = A = A^*$ and $\operatorname{trace} A = p$. Conversely, any **F**-linear map $A: \mathbf{F}^n \to \mathbf{F}^n$ satisfying these conditions is the orthogonal projection of $\mathbf{F}^n$ onto the p-dimensional **F**-vector subspace $A(\mathbf{F}^n)$ of $\mathbf{F}^n$.

For $p \leq n$ define

$$G_{n,p}(\mathbf{F}) = \{A \in M(n, \mathbf{F}) \mid A^2 = A = A^*, \operatorname{trace} A = p\}.$$

The above remarks imply that $G_{n,p}(\mathbf{F})$ can be considered as the Grassmannian of p-dimensional **F**-vector subspaces of $\mathbf{F}^n$. Canonically identifying $M(n, \mathbf{F})$ with $\mathbf{R}^{d(\mathbf{F})n^2}$ one can consider $G_{n,p}(\mathbf{F})$ as a real algebraic subset of $\mathbf{R}^{d(\mathbf{F})n^2}$. Moreover, one checks easily that $G_{n,p}(\mathbf{F})$ is nonsingular. We shall always regard $G_{n,p}(\mathbf{F})$ as a real algebraic nonsingular set with real algebraic structure described above.

REMARK 2.1. One can consider the category of real algebraic varieties and regular maps in the sense of Serre [15] (Serre considers algebraic varieties over an algebraically closed field but his definitions make sense over any field). One shows easily [6] that if X and Y are algebraic subsets of $\mathbf{R}^n$ and $\mathbf{R}^p$, respectively, then a map from X to Y is regular if and only if it is entire rational. The Grassmannians $G_{n,p}(\mathbf{F})$ have the natural structure of a nonsingular abstract real algebraic variety. One can identify this abstract real algebraic structure with the affine structure described above (cf. [6]).

The results of this paper depend on the knowledge of the structure of algebraic vector bundles over real algebraic sets. We shall briefly recall a few definitions and properties. Details can be found in [4] and [6].

An analytic **F**-vector bundle $\xi=(E,\pi,X)$ of rank p over a real algebraic set X is said to be an *algebraic* **F**-*vector bundle* if the total space E is a real algebraic variety in the sense of Serre [15], the projection $\pi: E\to X$ is a real regular map in the sense of Serre [15], and there exists a Zariski open covering $\{U_i\}$ of X and, for each i, a commutative diagram

$$\begin{array}{ccc} U_i\times \mathbf{F}^p & \xrightarrow{\varphi_i} & \pi^{-1}(U_i) \\ & \text{proj.}\searrow \quad \swarrow \pi & \\ & U_i & \end{array}$$

where φ_i is a real algebraic isomorphism which is **F**-linear on each fiber. One can define in the natural way the notions of a homomorphism, monomorphism, isomorphism, etc., of algebraic **F**-vector bundles. The experience shows that the class of all algebraic vector bundles over X is too large ([5], [6]). We shall only consider so-called strongly algebraic vector bundles ([4], [6]). An algebraic **F**-vector bundle ξ over X is said to be *strongly algebraic* if there exists an algebraic **F**-vector bundle η over X such that the direct sum $\xi\oplus\eta$ is algebraically isomorphic to a trivial **F**-vector bundle over X. Note that the total space of a strongly algebraic **F**-vector bundle over X is an affine variety.

EXAMPLE 2.2. The natural **F**-vector bundle $\gamma_{n,p}(\mathbf{F})=(E(\gamma_{n,p}(\mathbf{F})),\pi,G_{n,p}(\mathbf{F}))$ over $G_{n,p}(\mathbf{F})$ defined by

$$E(\gamma_{n,p}(\mathbf{F}))=\{(A,x)\in G_{n,p}(\mathbf{F})\times\mathbf{F}^n \mid Ax=x\},\qquad \pi(A,x)=A$$

is strongly algebraic ([4], [6]). More generally, if $f: X\to G_{n,p}(\mathbf{F})$ is an entire rational map, then the pullback vector bundle $f^*\gamma_{n,p}(\mathbf{F})$ is strongly algebraic. In fact, each strongly algebraic vector bundle over X is of this type ([4], [6]).

We need to collect a few results concerning strongly algebraic vector bundles.

