THE SPECTRUM OF THE LAPLACIAN ON RIEMANNIAN HEISENBERG MANIFOLDS

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1. Introduction. For any compact Riemannian manifold (M, g) let $\text{spec}(M, g)$ denote the collection of eigenvalues, with multiplicities, of the associated Laplace-Beltrami operator acting on $C^\infty(M)$. Two manifolds (M, g) and (M', g') are said to be isospectral if $\text{spec}(M, g) = \text{spec}(M', g')$. Many examples exist of pairs of isospectral, non-isometric Riemannian manifolds ([3], [6], [10], [12], [15], [17], [18]). Vigńeras gave the first examples of isospectral manifolds with non-isomorphic fundamental groups. In contrast, some manifolds such as the canonical sphere S^n and real projective space P^n, $n \leq 6$, are uniquely determined up to isometry by $\text{spec}(M, g)$. (See e.g. [1], [9].)

In this paper we study the spectrum of the Laplacian of compact Riemannian Heisenberg manifolds; that is, manifolds of the form $(\Gamma \backslash H_n, g)$, where H_n is the $(2n+1)$-dimensional Heisenberg group, Γ is a uniform discrete subgroup, and g is a Riemannian metric on $\Gamma \backslash H_n$ whose lift to H_n is left-invariant. The Heisenberg manifolds are among the few manifolds for which $\text{spec}(M, g)$ can be explicitly computed. By comparing the spectra of various Heisenberg manifolds, we find:

(A) If $n = 1$, $(\Gamma \backslash H_n, g)$ is uniquely determined by its spectrum.

(B) If $n > 1$, there exist many choices of pairs $(\Gamma \backslash H_n, g)$ and $(\Gamma' \backslash H_n, g')$ that are isospectral but not isometric.

More specifically, we associate with every uniform discrete subgroup Γ of H_n a positive integer denoted $|\Gamma|$. Whenever $n > 1$ and $|\Gamma| = |\Gamma'|$, there exist continuous families of metrics g_t and g_t' such that for each t, $(\Gamma \backslash H_n, g_t)$ is isospectral to $(\Gamma' \backslash H_n, g_t')$. (Note that we are *not* asserting the existence of continuous isospectral deformations of a metric.) Since $|\Gamma|$ does not always determine the isomorphism class of Γ, we thus obtain examples of isospectral manifolds with non-isomorphic fundamental groups. In some cases the manifolds are also isospectral on p-forms for all $p \geq 0$.

This paper was partly motivated by the following result of [6]. Let G be a nilpotent Lie group. In [6] we defined a group $\text{AIA}(G)$ of "almost inner" automorphisms, and showed that $(\varphi(\Gamma) \backslash G, g)$ is isospectral to $(\Gamma \backslash G, g)$ for all $\varphi \in \text{AIA}(G)$ whenever Γ is any uniform discrete subgroup of G and g any metric arising from a left-invariant metric on G. The manifolds are isometric if φ lies in the group $\text{Inn}(G) \subset \text{AIA}(G)$ of inner automorphisms but are rarely isometric otherwise. We thus obtained continuous families of non-isometric manifolds all isospectral to $(\Gamma \backslash G, g)$ under the condition $\text{Inn}(G) \neq \text{AIA}(G)$. We do not know whether this condition is necessary as well as sufficient for the existence of a non-

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trivial isospectral deformation of $(\Gamma\backslash G, g)$. The Heisenberg groups are among the simplest examples of nilpotent groups for which $\mathrm{Inn}(G)=\mathrm{AIA}(G)$. Certainly, by (A) every 3-dimensional Heisenberg manifold is spectrally rigid; we give evidence supporting (but not proving) our conjecture that no Heisenberg manifold of any dimension admits a non-trivial continuous isospectral deformation.

The organization of this paper is as follows: After classifying all Riemannian Heisenberg manifolds in Section 2, we compute their spectra in Section 3. We construct the examples (B) of isospectral manifolds in Section 4. In Section 5 we prove (A) and address the question of spectral rigidity in higher dimensions. We discuss the spectra of the Laplacian on p-forms in an appendix.

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2. Classification of Riemannian Heisenberg manifolds.

(2.1) DEFINITIONS AND NOTATION. (a) For x, y row vectors in $\mathbf{R}^n$, let

$$(1)\qquad \gamma(x,y,t)=\begin{bmatrix}1 & x & t\\ 0 & I_n & {}^t y\\ 0 & 0 & 1\end{bmatrix},\qquad X(x,y,t)=\begin{bmatrix}0 & x & t\\ 0 & 0 & {}^t y\\ 0 & 0 & 0\end{bmatrix},$$

where ${}^t y$ is the transpose of y and I_n is the $n\times n$ identity matrix. The real $(2n+1)$-dimensional Heisenberg group H_n is the Lie subgroup of $\mathrm{GL}(n+2,\mathbf{R})$ consisting of all matrices of the form $\gamma(x,y,t)$ and its Lie algebra $\mathfrak{h}_n$ is the Lie subalgebra of $\mathfrak{gl}(n+2,\mathbf{R})$ consisting of all matrices of the form $X(x,y,t)$. The matrix exponential maps $\mathfrak{h}_n$ diffeomorphically onto H_n and satisfies

$$\exp X(x,y,t)=\gamma(x,y,t+\tfrac{1}{2}x\cdot y),$$

where $x\cdot y$ is the usual dot product in $\mathbf{R}^n$. The product operation in H_n and Lie bracket in $\mathfrak{h}_n$ are given by

$$(2)\qquad \begin{aligned}\gamma(x,y,t)\gamma(x',y',t')&=\gamma(x+x',y+y',t+t'+x\cdot y'),\\ [X(x,y,t),X(x',y',t')]&=X(0,0,x\cdot y'-x'\cdot y).\end{aligned}$$

Let $\mathfrak{z}_n=\{X(0,0,t): t\in\mathbf{R}\}$. Then $\mathfrak{z}_n$ is both the center and the derived subalgebra of $\mathfrak{h}_n$. It is convenient to identify the subspace $\{X(x,y,0): x,y\in\mathbf{R}^n\}$ of $\mathfrak{h}_n$ with $\mathbf{R}^{2n}$. Thus $\mathfrak{h}_n$ is the vector space direct sum $\mathfrak{h}_n=\mathbf{R}^{2n}+\mathfrak{z}_n$. The bracket operation defines a non-singular alternating bilinear form $A:\mathbf{R}^{2n}\times\mathbf{R}^{2n}\to\mathbf{R}$ by

$$(3)\qquad A(X,Y)Z=[X,Y]$$

for $X, Y\in\mathbf{R}^{2n}$ and $Z=X(0,0,1)$. By the standard basis of $\mathfrak{h}_n$ we shall mean

$$\mathcal{S}=\{X_1,\dots,X_n,Y_1,\dots,Y_n,Z\},$$

where the first $2n$ elements of $\mathcal{S}$ are the standard basis of $\mathbf{R}^{2n}$. The non-zero brackets among the elements of $\mathcal{S}$ are thus given by $[X_i, Y_i]=Z$ for $1\le i\le n$.

(b) A *Riemannian Heisenberg* manifold is a pair $(\Gamma\backslash H_n, g)$, where Γ is a uniform discrete subgroup of H_n ("uniform" means that $\Gamma\backslash H_n$ is compact) and g is a Riemannian metric on $\Gamma\backslash H_n$ whose lift to H_n, also denoted g, is left H_n-invariant.

(2.2) PROPOSITION. *Let Γ be a uniform discrete subgroup of H_n and let g and g' be left-invariant Riemannian metrics on H_n. $(\Gamma\backslash H_n, g)$ is isometric to $(\Gamma\backslash H_n, g')$ if and only if there exists an automorphism φ of H_n such that $\varphi^* g = g'$ and $\varphi(\Gamma) = \gamma\Gamma\gamma^{-1}$ for some $\gamma \in H_n$.*

Proof. This is a special case of a result proved in [5]. □

We first classify the uniform discrete subgroups of H_n.

(2.3) DEFINITION. For $r = (r_1, r_2, \ldots, r_n) \in (\mathbf{Z}^+)^n$ such that r_j divides r_{j+1}, $1 \le j \le n$, let $r\mathbf{Z}^n$ (respectively, $(1/r)\mathbf{Z}^n$) denote the n-tuples $x = (x_1, \ldots, x_n)$ for which $x_i \in r_i\mathbf{Z}$ (respectively, $x_i \in (1/r_i)\mathbf{Z}$), $1 \le i \le n$. Define

$$\Gamma_r = \{\gamma(x, y, t) : x \in r\mathbf{Z}^n, y \in \mathbf{Z}^n, t \in \mathbf{Z}\}.$$

It follows easily from (2) that Γ_r is a uniform discrete subgroup of H_n.

Define $\mathcal{L}_r = \{X(x, y, 0) : x \in r\mathbf{Z}^n, y \in \mathbf{Z}^n\}$. Then $\mathcal{L}_r$ is a lattice in $\mathbf{R}^{2n}$ with lattice basis $\{r_1X_1, \ldots, r_nX_n, Y_1, \ldots, Y_n\}$. Note that $X \in \mathcal{L}_r$ if and only if $\exp(X+W) \in \Gamma_r$ for some $W \in \mathfrak{z}_n$.

