PRIMES IN SHORT INTERVALS

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1. Introduction. The distribution of primes in short intervals is an important problem in the theory of prime numbers. The following question is suggested by the prime number theorem: for which functions Φ is it true that

$$\pi(x+\Phi(x))-\pi(x)\sim\frac{\Phi(x)}{\log x}\quad(x\to\infty)?\tag{1.1}$$

Heath-Brown [4] proved that one can choose $\Phi(x)=x^{7/12-\epsilon(x)}$ ($\epsilon(x)\to 0$, as $x\to\infty$), which is a slight improvement of Huxley's result [5], $\Phi(x)=x^{7/12+\epsilon}$ ($\epsilon>0$ fixed). The Riemann hypothesis implies that one can take $\Phi(x)=x^{1/2+\epsilon}$. There is a large gap between these upper bounds and the known lower bounds of $\Phi(x)$. It follows from [9] that (1.1) is wrong if

$$\Phi(x)=\log x(\log\log x\log\log\log\log x/(\log\log\log x)^2).$$

A slight improvement is implicit in the author's paper [7].

On assumption of the Riemann hypothesis this gap can be narrowed considerably if an exceptional set of x-values is admitted. In 1943, A. Selberg [10] proved that, on assumption of the Riemann hypothesis, (1.1) is true for almost all x if $\Phi(x)/(\log x)^2\to\infty$ ($x\to\infty$). By "for almost all values of x" is meant that $x\to\infty$ through any sequence lying outside a certain *exceptional set* $\mathcal{E}$ of x-values, for which the Lebesgue measure of $\mathcal{E}\cap(0,u]$ is $o(u)$ for $u\to\infty$. It is known unconditionally that (1.1) is true for almost all x if $\Phi(x)=x^{1/6+\epsilon}$. This is implicit in the work of Huxley [5].

A natural question is whether Selberg's result is true without exceptions. The purpose of this paper is to show that *exceptions do exist* even for functions $\Phi(x)$ growing considerably faster than $(\log x)^2$.

We prove the following.

THEOREM. *Let* $\Phi(x)=(\log x)^{\lambda_0}$, $\lambda_0>1$. *Then*

$$\limsup_{x\to\infty}\frac{\pi(x+\Phi(x))-\pi(x)}{\Phi(x)/\log x}>1\quad\textit{and}\quad\liminf_{x\to\infty}\frac{\pi(x+\Phi(x))-\pi(x)}{\Phi(x)/\log x}<1.$$

For the range $1<\lambda_0<e^{\gamma}$ *we have even*

$$\limsup_{x\to\infty}\frac{\pi(x+\Phi(x))-\pi(x)}{\Phi(x)/\log x}\geq\frac{e^{\gamma}}{\lambda_0},$$

where γ *denotes Euler's constant.*

Most of the principles of the proof already appear in [7]. At some places however we need sharper estimates.

Received May 11, 1984.
Michigan Math. J. 32 (1985).

2. Basic lemmas. Following [7] we call an integer $q>1$ a "good" modulus if $L(s,\chi|\neq 0$ for all characters $\chi \bmod q$ and all s with

$$\sigma>1-C/\log|q(|t|+1)|.$$

This definition depends on $C>0$. However, if C is sufficiently small, then we have: For all $q>1$, either q is good or there is an exceptional real zero of some quadratic character mod q. In the latter case the exceptional zero and character are unique (Page's theorem, cf. [8, Satz 6.9b]).

We define $P(z)=\prod_{p<z} p$.

LEMMA 1. *There is a constant $C>0$ such that, in terms of C, there exist arbitrarily large values of z for which the modulus $P(z)$ is good.*

Proof. This is Lemma 1 of [7]. □

The constant C will be fixed throughout the rest of the paper such that the conclusion of Lemma 1 is true.

LEMMA 2 (Gallagher). *Let q be a good modulus. Then*

$$\pi(x+h,q,a)-\pi(x,q,a)=\frac{1}{\varphi(q)}(\mathrm{li}(x+h)-\mathrm{li}\,x)(1+O(e^{-cD}+e^{-\sqrt{\log x}})),$$

provided $(a,q)=1$, $x\geq q^D$, and $x/2\leq h\leq x$, where $\log q\geq D\geq D_0$. (Here the constant $D_0>0$ and the constant implied in $O(\ )$ depend only on C in Lemma 1; $c>0$ is an absolute constant.)

Proof. This follows if we combine the proof of Lemma 2 in [7] with the prime number theorem in the form $\pi(x)=\mathrm{li}\,x(1+O(e^{-\sqrt{\log x}}))$. □

We set

$$\Phi(x,y)=|\{n\leq x\colon (n,P(y))=1\}|, \qquad W(z)=\prod_{p<z}\left(1-\frac{1}{p}\right).$$

LEMMA 3 (Buchstab). *Let $\lambda>1$. Then*

$$\lim_{z\to\infty} z^{-\lambda}W(z)^{-1}\Phi(z^\lambda,z)=e^\gamma\omega(\lambda),$$

where $\omega(u)$ is defined by

$$\omega(u)=u^{-1},\quad 1\leq u\leq 2,$$

(2.1)

$$\frac{d}{du}(u\omega(u))=\omega(u-1),\quad u\geq 2,$$

and where the right-hand derivative has to be taken at $u=2$.

Proof. This follows from [2] and from Mertens' formula

$$\prod_{p<z}\left(1-\frac{1}{p}\right)\sim\frac{e^{-\gamma}}{\log z}.$$

A uniform result has been proven in [1]. □

LEMMA 4. *The function* $\omega(u)-e^{-\gamma}$ *changes sign in any interval* $[a-1, a]$, $a \geq 2$.