PROPOSITION 2.3. *Let X be a compact nonsingular real algebraic set.*

(1) *If ξ is a strongly algebraic* **F**-*vector bundle over X, then every smooth section of ξ can be approximated in the C^∞ topology by algebraic (i.e., regular) sections.*

(2) *If ξ and η are strongly algebraic* **F**-*vector bundles over X and $\varphi:\xi\to\eta$ is an algebraic monomorphism of* **F**-*vector bundles, then $\varphi(\xi)$ is a strongly algebraic vector bundle over X. Moreover, if* rank $\xi=p$ *and $\eta=X\times\mathbf{F}^n$ is a trivial* **F**-*vector bundle, then the map $f: X\to G_{n,p}(\mathbf{F})$ defined by $f(x)=\rho(\varphi(\xi_x))$ for all x in X, where ξ_x is the fiber of ξ over x and $\rho: X\times\mathbf{F}^n\to\mathbf{F}^n$ is the natural projection, is entire rational.*

(3) If *ξ and η are strongly algebraic* **F**-*vector bundles over X, then the* **F**-*vector bundle* Hom(ξ,η) *is strongly algebraic.*

(4) *If a continuous* **F**-*vector bundle ξ over X is stably C^0 isomorphic to a strongly algebraic* **F**-*vector bundle, then ξ is C^0 isomorphic to a strongly algebraic* **F**-*vector bundle.*

Proof. See [4] or [6].

EXAMPLE 2.4. Every continuous **F**-vector bundle over the unit n-sphere S^n is C^0 isomorphic to a strongly algebraic **F**-vector bundle (cf. [9] for $\mathbf{F}=\mathbf{R}$ or $\mathbf{C}$ and [17] for $\mathbf{F}=\mathbf{H}$ and Proposition 2.3(4)).

The next result plays an important role.

THEOREM 2.5. *Let X be a compact nonsingular real algebraic set and let $f: X \to G_{n,p}(\mathbf{F})$ be a smooth map. Then the following conditions are equivalent:*

(i) *f can be approximated in the C^∞ topology by entire rational maps from X to $G_{n,p}(\mathbf{F})$.*
(ii) *f is homotopic to an entire rational map from X to $G_{n,p}(\mathbf{F})$.*
(iii) *The pullback smooth **F**-vector bundle $f^*\gamma_{n,p}(\mathbf{F})$ is C^0 isomorphic to a strongly algebraic **F**-vector bundle over X.*

Proof. The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are well known. We shall show (iii)$\Rightarrow$(i) using Proposition 2.3.

Let $\gamma=\gamma_{n,p}(\mathbf{F})$ and let ξ be a strongly algebraic **F**-vector bundle over X which is C^0 (hence also C^∞) isomorphic to $f^*\gamma$. Consider $f^*\gamma$ as a smooth **F**-vector subbundle of the trivial vector bundle $\epsilon^n = X \times \mathbf{F}^n$. Clearly, there exists a C^∞ monomorphism $\varphi: \xi \to \epsilon^n$ of **F**-vector bundles mapping the fiber ξ_x of ξ onto the fiber $(f^*\gamma)_x = \{x\} \times \gamma_{f(x)}$ of $f^*\gamma$ for all x in X. The **F**-vector bundle $\mathrm{Hom}(\xi, \epsilon^n)$ is strongly algebraic and φ defines the C^∞ section s_φ of $\mathrm{Hom}(\xi,\epsilon^n)$ satisfying $s_\varphi(x)(e)=\varphi(e)$ for all x in X and e in ξ_x. Let u be an algebraic section of $\mathrm{Hom}(\xi,\epsilon^n)$ approximating s_φ. If u is sufficiently close to s_φ, then $u=s_\psi$ for some uniquely determined algebraic monomorphism of **F**-vector bundles $\psi: \xi \to \epsilon^n$. It follows that the map $g: X \to G_{n,p}(\mathbf{F})$ defined by $g(x)=\rho(\psi(\xi_x))$ for x in X, where $\rho: X \times \mathbf{F}^n \to \mathbf{F}^n$ is the standard projection, is entire rational. Clearly, g approximates f. □

REMARK 2.6. A different proof of Theorem 2.5 for $\mathbf{F}=\mathbf{R}$ can be found in [12].