(2.4) THEOREM. *The subgroups Γ_r defined in 2.3 classify the uniform discrete subgroups of H_n up to automorphism; that is, if Γ is any uniform discrete subgroup of H_n, then there exists a unique r for which some automorphism of H_n maps Γ to Γ_r. Moreover, for r and s as in 2.3, Γ_r and Γ_s are isomorphic groups if and only $r = s$.*

Proof. Suppose Γ is a uniform discrete subgroup of H_n. Let $\log : H_n \to \mathfrak{h}_n$ be the inverse of $\exp : \mathfrak{h}_n \to H_n$. For $X, Y \in \mathfrak{h}_n$,

$$\log(\exp X \exp Y \exp(-X) \exp(-Y)) = [X, Y].$$

It follows that $\log\Gamma$ is a discrete spanning set of $\mathfrak{h}_n$, $\log\Gamma$ is closed under the bracket operation, $\log\Gamma \cap \mathfrak{z}_n = \mathbf{Z}W$ for some $W \ne 0$ in $\mathfrak{z}_n$, and

$$\mathcal{L} = \{X \in \mathbf{R}^n : \text{there exists } X' \in \log\Gamma \text{ such that } X - X' \in \mathfrak{z}_n\}$$

is a lattice in $\mathbf{R}^{2n}$. Since, for every $a \in \mathbf{R} - \{0\}$, $X(x, y, t) \to X(ax, ay, a^2t)$ defines an automorphism of $\mathfrak{h}_n$ moving $(1/a^2)Z$ to Z, replacement of Γ by a suitable automorphic image permits us to assume $W = Z$. With this assumption, we claim that there exists $r = (r_1, r_2, \ldots, r_n)$ as in (2.3) and a lattice basis $\{U_1, \ldots, U_n, V_1, \ldots, V_n\}$ for $\mathcal{L}$ such that

$$0 = A(U_i, U_j) = A(V_i, V_j) = A(U_i, V_j) - \delta_{ij}r_i \quad \text{for } 1 \le i, j \le n, \tag{4}$$

where δ_{ij} is the Kronecker symbol. To see this, note that $\mathcal{I}_1 = \{A(X, Y) : X, Y \in \mathcal{L}\}$ is an ideal of integers. Let r_1 be its positive generator and choose $U_1, V_1 \in \mathcal{L}$ such that $A(U_1, V_1) = r_1$. For $\mathfrak{a} \subset \mathbf{R}^{2n}$ the annihilator of $\mathbf{R}U_1 + \mathbf{R}V_1$ relative to A, we have $\mathfrak{a} \cong \mathbf{R}^{2n-2}$, $\mathfrak{a} + \mathfrak{z}_n \cong \mathfrak{h}_{n-1}$, and $\mathcal{L} = \mathbf{Z}U_1 + \mathbf{Z}V_1 + \mathcal{L} \cap \mathfrak{a}$. Indeed, expressing $Y \in \mathcal{L}$ in the form $aU_1 + bV_1 + X$ with $X \in \mathfrak{a}$, then $ar_1 = A(Y, V_1)$, $br_1 = A(U_1, Y)$ are in $\mathcal{I}$, whence $a, b \in \mathbf{Z}$ and $X \in \mathcal{L}$. By induction on n, we obtain (4). The ideal $\mathcal{I}_j$ constructed in the jth step of the inductive process is a subideal of $\mathcal{I}_{j-1}$, so $r_{j-1} \mid r_j$. Now choose, for $1 \leq i \leq n$, elements X_i^1, $Y_i^{\,1}$ in $\log \Gamma$ such that $U_i - X_i^1$, $V_i - Y_i^1$ are in $\mathfrak{z}_n$. By (4) and (2.3), the unique linear map of $\mathfrak{h}_n$ which sends Z to Z, X_i^1 to $r_i X_i$, and Y_i^1 to Y_i is an automorphism of $\mathfrak{h}_n$ mapping $\mathcal{L}$ to $\mathcal{L}_r$ and $\log \Gamma$ to $\log \Gamma_r$. It therefore lifts to an automorphism φ of H_n satisfying $\varphi(\Gamma) = \Gamma_r$.

It remains to show that r is uniquely determined by the isomorphism class of Γ_r. As an abstract group, Γ_r is prescribed by generators $\{\alpha_i, \beta_i, \gamma : 1 \leq i \leq n\}$ where γ generates the center of Γ_n and the relations $\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = \gamma^{r_i}$ are satisfied. Since γ is unique up to inverse and $[\gamma^{r_1}]$ is the commutator subgroup of Γ_r, r_1 is uniquely determined. Let $J = \{j : r_j > r_{j-1}\}$. For $j \in J$, $\Gamma_r/[\gamma^{r_j}]$ contains as a direct factor the free abelian group of rank $2(n-j+1)$ with generators $\{\bar{\alpha}_k, \bar{\beta}_k : j \leq k \leq n\}$. If $t \in \mathbf{Z}r_1$ is larger than r_j, one can check that the maximal free abelian direct factor of $\Gamma_r/[\gamma^t]$ has rank $< 2(n-j+1)$. Thus J, $\{r_j : j \in J\}$, and hence r are uniquely determined by the isomorphism class of Γ_r. □

(2.5) COROLLARY.

(i) *Given any Riemannian Heisenberg manifold* $M = (\Gamma \backslash H_n, g)$, *there is a unique* r *as in Definition* 2.3 *and a left-invariant metric* $\tilde{g}$ *on* H_n *such that* M *is isometric to* $(\Gamma_r \backslash H_n, \tilde{g})$.

(ii) *If* $r \neq r'$, *the manifolds* $\Gamma_r \backslash H_n$ *and* $\Gamma_{r'} \backslash H_n$ *have distinct fundamental groups.*

Proof. (i) and (ii) follow from Theorem 2.4 and the fact that the map

$$\Gamma\gamma \to \varphi(\Gamma)\varphi(\gamma)$$

is an isometry from $(\Gamma \backslash H_n, g)$ to $(\varphi(\Gamma) \backslash H_n, (\varphi^*)^{-1} g)$ for any automorphism φ. □

(2.6) REMARKS AND NOTATION. (a) We will identify each automorphism φ of H_n with the matrix of its differential φ_* relative to the standard basis $\mathcal{S}$ (see (2.1)) of $\mathfrak{h}_n$. Let $\widetilde{\mathrm{Sp}}(n, \mathbf{R}) = \{\beta \in \mathrm{GL}(2n, \mathbf{R}) : {}^t\beta J \beta = \epsilon J \text{ with } \epsilon = \pm 1\}$, where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

We identify $\beta \in \widetilde{\mathrm{Sp}}(n, \mathbf{R})$ with the $(2n+1) \times (2n+1)$ matrix

$$\begin{bmatrix} \beta & 0 \\ 0 & \epsilon \end{bmatrix}.$$

It is then routine to check that, with these identifications, $\widetilde{\mathrm{Sp}}(n\, \mathbf{R})$ is a subgroup of $\mathrm{Aut}(H_n)$. The full group $\mathrm{Aut}(H_n)$ is the set of all matrices of the form $\alpha\beta$, with

$$\beta \in \widetilde{\mathrm{Sp}}(n, \mathbf{R}) \quad \text{and} \quad \alpha = \begin{bmatrix} aI_{2n} & 0 \\ w & a^2 \end{bmatrix} \quad \text{for some } a \in \mathbf{R},\ w \in \mathbf{R}^{2n}.$$

The inner automorphisms are those for which $a=1$ and $\beta=\mathrm{Id}$.

(b) A left-invariant metric g on H_n is uniquely determined by the induced inner product $\langle\,,\rangle$ on $\mathfrak{h}_n$, where $\mathfrak{h}_n$ is viewed as the tangent space to H_n at the identity. Conversely, every inner product on $\mathfrak{h}_n$ determines a left-invariant metric on H_n. We will identify g with the matrix of $\langle\,,\rangle$ relative to the standard basis $\mathcal{S}$ of $\mathfrak{h}_n$. For any g, we can choose an inner automorphism φ such that $\mathbf{R}^{2n}$ is orthogonal to $\mathfrak{z}_n$ relative to φ^*g. By Proposition 2.2, $(\Gamma\backslash H_n, g)$ is isometric to $(\Gamma\backslash H_n, \varphi^*g)$ for every Γ. Replacing g by φ^*g we may then assume that g has the form

$$g=\begin{bmatrix} h & 0 \\ 0 & g_{2n+1}\end{bmatrix}, \tag{5}$$

with h a positive-definite $2n\times 2n$ matrix and $g_{2n+1}>0$. From now on, every metric g will be assumed to have the form (5).

(c) Let δ_r be the $2n\times 2n$ matrix with diagonal entries $r_1,\dots,r_n,1,\dots,1$, and let $\widetilde{\mathrm{SL}}(2n,\mathbf{Z})$ be the group of $2n\times 2n$ matrices with integer entries and determinant equal to ± 1. Note that for $\varphi\in\mathrm{Aut}(H_n)$, $\varphi(\Gamma_r)=\Gamma_r$ if and only if

$$\varphi=\begin{bmatrix}\beta & 0\\ w & \epsilon\end{bmatrix} \text{ for some } w\in\mathbf{Z}^n \quad\text{and}\quad \beta\in\widetilde{\mathrm{Sp}}(n,\mathbf{R})\cap\delta_r\,\widetilde{\mathrm{SL}}(2n,\mathbf{Z})\delta_r^{-1}.$$

(2.7) THEOREM. *Let*

$$\mathcal{G}_n=\{(r,g): r\in(\mathbf{Z}^+)^n \textit{ satisfies } 2.3 \textit{ and } g \textit{ is of the form } (10)\}.$$

Define an equivalence relation on $\mathcal{G}_n$ *by* $(r,g)\sim(r',g')$ *if and only if* $r=r'$ *and* $g'=\beta^*g$ *(see* 2.6(a) *for notations), with* $\beta\in\widetilde{\mathrm{Sp}}(n,\mathbf{R})\cap\delta_r\,\widetilde{\mathrm{SL}}(2n,\mathbf{Z})\delta_r^{-1}$. *(Note that each equivalence class is discrete.) Using* (r,g) *to parameterize* $(\Gamma_r\backslash H_n, g)$, $\mathcal{G}_n/\!\sim$ *parameterizes the collection of isometry classes of Riemannian Heisenberg manifolds of dimension* $2n+1$.

Proof. If g has the form (5) and $\varphi=\alpha\beta$ as in 2.6(a), then $g'=\varphi^*g$ again has the form (5) precisely when $w=0$. But then $\varphi(\Gamma_r)=\gamma\Gamma_r\gamma^{-1}$ is possible only if $\varphi(\Gamma_r)=\Gamma_r$. Thus the theorem follows from Proposition 2.2, Theorem 2.4, and the remarks in 2.6(b). □

(2.8) DEFINITION. Let Γ be a uniform discrete subgroup of H_n. Define $|\Gamma|=r_1r_2\cdots r_n$ for $r=(r_1,\dots,r_n)$ the unique n-tuple as in Definition 2.3 for which Γ is isomorphic to Γ_r.

(2.9) PROPOSITION. *The Riemannian volume of a Heisenberg manifold* $(\Gamma_r\backslash H_n, g)$ *is given by* $|\Gamma_r|(\det(g))^{1/2}$, *where the conventions of* 2.6(b) *are used to identify* g *with a positive-definite matrix.*

Proof. Standard computation using the coordinates defined in (2.1). □

3. The spectrum of a Riemannian Heisenberg manifold.

(3.1) DEFINITIONS AND NOTATIONS. Let $M=(\Gamma\backslash H_n, g)$ be a Riemannian Heisenberg manifold and $E^\circ(M)$ the space of smooth functions on M. Viewing functions on M as left Γ-invariant functions on H_n, the Laplace–Beltrami operator on $E^\circ(M)$ is given by

(1) $$\Delta f = -\sum_{i=1}^{2n+1} U_i^2 f,$$

where $U_1, \ldots, U_{2n+1}$ is any g-orthonormal basis of $\mathfrak{h}_n$ (see [16]). But

$$U_i f(\gamma) = \left(\frac{d}{dt}\right)_{t=0} f(\gamma \exp tU_i) = (R_* U_i) f(\gamma),$$

where R is the quasi-regular representation of H_n on $L^2(\Gamma \backslash H_n)$, that is,

$$R(\gamma')f(\gamma) = f(\gamma\gamma').$$

Thus the extension of Δ to an unbounded operator on $L^2(\Gamma \backslash H_n)$ is given by $\Delta = -\sum_{i=1}^{2n+1}(R_*(U_i))^2$.

By $\Sigma(M)$ we mean the spectrum of M in $E^\circ(M)$, that is, the collection of all eigenvalues, with multiplicities, of Δ. For $\Gamma = \Gamma_r$ as in 2.3, we write $\Sigma(r, g)$ for $\Sigma(M)$. Two Riemannian Heisenberg manifolds M and M' are said to be *isospectral* if $\Sigma(M) = \Sigma(M')$.

(3.2) NOTATION. (a) Let $r = (r_1, \ldots, r_n)$ be as in 2.3, δ_r the $2n \times 2n$ matrix defined in 2.6(c), and h a positive-definite $2n \times 2n$ matrix. For $a, b \in \mathbf{Z}^n$, define

(2) $$\lambda(a, b) = 4\pi^2 [a, b](\delta_r h \delta_r)^{-1}\, {}^t[a, b].$$

Then by $\Sigma_1(r, h)$ we shall mean the collection of numbers λ which may be described in the form $\lambda(a, b)$, with the understanding that λ is counted once in $\Sigma_1(r, h)$ for each pair $(a, b) \in \mathbf{Z}^{2n}$ such that $\lambda = \lambda(a, b)$.

(b) Let g be of the form 2.6(5) and J the $2n \times 2n$ matrix

$$\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Since $h^{-1}J$ is similar to the skew-symmetric matrix $h^{-1/2}Jh^{-1/2}$, it has pure imaginary eigenvalues; we denote them by $\pm\sqrt{-1}\, d_j^2$, $1 \le j \le n$. For c a positive integer and $k = (k_1, \ldots, k_n)$ an n-tuple of non-negative integers, define

(3) $$\mu(c, k) = \frac{4\pi^2 c^2}{g_{2n+1}} + \sum_{i=1}^{n} 2\pi c d_i^2 (2k_i + 1).$$

By $\Sigma_2(r, g)$ we shall mean the collection of numbers μ which can be written in the form $\mu(c, k)$, with the understanding that μ occurs $2c^n|\Gamma_r| = 2c^n r_1 \cdots r_n$ times for each pair (c, k) such that $\mu = \mu(c, k)$.

(c) Let $\mathfrak{h}_n^*$ denote the dual space of $\mathfrak{h}_n$. Given a metric g as in 2.6(5), define $\#: \mathfrak{h}_n^* \to \mathfrak{h}_n$ by $\tau(X) = g(\#\tau, X)$ and define an inner product $\langle\, ,\, \rangle$ on $\mathfrak{h}_n^*$ by $\langle \sigma, \tau \rangle = g(\#\sigma, \#\tau)$. Define $\eta: \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ by $[X, Y] = h(X, \eta Y)Z$, that is, $h(X, \eta Y) = A(X, Y)$ for A the alternating form defined in 2.1. Since J is the matrix of A relative to the standard basis and since we are identifying the inner product h with its matrix in the standard basis, the matrix of η in the standard basis is given by $h^{-1}J$. Given Γ_r as in 2.3, let $\mathfrak{A}_r = \{\tau \in \mathfrak{h}_n^* : \tau(Z) = 0 \text{ and } \tau(\log \Gamma_r) \subset \mathbf{Z}\}$.

(3.3) THEOREM. *For r as in* 2.3 *and g of the form* 2.6(5), *the spectrum* $\Sigma(r,g)$ *of* $(\Gamma_r\backslash H_n, g)$ *is the join of* $\Sigma_1(r,h)$ *and* $\Sigma_2(r,g)$ (*see* 3.2(a, b)) *in the sense that the multiplicity of λ in $\Sigma(r,g)$ is the sum of the multiplicities of λ in $\Sigma_1(r,h)$ and* $\Sigma_2(r,g)$.

(3.4) LEMMA. *We use the notation of* 3.2. *Then:*

(a) *λ occurs in $\Sigma_1(r,h)$ with multiplicity k if and only if $\lambda=4\pi^2\langle\tau,\tau\rangle$ for exactly k elements τ in $\mathfrak{A}_r$.*

(b) *$\Sigma_1(r,h)$ is the spectrum of the Laplace-Beltrami operator on the flat torus $T_{r,h}=(\mathcal{L}_r\backslash\mathbf{R}^{2n},h)$. (See* 2.3 *for the definition of the lattice $\mathcal{L}_r$.)*

Proof. (a) follows easily from the definitions. If we identify elements of $\mathfrak{A}_r$ with their restrictions to $\mathbf{R}^{2n}$, then $\mathfrak{A}_r$ is the dual lattice of $\mathcal{L}_r$. Thus (b) follows from (a) and the classical description of the spectrum of a flat torus (see [1]). □

(3.5) LEMMA. *If $h^{-1}J$ has eigenvalues $\pm\sqrt{-1}\,d_1^2,\dots,\pm\sqrt{-1}\,d_n^2$, then there exists an h-orthonormal basis $\{X_1',\dots,X_n',Y_1',\dots,Y_n'\}$ of $\mathbf{R}^{2n}$ such that $[X_i',Y_i']=d_i^2Z$ and all other brackets of basis vectors are zero. In particular, the isometry class of (H_n,g) is uniquely determined by $d_1^2\sqrt{g_{2n+1}},\dots,d_n^2\sqrt{g_{2n+1}}$.*

Proof. Since η (as defined in 3.2(c)) is skew-symmetric relative to h with eigenvalues $\pm\sqrt{-1}\,d_1^2,\dots,\pm\sqrt{-1}\,d_n^2$, there exists an h-orthonormal basis $\{X_1',\dots,X_n',Y_1',\dots,Y_n'\}$ of $\mathbf{R}^{2n}$ such that $\eta X_i'=-d_i^2Y_i'$, $\eta Y_i'=d_i^2X_i'$. Since $[X,Y]=h(X,\eta Y)Z$ for all $X,Y\in\mathbf{R}^{2n}$, the bracket relations follow. For the second statement, let $Z'=(g_{2n+1})^{-1/2}Z$. Then $\mathfrak{B}=\{X_1',\dots,X_n',Y_1',\dots,Y_n',Z'\}$ is a g-orthonormal basis of $\mathfrak{h}_n$ with $[X_i',Y_i']=d_i^2\sqrt{g_{2n+1}}\,Z'$ and with all other brackets trivial. If $\tilde g$ is a second metric such that $(\tilde d_1^2(\tilde g_{2n+1})^{-1/2},\dots,\tilde d_n^2(\tilde g_{2n+1})^{-1/2})=(d_1^2g_{2n+1}^{-1/2},\dots,d_n^2g_{2n+1}^{-1/2})$ up to order, then $\mathfrak{h}_n$ admits a $\tilde g$-orthonormal basis $\tilde{\mathfrak{B}}$ whose elements satisfy the same bracket relations as $\mathfrak{B}$. Thus there exists $\varphi\in\operatorname{Aut}(H_n)$ with $\tilde g=\varphi^*g$. □

By (1) and (2), any subspace of $L^2(\Gamma\backslash H_n,\Omega)$ invariant under the right action R of H_n is also Δ-invariant. The proof of Theorem 3.3 requires decomposing $L^2(\Gamma\backslash H_n)$ into irreducible subspaces under R and examining the action of Δ on each such subspace.

(3.6) IRREDUCIBLE UNITARY REPRESENTATIONS OF H_n. (a) For $\tau\in\mathfrak{h}_n^*$ with $\tau\,|\,_{\mathfrak{z}_n}=\{0\}$, define $f_\tau\colon H_n\to\mathbf{C}$ by $f_\tau(\exp X)=\exp[2\pi\sqrt{-1}\,\tau(X)]$. Then f_τ is a character of H_n and every character of H_n is of this form. If $\tau\in\mathfrak{A}_r$ (see 3.2(c)), then f_τ may be viewed as a function on $\Gamma_r\backslash H_n$.

(b) For $c\in\mathbf{R}-\{0\}$, define a representation π_c of H_n on $L^2(\mathbf{R}^n)$ by

$$(\pi_c(\gamma(x,y,t))f)(u)=\exp[2\pi\sqrt{-1}\,c(t+u\cdot y)]f(x+u)\quad\text{for all } \gamma(x,y,t)\in H_n$$

(see 2.1), where $u\cdot y$ denotes the standard dot product on $\mathbf{R}^n$. One can check that π_c is an irreducible unitary representation of H_n.