COROLLARY 2.7. *The set $\mathcal{R}(S^m, G_{n,p}(\mathbf{F}))$ is dense in $\mathcal{E}(S^m, G_{n,p}(\mathbf{F}))$ for all m, n, p with $n \geq p$.*

Proof. It follows from Theorem 2.5 and Example 2.4. □

We need one more preliminary observation to prove the results of Section 1.

LEMMA 2.8. *The $d(\mathbf{F})$-dimensional unit sphere $S^{d(\mathbf{F})}$ and the Grassmannian $G_{2,1}(\mathbf{F})$ are algebraically isomorphic, that is, there exists an entire rational bijection $\phi(\mathbf{F}): S^{d(\mathbf{F})} \to G_{2,1}(\mathbf{F})$ such that the inverse map $\phi(\mathbf{F})^{-1}$ is also entire rational.*

Proof. Note that

$$G_{2,1}(\mathbf{F}) = \left\{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & 1-\alpha \end{pmatrix} \middle| \alpha \in [0,1],\ \beta \in \mathbf{F},\ \|\beta\|^2 = \alpha(1-\alpha) \right\}$$

and define $\phi(\mathbf{F}): S^{d(\mathbf{F})} \to G_{2,1}(\mathbf{F})$ by

$$\phi(\mathbf{F})(\alpha, u) = \frac{1}{2}\begin{pmatrix} 1-\alpha & \bar{u} \\ u & 1+\alpha \end{pmatrix}$$

for (α, u) in $S^{d(\mathbf{F})} = \{x \in \mathbf{R} \times \mathbf{F} \mid \|x\| = 1\}$. Then $\phi(\mathbf{F})$ is a polynomial map and the inverse map given by

$$\phi(\mathbf{F})^{-1}(A) = (1-2\alpha, 2\beta) \quad \text{for } A = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & 1-\alpha \end{pmatrix} \text{ in } G_{2,1}(\mathbf{F})$$

is also polynomial. □

Denote by $\gamma_{d(\mathbf{F})}$ the strongly algebraic vector bundle $\phi(\mathbf{F})^*\gamma_{2,1}(\mathbf{F})$ over $S^{d(\mathbf{F})}$.

Proof of Theorems 1.1 *and* 1.3. The conclusion follows from Theorem 2.5, Lemma 2.8, and Example 2.4. □

Proof of Theorem 1.4. The **R**-line bundle $f^*\gamma_1$ is C^0 isomorphic to a strongly algebraic **R**-line bundle over X if and only if its first Stiefel–Whitney characteristic class $w_1(f^*\gamma_1) = f^*(u)$ belongs to $H^1_{\text{alg}}(X, \mathbf{Z}_2)$ ([5], [16]). Thus the equivalence (i)⇔(iii) follows from Theorem 2.5 and Lemma 2.8. If y is a regular value of f, then the homology class represented by $f^{-1}(y)$ in $H_{n-1}(X, \mathbf{Z}_2)$ is Poincaré dual to $f^*(u)$. Thus (i)⇔(ii) is obvious. □

Proof of Theorem 1.6. Note that the **C**-line bundle $f^*\gamma_2$ over X is C^0 trivial. This is so since the first Chern characteristic class of $f^*\gamma_2$ vanishes [11]. We conclude the proof by applying Theorem 2.5 and Lemma 2.8. □

Proof of Theorem 1.7. (i) Let ξ be a continuous **R**-line bundle over X whose first Stiefel–Whitney characteristic class $w_1(\xi)$ corresponds, via Poincaré duality, to the homology class represented by C in $H_1(X, \mathbf{Z}_2)$. Since $w_1(\xi)$ belongs to $H^1_{\text{alg}}(X, \mathbf{Z}_2)$, we may assume that ξ is a strongly algebraic **R**-vector bundle ([5], [16]). Note that the second Stiefel-Whitney characteristic class $w_2(\xi \oplus \xi) = w_1(\xi) \cup w_1(\xi)$ of $\xi \oplus \xi$ corresponds, via Poincaré duality, to the intersection of the homology class represented by C with itself. Thus $w_2(\xi \oplus \xi)$ is different from zero and the **R**-vector bundle $\xi \oplus \xi$ is not C^0 trivial. Clearly, $\xi \otimes \mathbf{C}$ is a strongly algebraic **C**-line bundle over X which is not C^0 trivial. Since $H^2(X, \mathbf{Z}) = \mathbf{Z}_2$, every continuous **C**-line bundle over X is either C^0 trivial or C^0 isomorphic to $\xi \otimes \mathbf{C}$ [11]. It suffices to apply Theorem 2.5 and Lemma 2.8.