(3.7) LEMMA. (a) *With the notations of* 3.6,

$$\{f_\tau:\tau\in\mathfrak{h}_n^*,\ \tau\,|\,_{\mathfrak{z}_n}=0\}\cup\{\pi_c:c\in\mathbf{R}-\{0\}\}$$

is a complete set of irreducible unitary representations of H_n. In particular, any *irreducible unitary representation of H_n which agrees with π_c on the center of H_n is unitarily equivalent to π_c.*

(b) *The representation R defined in* 3.1 *of H_n on $L^2(\Gamma_r\backslash H_n)$ decomposes discretely as the orthogonal direct sum of irreducible unitary representations of H_n, with f_τ occurring once for each $\tau \in \mathcal{A}_r$ and with π_c occurring $|c^n||\Gamma_r| = |c^n|r_1 \cdots r_n$ times for each non-zero integer c.*

Proof. (a) For π an arbitrary irreducible unitary representation, $\pi(\gamma)$ is a scalar multiple of the identity operator Id for each γ in the center of H_n. Hence there exists $c \in \mathbf{R}$ such that $\pi(\gamma(0,0,t)) = \exp[2\pi\sqrt{-1}ct]$ Id for every $t \in \mathbf{R}$. If $c=0$, π must be a character and otherwise the Stone–von Neumann Theorem [19] states that π is unitarily equivalent to π_c (see [8, pp. 824–825] for a discussion of this theorem). Alternatively, one may prove (a) by applying the Kirillov theory of representations of nilpotent Lie groups (see [13], [14], or [11]).

(b) follows either from the general results on compact nilmanifolds described in [13] (in particular Theorem 37), or by carrying out a straightforward Fourier analysis of $L^2(\Gamma_r\backslash H_n)$. □

Proof of Theorem 3.3. Given any unitary representation π of H_n on a Hilbert space $\mathcal{H}$, one may define on the space of analytic vectors in $\mathcal{H}$ the Laplacian

$$\Delta_{\pi,g} = -\sum_{i=1}^{2n+1} [\pi_*(U_i)]^2, \tag{4}$$

where $\{U_i : 1 \le i \le 2n+1\}$ is any g-orthonormal basis of $\mathfrak{h}_n$. By the remarks in 3.1, $\Sigma(r,g)$ is the compilation of the spectra of the operators $\Delta_{\pi,g}$ as π ranges over all representations occurring in the direct sum decomposition of R. $L^2(\mathcal{L}_r\backslash\mathbf{R}^{2n})$ may be identified with the subspace $\mathcal{H}_1$ of $L^2(\Gamma_r\backslash H_n)$ spanned by the characters f_τ, $\tau \in \mathcal{A}_r$. Since the center of H_n acts trivially on this space, (1) implies that the spectrum of Δ on $\mathcal{H}_1$ is just the spectrum of the torus $T_{r,h}$ and thus is given by $\Sigma_1(r,h)$ (see Lemma 3.4). To complete the proof, it therefore suffices to show that the eigenvalues of both $\Delta_{\pi_c,g}$ and $\Delta_{\pi_{-c},g}$, $c \in \mathbf{R}-\{0\}$, are the numbers $\mu(c,k)$ defined in 3.2(b).

Let $\{X_1', \ldots, X_n', Y_1', \ldots, Y_n'\}$ be the orthonormal basis of $(\mathbf{R}^{2n}, h)$ defined in Lemma 3.5 and let ψ be the unique linear map which fixes Z and maps X_i' to $d_i X_i$, Y_i' to $d_i Y_i$, $1 \le i \le n$. By 2.1 and 3.5, ψ is an automorphism of $\mathfrak{h}_n$. Continuing to denote by ψ the corresponding automorphism of H_n, $\pi_c' = \pi_c \circ \psi$ is an irreducible unitary representation of H_n which agrees with π_c on the center of H_n and hence, by Lemma 3.7(a), is unitarily equivalent to π_c. By (4), $\Delta_{\pi_c',g}$ is similar to $\Delta_{\pi_c,g}$ and thus has the same eigenvalues. Since $\{X_1', \ldots, X_n', Y_1', \ldots, Y_n', g_{2n+1}^{-1/2} Z\}$ is an orthonormal basis of $(\mathfrak{h}_n, g)$ and $(\pi_c')_*(X_i') = d_i \pi_c(X_i)$, etc.,

$$\Delta_{\pi_c',g} = -\frac{1}{g_{2n+1}}[\pi_c(Z)]^2 - \sum_{i=1}^{n} d_i^2\{\pi_c(X_i)^2 + \pi_c(Y_i^2)\}.$$

By 3.6(b), for $u = (u_1, \ldots, u_n) \in \mathbf{R}^n$ and $f(u)$ a smooth square integrable function, we have

$$(\Delta_{\pi_c',g} f)(u) = \left\{\frac{4\pi^2c^2}{g_{2n+1}} + \sum_{i=1}^{n} d_i^2(2\pi c u_i)^2\right\} f(u) - \sum_{i=1}^{n} d_i^2 \frac{\partial^2 f}{\partial u_i^2}(u). \tag{5}$$

Now recall that as $k=(k_1,k_2,\ldots,k_n)$ ranges over all n-tuples of non-negative integers, the Hermite functions

$$h_k(v)=\exp[|v|^2/2]\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial v_1^{k_1}\cdots\partial v_n^{k_n}}\exp[-|v|^2]$$

form an orthogonal Hilbert basis of $L^2(\mathbf{R}^n)$ and satisfy

$$(v_i^2-\partial^2/\partial v_i^2)h_k=(2k_i+1)h_k \quad \text{for } 1\le i\le n.$$

For $\tilde{h}_k(u)=h_k(\sqrt{2\pi|c|}\,u)$, it follows from (5) that $\tilde{h}_k$ is an eigenfunction of $\Delta_{\pi_c',g}$ with eigenvalue $\mu(c,k)$ given by 3.2(b). By our remarks above, the proof is now complete. □

4. Isospectral Heisenberg manifolds which are not isometric.

(4.1) THEOREM. *Let* $M=(\Gamma_r\backslash H_n,g)$ *and* $M'=(\Gamma_{r'}\backslash H_n,g')$, *where*

$$g=\begin{bmatrix} h & 0\\ 0 & g_{2n+1}\end{bmatrix} \quad \textit{and} \quad g'=\begin{bmatrix} h' & 0\\ 0 & g'_{2n+1}\end{bmatrix}.$$

Then M and M′ are isospectral if the following four conditions are satisfied:

(a) $g_{2n+1}=g'_{2n+1}$;

(b) $|\Gamma_{r'}|=|\Gamma_r|$ (*see* 2.8);

(c) $h'={}^t\alpha h\alpha$ *for some* $\alpha\in\widetilde{\mathrm{Sp}}(n,\mathbf{R})$ (*see* 2.6(a) *for the definition of* $\widetilde{\mathrm{Sp}}(n,\mathbf{R})$);

(d) $\mathcal{L}_{r'}=\alpha^{-1}\sigma\mathcal{L}_r$ *for some* $2n\times 2n$ *matrix* σ *satisfying* ${}^t\sigma h\sigma=h$ ($\mathcal{L}_r$ *and* $\mathcal{L}_{r'}$ *defined as in* 2.3).

Under conditions (a)–(d), *M and M′ are isometric if and only if* $r=r'$ *and it is possible to choose* α *in* (c) *and* σ *in* (d) *so that* σ *is the identity.*

Proof. Condition (c) implies that $h^{-1}J$ and $(h')^{-1}J$ have the same eigenvalues. Indeed, ${}^t\alpha J\alpha=\epsilon J$ ($\epsilon=\pm1$) and $h'={}^t\alpha h\alpha$, so $(h')^{-1}J=\alpha^{-1}(\epsilon h^{-1}J)\alpha$. Hence $(h')^{-1}J$ has the same eigenvalues as $\epsilon h^{-1}J$ and hence as $h^{-1}J$. Consequently (a)–(c) and Definition 3.2(b) imply that $\Sigma_2(r,g)=\Sigma_2(r',g')$.

Using (c) and (d) with $\cong$ denoting isometry, we have

$$(\mathcal{L}_{r'}\backslash\mathbf{R}^{2n},h')\cong(\mathcal{L}_r\backslash\mathbf{R}^{2n},(\alpha^{-1}\sigma)^*\alpha^*h))\cong(\mathcal{L}_r\backslash\mathbf{R}^{2n},h)$$

and thus $\Sigma_1(r',h')=\Sigma_1(r,h)$ by Lemma 3.4. By Theorem 3.3, $\Sigma(r,g)=\Sigma(r',g')$.

By Theorem 2.7, M and M' are isometric if and only if $r=r'$ and $g'=\varphi^*g$ for some φ of the form

$$\varphi=\begin{bmatrix}\beta & 0\\ 0 & \epsilon\end{bmatrix},\quad \beta\in\widetilde{\mathrm{Sp}}(n,\mathbf{R}),$$

satisfying $\varphi(\Gamma_r)=\Gamma_r$. Choosing $\alpha=\beta$ in (c), the condition $\varphi(\Gamma_r)=\Gamma_r$ means σ can be chosen to be the identity. □

(4.2) REMARKS. (i) Conditions (c) and (d) of Theorem 4.1 are equivalent to:

(c′) $(h')^{-1}J$ and $h^{-1}J$ have the same eigenvalues;

(d′) $\delta_{r'}h'\delta_{r'}={}^t\psi(\delta_r h\delta_r)\psi$ for some $\psi\in\widetilde{\mathrm{SL}}(2n,\mathbf{Z})$, with $\delta_r,\delta_{r'}$ the diagonal matrices defined in 2.6(c).

Indeed, we showed that (c) implies (c′) in the proof of (4.1), and the converse follows trivially with $\alpha\in\mathrm{Sp}(n,\mathbf{R})$ the transformation mapping the h-orthonormal basis of $\mathbf{R}^{2n}$ (given by Lemma 3.5) to the corresponding h'-orthonormal basis. We could therefore have simplified (c) by stipulating that $\alpha \in \mathrm{Sp}(n, \mathbf{R})$, but this would entail replacing the isometry condition $\sigma = \mathrm{Id}$ by a more cumbersome condition. The elements σ in (d) and ψ in (d′) are related by $\sigma = \alpha\delta_{r'}\psi^{-1}\delta_r^{-1}$. Using (d′), the isometry conditions become

$$r = r' \quad \text{and} \quad \delta_r \psi \delta_r^{-1} = \alpha \in \mathrm{Sp}(n, \mathbf{R}) \cap \delta_r \, \mathrm{SL}(2n, \mathbf{Z})\delta_r^{-1}.$$

(ii) We do not know the extent to which the conditions (a)–(d) of Theorem 4.1 are necessary as well as sufficient for M and M' to be isospectral. We will see later (in proving Theorem 5.1) that (a) is necessary if one is to separately have $\Sigma_1(r, h) = \Sigma_1(r', h')$ and $\Sigma_2(r, g) = \Sigma_2(r', g')$. Given (c), (d) is equivalent to asserting that the tori $T_{r,h}$ and $T_{r',h'}$ are not only isospectral but isometric. It is difficult to imagine how the multiplicities in $\Sigma_2(r, g)$ could match those of $\Sigma_2(r', g')$ if (b) were not satisfied.

(4.3) THEOREM. *Let $n > 1$ and let Γ_r and Γ_s be uniform discrete subgroups of H_n as in 2.3. If $r \neq s$ but $|\Gamma_r| = |\Gamma_s|$ (i.e., $r_1 \cdots r_n = s_1 \cdots s_n$), then there exist continuous families $\{g_t : t \geq 0\}$ and $\{g_t' : t \geq 0\}$ of non-isometric left-invariant Riemannian metrics on H_n such that $(\Gamma_r \backslash H_n, g_t)$ is isospectral to $(\Gamma_s \backslash H_n, g_t')$ for every $t \geq 0$.*

We note that by Corollary 2.5 the manifolds $\Gamma_r \backslash H_n$ and $\Gamma_s \backslash H_n$ have different fundamental groups, so the isospectral manifolds of Theorem 4.3 are not homeomorphic.

Proof. Consider diagonal matrices g and g' with diagonal entries $a_1, \ldots, a_n$, $b_1, \ldots, b_n, g_{2n+1}$ and $a_1', \ldots, a_n', b_1', \ldots, b_n', g_{2n+1}'$, where $a_1, b_1, b_2, \ldots, b_n, g_{2n+1}$ are arbitrary in $\mathbf{R}^+$ and the remaining entries are defined by $g_{2n+1}' = g_{2n+1}$, $b_n' = b_1$, $a_n' = a_1$, and (for $1 \leq i \leq n-1$) $b_i' = b_{i+1}$, $a_i' = a_{i+1} = (r_i/s_i)^2 a_i$. We claim that $(\Gamma_r \backslash H_n, g)$ is isospectral to $(\Gamma_s \backslash H_n, g')$ for every choice of the $n+2$ parameters $a_1, b_1, \ldots, b_n, g_{2n+1}$. Indeed, we have conditions (a) and (b) of Theorem (4.1), so it suffices to check conditions (c′) and (d′) of (4.2). By definition, $a_n' b_n' = a_1 b_1$ and, for $1 \leq i \leq n-1$, $a_i' b_i' = a_{i+1} b_{i+1}$. This verifies (c′) since, in the notation of 2.6(5), $h^{-1}J$ has eigenvalues $\pm(-a_i b_i)^{-1/2}$, etc. Next note that $\delta_r h \delta_r$ has diagonal entries $r_1^2 a_1, \ldots, r_n^2 a_n, b_1, \ldots, b_n$. By definition, $b_1', \ldots, b_n'$ is a permutation of $b_1, \ldots, b_n$ and $s_i^2 a_i' = r_i^2 a_i$ for $1 \leq i \leq n-1$. From $|\Gamma_r| = |\Gamma_s|$, it follows that $s_n^2 a_n' = r_n^2 a_n$ as well. The entries of $\delta_s h' \delta_s$ are thus a permutation of those of $\delta_r h \delta_r$ and this verifies (d′).

Using Theorem (2.7), it follows that for one-parameter families of metrics g_t and g_t' arising in this way from a suitably chosen path in our $(n+2)$-dimensional parameter space, g_{t_1} is not isometric to g_{t_2} for $t_1 \neq t_2$ and similarly for g_{t_1}' and g_{t_2}'. □

(4.4) REMARK. The metric g constructed in the proof of 4.3 depends on $n+2$ positive parameters $b_1, \ldots, b_n$, a_1, and g_{2n+1}. Following 3.2(c), denote the eigenvalues of $h^{-1}J$ by $\pm\sqrt{-1}\, d_1^2, \ldots, \pm\sqrt{-1}\, d_n^2$. As the b_i's vary (with a_1 arbitrary but fixed), $(d_1, \ldots, d_n)$ takes on every value in $\mathbf{R}^+ \times \cdots \times \mathbf{R}^+$.

(4.5) THEOREM. *Let* $M=(\Gamma_r\backslash H_n, g)$ *and* $M'=(\Gamma_s\backslash H_n, g')$ *be the isospectral manifolds constructed as in the proof of Theorem* 4.3. *If* $d_1=d_2=\cdots=d_n$ *(see* 4.4), *then M and M' are isospectral on p-forms for every* $p\geq 0$.

Proof. See the appendix. □

(4.6) REMARK. Let $n>1$. One can construct continuous families of metrics g_t and g_t' on the same manifold $\Gamma_r\backslash H_n$ such that $(\Gamma_r\backslash H_n, g_t)$ is isospectral (on functions) but not isometric to $(\Gamma_r\backslash H_n, g_t')$. To see this, use the same notations as in Theorem (4.3) for g and g' but now take $a_1, \ldots, a_n$, b_1, and g_{2n+1} as free parameters and define the remaining entries by $g_{2n+1}'=g_{2n+1}$, $a_1'=(r_n/r_1)^2a_n$, $b_1'=b_1$, and (for $1\leq i\leq n-1$) $a_{i+1}'=(r_i/r_{i+1})^2a_i$, $b_{i+1}=b_{i+1}'=a_i'b_i'/a_{i+1}$. As in the proof of Theorem (4.3), one checks that (c′) and (d′) are satisfied; for example, from the defining relations, $a_{i+1}b_{i+1}=a_i'b_i'$ $(1\leq i\leq n-1)$ and

$$a_1'\cdots a_n'b_1'\cdots b_n'=a_1\cdots a_nb_1\cdots b_n,$$

whence $a_1b_1=a_n'b_n'$. Hence $(\Gamma_r\backslash H_n, g)$ and $(\Gamma_r\backslash H_n, g')$ are isospectral. Since the manifolds are in this case diffeomorphic, one must check directly, using the last statement of Theorem (4.1), that for generic choices of the parameters, $(\Gamma_r\backslash H_n, g)$ and $(\Gamma_r\backslash H_n, g')$ are not isometric.

5. Spectral rigidity. A continuous isospectral deformation of a Riemannian manifold (M, g) is a continuous family g_t, $t\geq 0$, of Riemannian metrics on M such that $g_0=g$ and (M, g_t) is isospectral to (M, g) for all t. The deformation is non-trivial if (M, g_t) is non-isometric to (M, g) for all $t>0$. We conjecture that no Heisenberg manifold admits a non-trivial continuous isospectral deformation. We will see (Theorem 5.4) that the conjecture is true in dimension 3; in fact, any two isospectral 3-dimensional Heisenberg manifolds are isometric. The following theorem supports the conjecture in higher dimensions.

(5.1) THEOREM. *As in Theorem* 2.7, *we parameterize n-dimensional Heisenberg manifolds by pairs* $(r, g)\in\mathcal{G}_n$. *For* $(r, g)\in\mathcal{G}_n$,

$$S=\{(r', g')\in\mathcal{G}_n: \Sigma_1(r, h)=\Sigma_1(r', h') \text{ and } \Sigma_2(r, g)=\Sigma_2(r', g')\}$$

is a countable set.

Proof. If $(r', g')\in S$ then $(\Gamma_r\backslash H_n, g)$ and $(\Gamma_{r'}\backslash H_n, g')$ are isospectral by Theorem 3.3 and, by Lemma 3.4., so are the flat tori $T_{r,h}$ and $T_{r',h'}$. It is well known that isospectral manifolds have the same volume. By Proposition 2.9 and its analog for flat tori, we therefore have

$$|\Gamma_r|(\det g)^{1/2}=|\Gamma_{r'}|(\det g')^{1/2} \quad\text{and}\quad |\Gamma_r|(\det h)^{1/2}=|\Gamma_{r'}|(\det h')^{1/2}.$$

Since $\det g=(\det h)g_{2n+1}$, it follows that $g_{2n+1}=g_{2n+1}'$.

Every flat torus is isometric to $T_{r,\tilde{h}}$ for some flat metric $\tilde{h}$ on $\mathbf{R}^{2n}$. Kneser, in unpublished work, proved that the number of isometry classes of flat tori which are isospectral to a given flat torus is finite. Thus there are metrics $h^{(1)}, \ldots, h^{(m)}$ such that for every $(r', g')\in S$, there exists j such that $T_{r',h'}$ is isometric to $T_{r,h^{(j)}}$;

that is, there exists a linear map ψ such that $h'=\psi^* h^{(j)}$ and $\psi(\mathcal{L}_{r'})=\mathcal{L}_r$. The last condition forces ψ to lie in the discrete set $\delta_r\,\mathrm{SL}(2n,\mathbf{Z})\delta_{r'}^{-1}$ and we conclude that there are only countably many possibilities for the pairs (r',h').

(5.2) REMARKS. Suppose g_t, $t\geq 0$, is a continuous family of isospectral Riemannian metrics on $(\Gamma_r\backslash H_n)$ with $g=g_0$. If the deformation is non-trivial, Theorem 5.1 implies that for all $t\neq 0$ sufficiently small, $\Sigma_1(r,h_t)\neq\Sigma_1(r,h)$ and $\Sigma_2(r,g_t)\neq\Sigma_2(r,g)$ even though $\Sigma(r,g_t)=\Sigma(r,g)$. It is easy to check that the asymptotic distribution of eigenvalues in $\Sigma_1(r,h)$ differs from that in $\Sigma_2(r,g)$. In dimensions 3 and 5, $\Sigma_2(r,g)$ has higher order than $\Sigma_1(r,h)$ (the proof in dimension 3 is given in 5.4); in dimensions 7 and higher the situation is reversed. Thus in dimension ≥ 7, for example, if $\Sigma(r,g_t)=\Sigma(r,g)$ then $\Sigma_1(r,h_t)$ and $\Sigma_1(r,h)$ must agree except for subsets of asymptotically lower order. Perhaps this is enough to imply that $\Sigma_1(r,h_t)=\Sigma_1(r,h)$ and hence that the deformation is trivial.