(ii) Clearly, there exists a closed smooth curve C in X having the band of Möbius as a tubular neighborhood. Of course, $\#_2(C, C; X) = 1$. By [5] or [16], we may assume that C is an algebraic nonsingular curve. Thanks to (i), the conclusion follows.

(iii) By [5], [7], or [16], there exists a nonsingular algebraic curve C in X whose homology class in $H_1(X, \mathbf{Z}_2)$ is Poincaré dual to the first Stiefel-Whitney characteristic class of X. Obviously, $\#_2(C, C; X) \equiv g (\text{mod } 2)$, where g is the genus of X. The proof is complete since g is odd. □

We conclude this paper by making the following observation.

REMARK 2.9. Theorems 1.3, 1.6, and 2.5 remain true if one drops the assumption "X is nonsingular," replaces smooth maps by continuous maps, and replaces the approximation in the C^∞ topology by the approximation in the C^0 topology.

REFERENCES

1. R. Abraham and J. Robbin, *Transversal mappings and flows,* Benjamin, New York, 1967.
2. S. Akbulut and H. King, *The topology of real algebraic sets,* Enseign. Math. (2) 29 (1983), 221–261.
3. ———, *Submanifolds and homology of nonsingular real algebraic varieties,* Amer. J. Math. 107 (1985), 45–83.
4. R. Benedetti and A. Tognoli, *On real algebraic vector bundles,* Bull. Sci. Math. (2) 104 (1980), 89–112.
5. ———, *Remarks and counterexamples in the theory of real algebraic vector bundles and cycles.* Real algebraic geometry and quadratic forms (Rennes, 1981), 198–211, Lecture Notes in Math., 959, Springer, Berlin, 1982.
6. J. Bochnak, M. Coste and M. F. Roy, *Géométrie algébrique réele,* Ergeb. Math., to appear.
7. J. Bochnak, W. Kucharz and M. Shiota, *Divisor class groups of some rings of global real analytic, Nash or rational regular functions.* Real algebraic geometry and quadratic forms (Rennes, 1981), 218–248, Lecture Notes in Math., 959, Springer, Berlin, 1982.
8. J. Bochnak and W. Kucharz, *Representation of homotopy classes by algebraic mappings,* J. Reine Angew. Math., to appear.
9. R. Fossum, *Vector bundles over spheres are algebraic,* Invent. Math. 8 (1969), 222–225.
10. M. Hirsch, *Differential topology,* Springer, New York, 1976.
11. D. Husemoller, *Fiber bundles,* 2nd ed., Springer, New York, 1975.
12. N. Ivanov, *Approximation of smooth manifolds by real algebraic sets,* Russian Math. Surveys 37 (1982), 1–59.
13. W. Kucharz, *On homology of real algebraic sets,* Invent. Math. 82 (1985), 19–25.
14. J. L. Loday, *Applications algébriques du tore dans la sphère et de $S^p \times S^q$ dans S^{p+q}.* Algebraic K-theory, II: classical algebraic K-theory and connections with arithmetic (Battele Memorial Inst., 1972), 79–91, Lecture Notes in Math., 342, Springer, Berlin, 1973.
15. J. P. Serre, *Faisceaux algébriques cohérents,* Ann. of Math. (2) 61 (1955), 197–278.
16. M. Shiota, *Real algebraic realization of characteristic classes,* Publ. Res. Inst. Math. Sci. 18 (1982), 995–1008.
17. R. Swan, *Topological examples of projective modules,* Trans. Amer. Math. Soc. 230 (1977), 201–234.
18. R. Wood, *Polynomial maps from spheres to spheres,* Invent. Math. 5 (1968), 163–168.

Vrije Universiteit
Department of Mathematics
P.O. Box 7161
1007 MC Amsterdam
The Netherlands

Department of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico 87131
USA