(5.3) PROPOSITION. *If M and M' are isospectral Riemannian Heisenberg manifolds of dimension* 3 *or* 5, *then M and M' are locally isometric.*

Proof. Let (H_n,g) be the simply-connected covering of a Riemannian Heisenberg manifold M. We may assume g is of the form 2.6(5). By Lemma 3.5, we may choose an orthonormal basis $\mathcal{B}'=\{X_1',\dots,X_n',Y_1',\dots,Y_n',Z'\}$ of $\mathfrak{h}_n$ relative to g such that $[X_i',Y_i']=a_iZ'$ $(i=1,\dots,n)$, with $0<a_1\leq\cdots\leq a_n$ and such that all other brackets of basis vectors equal zero. ($a_i=d_i^2(g_{2n+1})^{1/2}$ after reordering.) Moreover, by 3.5, the a_i's uniquely determine the isometry class of (H_n,g) and hence determine M up to local isometry. Thus we need only show that when $n=1$ or 2, $\Sigma(M)$ determines the a_i's. Recall that $\Sigma(M)$ determines the volume of M and the integrals over M of τ and of $2|R|^2-2|\rho|^2+5\tau^2$, where R and ρ are the curvature tensor and Ricci tensor of M and τ is the scalar curvature. (See [1].) In our case M is locally homogeneous, so τ is constant and R_p and ρ_p are independent of $p\in M$; thus $\Sigma(M)$ determines τ and $2|R|^2-2|\rho|^2+5\tau^2$. By a standard computation, one finds that

$$\tau=-\frac{3}{2}\sum_{i=1}^{n}a_i^2,\quad |R|^2=\frac{3}{4}\sum_{i=1}^{n}a_i^4+\frac{3}{2}\sum_{i\neq j}a_i^2a_j^2,\quad\text{and}\quad |\rho|^2=\frac{3}{4}\sum_{i=1}^{n}a_i^4+\sum_{i\neq j}\frac{1}{4}a_i^2a_j^2.$$

In particular, when $n=1$, τ uniquely determines a_1; when $n=2$,

$$\tau\quad\text{and}\quad 2|R|^2-2|\rho|^2+5\tau^2$$

together determine a_1 and a_2. □

(5.4) THEOREM. *A three-dimensional Heisenberg manifold M is uniquely determined up to isometry by the spectrum $\Sigma(M)$.*

Proof. Let M and M' be isospectral 3-dimensional Heisenberg manifolds. By Theorem 2.7, we may assume that $M=(\Gamma_r\backslash H_1,g)$ and $M'=(\Gamma_{r'}\backslash H_1,g')$ for some $r,r'\in\mathbf{Z}^+$ and g,g' of the form 2.6(5). By Lemma 3.5, there exists an orthonormal basis $\{X_0,Y_0,Z_0\}$ of $\mathfrak{h}_1$ relative to g such that $Z_0=g_3^{-1/2}Z$ and $[X_0,Y_0]=d^2Z=d^2(g_3)^{1/2}Z_0=(g_3/\det h)^{1/2}Z_0$, since $d^4=\det(h^{-1}J)=\det h^{-1}$. Hence the proof of Proposition 5.3 shows that $\Sigma(M)$ determines $g_3/\det(h)$ as well as the volume $r(\det g)^{1/2} = r(\det(h)g_3)^{1/2}$. Therefore

$$g_3/g_3' = \det(h)/\det(h') = r'/r. \tag{1}$$

We claim that M is isometric to M' provided that $r = r'$. Indeed, if $r = r'$, (1) says that $g_3 = g_3'$ and $\det(h) = \det(h')$; consequently $\Sigma_2(r, g) = \Sigma_2(r, g')$ by 3.2. Since $\Sigma(r, g) = \Sigma(r', g')$, it follows that $\Sigma_1(r, h) = \Sigma_1(r, h')$; that is, by Lemma 3.4 the tori $T_{r,h}$ and $T_{r,h'}$ are isospectral. But any two isospectral flat two-dimensional tori are necessarily isometric (see [1]). Hence $h' = {}^t\beta h\beta$ for some β satisfying $\beta(\mathcal{L}_r) = \mathcal{L}_r$, where $\mathcal{L}_r$ is defined by (2.3). Since $\det h = \det h'$, $\beta \in \widetilde{\mathrm{SL}}(2, \mathbf{R})$. But $\widetilde{\mathrm{SL}}(2, \mathbf{R}) \subset \widetilde{\mathrm{Sp}}(1, \mathbf{R})$, so β extends to an automorphism

$$\varphi = \begin{pmatrix} \beta & 0 \\ 0 & \epsilon \end{pmatrix} \tag{2}$$

of H_1, where $\epsilon = \det(\beta)$. $\varphi(\Gamma_r) = \Gamma_r$ and $g' = {}^t\varphi g \varphi$, so M is isometric to M' as claimed.

We are left to show that the condition $\Sigma(r, g) = \Sigma(r', g')$ implies $r = r'$. Consider the asymptotic distribution of eigenvalues. By a subset Λ of the join of $\Sigma(r, g)$ and $\Sigma(r', g')$, we shall mean a subcollection of elements with possible repetitions. For $s > 0$, $n_s(\Lambda)$ will denote the number of elements of Λ, counted with multiplicities, which are less than s. $n_s(\Sigma_1(r, h))$ is the number of points of $\mathbf{Z}^2$ whose norm relative to the inner product $(\delta_r h \delta_r)^{-1}$ is less than $s^{1/2}$. Hence $n_s(\Sigma_1(r, h)) = O(s)$ and $n_s(\Sigma_1(r', h')) = O(s)$. To estimate $n_s(\Sigma_2(r, g))$, set $A = 4\pi^2/g_3$ and $B = 2\pi/(\det(h))^{1/2}$. It follows from (1) and 3.2 that the elements of $\Sigma_2(r, g)$ and of $\Sigma_2(r', g')$ are of the form

$$\begin{aligned} &\mu(c, k) = Ac^2 + Bc(2k+1) \quad \text{and} \\ &\mu'(c, k) = A(r'/r)c^2 + B(r'/r)^{1/2}c(2k+1), \end{aligned} \tag{3}$$

respectively. $\mu(c, k) < s$ if and only if $c < (s/A)^{1/2}$ and $2k+1 < (s - Ac^2)/(Bc)$. Hence

$$\begin{aligned} n_s(\Sigma_2(r, g)) &= \sum_{c=1}^{[(s/A)^{1/2}]} \sum_{\substack{j=1 \\ j \text{ odd}}}^{[(s-Ac^2)/(Bc)]} 2cr \sim \int_0^{(s/A)^{1/2}} \frac{s - Ac^2}{B} r\, dc \\ &= \tfrac{2}{3} r s^{3/2}/(\sqrt{A}B) = O(s^{3/2}), \end{aligned}$$

and similarly $n_s(\Sigma_2(r', g')) = O(s^{3/2})$. (We note that the first-order approximations do not immediately distinguish $\Sigma_2(r, g)$ and $\Sigma_2(r', g')$ when $r \neq r'$, since for $A' = A(r'/r)$ and $B' = B(r'/r)^{1/2}$, $r'/((A')^{1/2}B') = r/(A^{1/2}B)$.) Let Λ be the "symmetric difference" of $\Sigma_2(r, g)$ and $\Sigma_2(r', g')$ in the sense that each element of $\Sigma_2(r, g) \cup \Sigma_2(r', g')$ occurs in Λ with multiplicity equal to the absolute value of the difference of its multiplicities in $\Sigma_2(r, g)$ and $\Sigma_2(r', g')$. Since $\Sigma(r, g) = \Sigma(r', g')$, Λ is contained in the join of $\Sigma_1(r, h)$ and $\Sigma_1(r', h')$ and hence must satisfy $n_s(\Lambda) \leq O(s)$. We will show that the assumption $r < r'$ implies $n_s(\Lambda)$ is at least $O(s \log(s))$, a contradiction. It will follow that $r \geq r'$ and, by symmetry, $r = r'$. We consider four cases.

Case 1. Suppose A and B are rationally independent. Then by (3), $\mu(c_1, k_1) = \mu(c_2, k_2)$ only if $c_1 = c_2$, $k_1 = k_2$. That is, $\mu(c, k)$ has multiplicity $2cr$ in $\Sigma_2(r, g)$. On the other hand, the multiplicity of $\mu(c, k)$ in $\Sigma_2(r', g')$ is a (possibly zero) multiple of $2r'$. For any c such that cr is not divisible by r', $\mu(c, k)$ occurs in Λ with multiplicity at least 2 for every k. Thus we obtain approximately $(s - Ac^2)/Bc$ eigenvalues in Λ for each such c. Λ therefore contains a subcollection of order approximately

$$(\text{const}) \int_{c=1}^{(s/A)^{1/2}} B^{-1}(s/c - Ac)\, dc = O(s \log(s)),$$

where $(\text{const}) \geq r/r'$.

Case 2. Suppose A and B are rationally dependent and $(r'/r)^{1/2}$ is irrational. Then by (3), $\Sigma_2(r, g) \cap \Sigma_2(r', g') = \varnothing$ and $n_s(\Lambda) = O(s^{3/2})$.

Case 3. Suppose A and B are rationally dependent and $(r'/r)^{1/2}$ is rational but not equal to 2. We may assume, after multiplying all elements of $\Sigma_2(r, g)$ and $\Sigma_2(r', g')$ by a suitable constant, that A and B are relatively prime positive integers. Write $(r'/r)^{1/2} = p/q$ with $(p, q) = 1$. By assumption, $p > q$. (3) implies

$$\begin{aligned} q^2\mu(c, k) &= Ac^2q^2 + Bc(2k+1)q^2, \\ q^2\mu'(c, k) &= Ac^2p^2 + Bc(2k+1)pq. \end{aligned} \tag{4}$$

Subcase a. Suppose $(p, B) = 1$. By (4) and the fact that $(p, q^2) = 1$, $p \mid \mu'(c, k)$ for all pairs (c, k). On the other hand, if p divides both $\mu(c, k)$ and $\mu(c, k+1)$, then p divides $\mu(c, k+1) - u(c, k) = 2Bc$. Thus $p \mid 2c$. By our assumptions, $p > 2$. For all c such that $p \nmid 2c$ and for all k, either $\mu(c, k)$ or $\mu(c, k+1)$ occurs in Λ with the same multiplicity as in $\Sigma_2(r, g)$. It follows that $n_s(\Lambda) = O(s^{3/2})$.

Subcase b. Suppose $(p, B) > 1$. Choose a common prime factor p_0 of p and B. $p_0 \mid q^2\mu'(c, k)$ for all (c, k). However, since $(p_0, Aq^2) = 1$, $p_0 \mid q^2\mu(c, k)$ only if $p_0 \mid c$. It again follows that $n_s(\Lambda) = O(s^{3/2})$.

Case 4. Suppose A and B are rationally dependent and $r'/r = 4$. As in Case 3 we may assume that A and B are relatively prime positive integers. By (3),

$$\begin{aligned} \mu(c, k) &= Ac^2 + Bc(2k+1), \\ \mu'(c, k) &= 4Ac^2 + 2Bc(2k+1). \end{aligned} \tag{5}$$

Thus $\mu'(c, k) = \mu(2c, k)$ for every c, k. However, since $r' = 4r$, μ is counted $8cr$ times in $\Sigma_2(r', g')$ for every c such that $\mu = \mu'(c, k)$, but is counted only $2(2c)r = 4cr$ times in $\Sigma_2(r, g)$ for each such c. Define Σ_{even} (resp. Σ_{odd}) to be the collection of all μ satisfying $\mu = \mu(c, k)$ for some even (resp. odd) positive integer c and some k, with the understanding that μ occurs $2cr$ times for each even (resp. odd) c such that $\mu = \mu(c, k)$. Then $\Sigma_2(r, g)$ is the join of Σ_{even} and Σ_{odd} while $\Sigma_2(r', g')$ is the join of two copies of Σ_{even}, so Λ is the symmetric difference of Σ_{even} and Σ_{odd}. If either A or B is even, then all elements of Σ_{odd} are odd while all elements of Σ_{even} are even; hence $\Lambda = \Sigma_2(r, g)$ and $n_s(\Lambda) = O(s^{3/2})$. Thus we may assume that A and B are both odd.

$\mu(c, k) = c(Ac + B + 2Bk)$. In particular, $c \mid \mu(c, k)$. We consider the elements $\mu(c, k)$ which satisfy the following conditions:

(a) c is a prime; $c \geq 3$. (Hence $Ac+B+2Bk$ is even.)
(b) $q = \frac{1}{2}(Ac+B+2Bk)$ is a prime.
(c) $q > mc$, where $m = \max\{\frac{1}{2}(A+B), 2\}$.

If $A \geq 2$, the definition of q in (b) implies that $Aq^2 > \mu(c, k)$; if $A = 1$, (b) and (c) together imply $Aq^2 > \mu(c, k)$. It follows that $\mu(q, \tilde{k}) > \mu(c, k)$ for every choice of $\tilde{k}$. Thus if $\mu(c, k) = \mu(\tilde{c}, \tilde{k})$ for some pair $(\tilde{c}, \tilde{k}) \neq (c, k)$, then $\tilde{c} < q$. But $\tilde{c}$ divides $\mu(c, k) = 2cq$, so $\tilde{c}$ must equal 2, c, or $2c$. Clearly $\mu(c, k) \neq \mu(c, \tilde{k})$ when $k \neq \tilde{k}$, so the multiplicity of $\mu(c, k)$ in Σ_{odd} is precisely $2cr$, and its multiplicity in Σ_{even} is one of 0, $4r$, $4cr$, or $4cr+4r$. In any case, $\mu(c, k)$ has multiplicity at least $2(c-2)r > \frac{2}{3}cr$ in Λ. Thus it suffices to show that the number of pairs (c, k) satisfying (a)–(c), each counted $\frac{2}{3}cr$ times and with $\mu(c, k) = 2cq \leq s$, has order greater than $O(s)$.

Condition (b) is equivalent to

(b′) $\quad \mu(c, k) = 2cq \quad$ where q is prime and $q \equiv \frac{1}{2}(Ac+B) \bmod B$.

Let $N(s)$ denote the number of elements $\mu(c, k) \leq s$ satisfying (a), (b′), and (c), each counted $\frac{2}{3}cr$ times. Note that these conditions imply that $c \leq (s/2m)^{1/2}$. Fix α with $0 < \alpha < \frac{1}{2}$. For s sufficiently large so that $s^\alpha \leq (s/2m)^{1/2}$, we have

$$N(s) \geq \sum_{\substack{3 \leq c \leq s^\alpha \\ c \text{ prime}}} \frac{2}{3} cr \left[\pi\left(\frac{s}{2c}, \frac{1}{2}(Ac+B), B\right) - \pi\left(mc, \frac{1}{2}(Ac+B), B\right) \right], \tag{6}$$

where $\pi(x, n, B)$ denotes the number of primes congruent to n modulo B which are less than x.

By the prime number theorem (see [2]), $\pi(x, n, B)$ is approximately

$$\frac{1}{\varphi(B)} \frac{x}{\log x} \quad \text{for large } x,$$

where φ is the Euler function. Hence there exist $b_1, b_2 \in \mathbf{R}^+$ depending only on B such that $b_1 x/\log(x) < \pi(x, n, B) < b_2 x/\log(x)$ for all $x \geq 3$. (Dependency of b_1 and b_2 on n can be avoided since n lies in one of only finitely many congruence classes modulo B.) In particular each term in (6) is greater than

$$\frac{2}{3} cr \left(b_1 \frac{s}{2c \log(s/2c)} - b_2 \frac{mc}{\log mc} \right) > b_1' \frac{s}{\log s} - b_2' s^\alpha \tag{7}$$

for some constants $b_1', b_2' \in \mathbf{R}^+$, since $3 \leq c < s^\alpha$. Again by the prime number theorem, the number of primes c in the interval $[3, x]$ is greater than $b_3 x/\log(x)$ for some constant b_3. Hence, by (6) and (7),

$$N(s) > b_1'' \frac{s^{1+\alpha}}{(\log s)^2} - b_2'' \frac{s^{2\alpha}}{\log s}$$

for some $b_1'', b_2'' \in \mathbf{R}^+$ independent of s. It follows that $n_s(\Lambda)$ is at least

$$O\left(\frac{s^{1+\alpha}}{(\log s)^2} \right).$$

Thus in all cases $n_s(\Lambda)$ is at least $O(s \log(s))$ as claimed. This completes the proof. □

Appendix. The spectrum on p-forms. If (M^n, g) is a compact Riemannian manifold, the Laplace–Beltrami operator Δ acts on the space $E^p(M)$ of smooth p-forms by

(1) $$\Delta = d\delta + \delta d,$$

where $\delta = (-1)^{n(p+1)+1} * d *$, $*$ being the Hodge-$*$ operator of (M, g). If $f \in C^\infty(M)$ and $\tau \in E^p(M)$, then ([6, Proposition 4.3]):

(2) $$\Delta(f\tau) = (\Delta f)\tau + f(\Delta \tau) - 2\nabla_{\mathrm{grad}\, f}\tau.$$

Now suppose that $M = \Gamma \backslash G$ (where G is a connected Lie group and Γ a uniform discrete subgroup) and that the metric g on M lifts to a left-invariant metric on G. Elements of the exterior algebra $\Lambda^p(\mathfrak{g}^*)$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra of $\mathfrak{g}$, may be viewed first as left-invariant p-forms on G and then as elements of $E^p(\Gamma \backslash G)$. With this interpretation,

(3) $$E^p(\Gamma \backslash G) = C^\infty(\Gamma \backslash G) \otimes \Lambda^p(\mathfrak{g}^*).$$

Note that if $U_1, \ldots, U_n$ is a basis of $\mathfrak{g}$ orthonormal with respect to g, then $\nabla_{\mathrm{grad}\, f}\tau = \sum_{i=1}^n (U_i f)\nabla_{U_i}\tau$. It therefore follows, from (2) and from equation (1) of Section 3 (with H_n replaced by G), that $\mathcal{H} \otimes \Lambda^p(\mathfrak{g}^*)$ is Δ-invariant whenever $\mathcal{H}$ is a subspace of $C^\infty(\Gamma \backslash G)$ invariant under the right action of G.

(A.1) LEMMA. *Let Γ and Γ' be uniform discrete subgroups of a Lie group G, let g' be a left-invariant Riemannian metric on G, let $\varphi \in \mathrm{Aut}(G)$, and set $g = \varphi^* g'$. Let Δ and Δ' denote the Laplacians of $(\Gamma \backslash G, g)$ and $(\Gamma' \backslash G, g')$. Denote by R and R' the right actions of G on $C^\infty(\Gamma \backslash G)$ and $C^\infty(\Gamma' \backslash G)$, respectively, and let $\mathcal{H} \subset C^\infty(\Gamma \backslash G)$ and $\mathcal{H}' \subset C^\infty(\Gamma' \backslash G)$ be subspaces invariant under R and R', respectively. If $R|_{\mathcal{H}}$ is unitarily equivalent to $R' \circ \varphi|_{\mathcal{H}'}$, then the action of Δ on $\mathcal{H} \otimes \Lambda^p(\mathfrak{g}^*)$ is equivalent to that of Δ' on $\mathcal{H}' \otimes \Lambda^p(\mathfrak{g}^*)$.*

Proof. This lemma is a straightforward application of (2). The details are given in [6, §4.4]. (The additional hypotheses in [6], that $\Gamma = \Gamma'$ and that φ is an "almost inner" automorphism, are not needed in the proof.) □

We now specialize to Heisenberg manifolds.

(A.2) NOTATION. For Γ_r the uniform discrete subgroup of H_n defined in 2.3, we will denote by $\mathcal{H}_{r,1}$ the space of all C^∞ functions on $\Gamma_r \backslash H_n$ which vanish on the center of H_n, and by $\mathcal{H}_{r,2}$ the complementary subspace of $C^\infty(\Gamma_r \backslash H_n)$ invariant under the right action of H_n. (Note that the action of H_n on any irreducible subspace of $\mathcal{H}_{r,2}$ is equivalent to the representation π_c defined in 3.6 for some c.) By the remarks above, if g is any left-invariant Riemannian metric on H_n and Δ is the Laplace–Beltrami operator of $(\Gamma_r \backslash H_n, g)$, then $\mathcal{H}_{r,i} \otimes \Lambda^p(\mathfrak{g}^*)$ is Δ-invariant, $i = 1, 2$. Denote by $\Sigma_i^p(r, g)$ the collection of eigenvalues, with multiplicities, of Δ on $\mathcal{H}_{r,i} \otimes \Lambda^p(\mathfrak{g}^*)$.

(A.3) PROPOSITION. *We use notation* A.2 *and let*

$$g = \begin{bmatrix} h & 0 \\ 0 & g_{2n+1} \end{bmatrix}$$

as in 2.6(5). *Then* $\Sigma_2^p(r, g)$ *is uniquely determined by* $|\Gamma_r|$, g_{2n+1}, *and the eigenvalues of* $h^{-1}J$ (*see* 3.2(b)).

Proof. Suppose $|\Gamma_r| = |\Gamma_s|$ and let

$$g' = \begin{bmatrix} h' & 0 \\ 0 & g_{2n+1} \end{bmatrix}$$

be another metric on H_n such that $h^{-1}J$ and $(h')^{-1}J$ have the same eigenvalues and $g_{2n+1} = g'_{2n+1}$. The argument given in remark 4.2(i) shows that $h' = {}^t\alpha h \alpha$ for some $\alpha \in \mathrm{Sp}(n, \mathbf{R})$ and hence

$$g' = \varphi^* g \quad \text{for } \varphi = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{Aut}(H_n).$$

Let R_r and R_s denote the action of H_n on $C^\infty(\Gamma_r \backslash H_n)$ and $C^\infty(\Gamma_s \backslash H_n)$ as defined in 3.1. By Lemma 3.7(b), for every integer c, π_c occurs in R_r with the same multiplicity as in R_s. Moreover, since φ acts as the identity on the center of H_n, $\pi_c \circ \varphi$ is unitarily equivalent to π_c. (See 3.7(a).) It follows that $R_s \circ \varphi \,|\, \mathcal{H}_{2,s}$ is unitarily equivalent to $R_r \,|\, \mathcal{H}_{2,r}$. Thus $\Sigma_2^p(r, g) = \Sigma_2^p(r', g')$ by Lemma A.1. □

$\Sigma_1^p(r, g)$ depends more intricately on Γ_r and g except in the special case of Theorem 4.5, that is, the case in which the eigenvalues $\pm\sqrt{-1}\, d_1^2, \dots, \pm\sqrt{-1}\, d_n^2$ of $h^{-1}J$ satisfy $d_1 = d_2 = \cdots = d_n$.

Proof of Theorem 4.5. From Proposition A.3, it suffices to prove that $\Sigma_1^p(r, g) = \Sigma_1^p(s, g')$. By Lemma 3.7(b),

(4) $$\mathcal{H}_{r,1} = \bigoplus_{\tau \in \mathcal{A}_r} \mathbf{R} f_\tau \quad \text{and} \quad \mathcal{H}_{s,1} = \bigoplus_{\tau \in \mathcal{A}_s} \mathbf{R} f_\tau,$$

where f_τ is defined in 3.6(a) and $\mathcal{A}_\tau$ in 3.2(c). Use g to define $\eta: \mathfrak{h}_n \to \mathfrak{h}_n$, $\#: \mathfrak{h}_n^* \to \mathfrak{h}_n$ and an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}_n^*$, with g' used to define analogous objects η', $\#'$, and $\langle \cdot, \cdot \rangle'$. Since $\Sigma_1(r, h) = \Sigma_1(s, h')$ by the proof of Theorem (4.3), Lemma (3.4) implies the existence of a bijection $\theta: \mathcal{A}_r \to \mathcal{A}_s$ such that $\langle \theta\tau, \theta\tau \rangle' = \langle \tau, \tau \rangle$. We will show below that for each $\tau \in \mathcal{A}_r$ there exists $\varphi \in \mathrm{Aut}(H_n)$ such that $\tau = \theta\tau \circ \varphi_*$. But then $f_\tau = f_{\theta\tau} \circ \varphi$, which means that $R_r \,|\, \mathbf{R} f_\tau$ is unitarily equivalent to $R_s \circ \varphi \,|\, \mathbf{R} f_{\theta\tau}$ (where R_r and R_s denote, as usual, the right actions of H_n on $C^\infty(\Gamma_r \backslash H_n)$ and $C^\infty(\Gamma_s \backslash H_n)$). By (4) and Lemma (4.1), it will follow that $\Sigma_1^p(r, g) = \Sigma_1^p(s, g')$.

Our assumption that $d_1 = d_2 = \cdots = d_n$ implies that $\eta^2 = -d_1^4(\mathrm{Id})$. Since the h-orthonormal basis of Lemma (3.5) arose by taking real and imaginary parts of eigenvectors in $\mathfrak{h}_n^{\mathbf{C}}$ of η, in the present case any unit vector in $\mathfrak{h}_n$ can serve as the first vector of such a basis. Thus we may choose an h-orthonormal basis $\mathcal{B} = \{U_1, V_1, \dots, U_n, V_n\}$ and an h'-orthonormal basis $\mathcal{B}' = \{U'_1, V'_1, \dots, U'_n, V'_n\}$ of $\mathbf{R}^{2n}$ such that

(6) $$[U_i, V_i] = d_1^2 Z = [U'_i, V'_i] \quad (1 \le i \le n),$$

with all other brackets of pairs of elements in $\mathcal{B}$ (resp., $\mathcal{B}'$) being trivial, and such that

(7) $$U_1 = \langle \tau, \tau \rangle^{-1/2} \# \tau, \qquad U'_1 = \langle \tau, \tau \rangle^{-1/2} \#'(\theta\tau).$$

By (6), there exists $\varphi \in \operatorname{Aut}(H_n)$ satisfying $\varphi_*(U_i)=U_i'$, $\varphi_*(V_i)=V_i'$, and $\varphi_*(Z)=Z$. Then $g=\varphi^*g'$ and by (7), $\theta\tau\circ\varphi_*=\tau$. □

In contrast to Theorem 4.5, the following computation suggests that $\Sigma_1^1(r,g)$ may often distinguish the isospectral manifolds of Theorem 4.3 when the d_i's are distinct.

(A.4) PROPOSITION. *We use the notation of* A.2 *and* 3.2(c). *For* $\tau\in\mathfrak{A}_r$, *let*

$$\alpha(\tau)=4\pi^2\langle\tau,\tau\rangle,\qquad A=\frac{g_{2n+1}}{2}\sum_{i=1}^n d_i^4,$$

$$B(\tau)=4\pi^2 g_{2n+1}\langle\#^{-1}\eta\#\tau,\#^{-1}\eta\#\tau\rangle,\qquad \beta_\pm(\tau)=\alpha(\tau)+A\pm\sqrt{A^2+B(\tau)}.$$

Then $\Sigma_1^1(r,g)$ *is the collection of numbers* λ *of the form* $\lambda=\alpha(\tau)$ *or* $\lambda=\beta_\pm(\tau)$ *for some* $\tau\in\mathfrak{A}_r$. λ *occurs in* $\Sigma_1^1(r,g)$ $2n-1$ *times for each* $\tau\in\mathfrak{A}_r$ *such that* $\lambda=\alpha(\tau)$, *and once for each* $\tau\in\mathfrak{A}_r$ *such that* $\lambda=\beta_+(\tau)$ *or* $\lambda=\beta_-(\tau)$.

Proof. Let $\tau\in\mathfrak{A}_r$, with $f_\tau(\exp X)=\exp[2\pi\sqrt{-1}\,\tau(x)]$ as in 3.6. Then $\operatorname{grad} f_\tau=2\pi\sqrt{-1}\,\#\tau$. For $\sigma\in\mathfrak{h}_n^*$, (2) yields

$$\Delta(f_\tau\sigma)=f_\tau\{4\pi^2\langle\tau,\tau\rangle\sigma+\Delta\sigma-4\pi\sqrt{-1}\,\nabla_{\#\tau}\sigma\}. \tag{8}$$

Let $\xi=(\#)^{-1}Z/g_{2n+1}$. Thus $\xi(Z)=1$ and $\xi\mid_{\mathbf{R}^{2n}}=0$. Using (1) and Lemma 3.5, an easy computation shows that for $\sigma\in\mathfrak{h}_n^*$,

$$\Delta\sigma=\sigma(Z)g_{2n+1}\left(\sum_{i=1}^n d_i^4\right)\xi=2A\sigma(Z)\xi. \tag{9}$$

Using the standard formula (see [7]),

$$g(\nabla_X Y,U)=\tfrac12\{g([X,Y],U)-g(Y,[X,U])-g(X,[Y,U])\}\quad\text{for } X,Y,U\in\mathfrak{h}_n,$$

together with $\nabla_X(\#\sigma)=\#\nabla_X\sigma$, routine computation yields

$$\nabla_{\#\tau}\sigma=-\tfrac12\sigma(Z)(\#)^{-1}\eta\#\tau+\tfrac12\sigma(\eta\#\tau)g_{2n+1}\xi. \tag{10}$$

From (8), (9), and (10), we see that if σ belongs to the $(2n-1)$-dimensional subspace orthogonal to both ξ and $\#^{-1}\eta\#\tau$, then $f_\tau\sigma$ is an eigenvector of Δ for the eigenvalue $\alpha(\tau)=4\pi^2\langle\tau,\tau\rangle$. Moreover, on the two-dimensional subspace spanned by $f_\tau\xi$ and $f_\tau(\#^{-1}\eta\#\tau)$, $\Delta-\alpha(\tau)\,\mathrm{Id}$ is described by the 2×2 matrix

$$\begin{bmatrix} g_{2n+1}\sum_{i=1}^n d_i^4 & -2\pi\sqrt{-1}\,g_{2n+1}\|\eta\#\tau\|^2\\ 2\pi\sqrt{-1} & 0\end{bmatrix}=\begin{bmatrix} 2A & \frac{-\sqrt{-1}}{2\pi}B(\tau)\\ 2\pi\sqrt{-1} & 0\end{bmatrix}.$$

Since the eigenvalues of this matrix are $A\pm\sqrt{A^2+B(\tau)}$, the Proposition now follows from (4). □

(A.5) REMARKS. In [4], the pairs of manifolds isospectral on functions which were constructed in the proof of Theorem 4.3 will be re-examined. It will be shown that in certain cases where the d_i's are not all equal, the spectrum of the Laplacian on 1-forms distinguishes the manifolds. Proposition (A.4) will be used heavily in this demonstration. To our knowledge, these examples provide the first known instance of manifolds isospectral on functions yet non-isospectral on p-forms for some p.

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