Proof. This has been established in [6] for the general sieve of Eratosthenes. For the convenience of the reader, however, we give a more specific argument which in [1] is already applied to the problem of the convergence of $\omega(u)$, $(u \to \infty)$.

We set

$$h(u) = \int_0^\infty \exp\left(-ux - x + \int_0^x \frac{e^{-t}-1}{t}\, dt\right) dx.$$

Then one easily checks that

$$\begin{gathered} h \text{ is analytic for } u > -1, \\ uh'(u-1) + h(u) = 0, \\ h(u) \sim \frac{1}{u} \quad \text{as } u \to \infty. \end{gathered} \tag{2.2}$$

Let $f(a) = \int_{a-1}^{a} \omega(u) h(u)\, du + a\omega(a) h(a-1)$. From (2.1) and (2.2) it is easily verified that

$$f'(a) = 0 \quad \text{for all } a \geq 2. \tag{2.3}$$

By [1] we have that $\lim_{u\to\infty} \omega(u) = e^{-\gamma}$. Therefore $f(a) \to e^{-\gamma}$ as $a \to \infty$ which together with (2.3) implies that

$$\int_{a-1}^{a} \omega(u) h(u)\, du + a\omega(a) h(a-1) = e^{-\gamma}. \tag{2.4}$$

Now let

$$g(a) = \int_{a-1}^{a} h(u)\, du + ah(a-1).$$

From (2.2) it is easily seen that $g'(a) = 0$ for all $a > 0$. Since $g(a) \to 1$ as $a \to \infty$, we have:

$$\int_{a-1}^{a} h(u)\, du + ah(a-1) = 1.$$

This together with (2.4) gives:

$$\int_{a-1}^{a} (\omega(u) - e^{-\gamma}) h(u)\, du + a(\omega(a) - e^{-\gamma}) h(a-1) = 0.$$

Since $h > 0$ it follows that either $\omega(u) \equiv e^{-\gamma}$ in $[a-1, a]$ or there is a sign-change of $\omega(u) - e^{-\gamma}$ in $[a-1, a]$. But the first alternative would imply that $\omega(u) \equiv e^{-\gamma}$ in $[1, \infty)$. This concludes the proof of Lemma 4. □

3. Proof of the theorem. We now fix an integer $D \geq D_0$ depending at most on $\epsilon > 0$ to be introduced later. *In the sequel we assume that* $z \to \infty$ *through a set* *of z for which $P(z)$ is a good modulus* in the sense of Lemma 1 and that $z \geq e^{cD}$, where c is the constant in Lemma 2. We also choose an integer $U = U(z)$, such that $U \leq P(z)$.

We consider the matrix

$$\mathfrak{M} = (a_{rs}), \quad \text{where}$$
$$a_{rs} = s + rP(z), \quad 1 \leq s \leq U, \quad P(z)^{D-1} < r \leq 2P(z)^{D-1}.$$

We want to estimate the number of primes in $\mathfrak{M}$ which we denote by $\pi(\mathfrak{M})$. The *rows* of $\mathfrak{M}$ are *intervals* of U consecutive integers, whereas the *columns* of $\mathfrak{M}$ are *arithmetic progressions* with common difference $P(z)$. Only those columns for which $(s, P(z)) = 1$ contain primes. We call such columns *admissible.*

The number of primes in an admissible column is, by Lemma 2:

$$\begin{aligned}\pi(2P(z)^D + s, P(z), s) &- \pi(P(z)^D + s, P(z), s)\\ &= \frac{P(z)^D}{\varphi(P(z))\log(P(z)^D)}(1 + O(e^{-cD}))\\ &= (P(z)^{D-1}/\log(P(z)^D))\,W(z)^{-1}(1 + O(e^{-cD})).\end{aligned}$$

Let the number of admissible columns be $UW(z)c(U,z)$. Then we have

$$\pi(\mathfrak{M}) = \frac{P(z)^{D-1}}{\log(P(z)^D)}\,Uc(U,z)(1 + O(e^{-cD})). \tag{3.1}$$

Now we can conclude the proof of the theorem. To prove the first part we fix $\lambda_1 > \lambda_0$ such that $\omega(\lambda_1) > e^{-\gamma}$, which is possible by Lemma 4. We choose $U = [z^{\lambda_1}]$. By (3.1) and Lemma 3 there is at least one row of $\mathfrak{M}$ with *at least*

$$\frac{U}{\log(P(z)^D)} e^{\gamma}\omega(\lambda_1)(1 + O(e^{-cD})) \text{ primes.}$$

We now set $l_0 = \{\log(P(z)^D)\}^{\lambda_0}$ and divide this row into $K_0 = [U/l_0] + 1$ subintervals of equal length $l_0(1 + o(1))$. At least one of these subintervals, say $(a_l, b_l]$, contains at least

$$r = \frac{U}{K_0(\log(P(z)^D))} e^{\gamma}\omega(\lambda_1)(1 + O(e^{-cD})) \text{ primes.}$$

We set $x = a_l$ and obtain $(a_l, b_l] \subseteq (x, x + \Phi(x)]$. Thus the interval $(x, x + \Phi(x)]$ contains at least

$$r = (\Phi(x)/\log x) e^{\gamma}\omega(\lambda_1)(1 + O(e^{-cD})) \text{ primes.}$$

For any given $\epsilon > 0$ we can fix D such that $r \geq (\Phi(x)/\log x)(e^{\gamma}\omega(\lambda_1) - \epsilon)$, which concludes the proof of the first part of the theorem.

The third part immediately follows by observing that $\omega(u) = u^{-1}$ for $1 \leq u \leq 2$. The proof of the second part is completely analogous to the proof of the first part. We choose $\lambda_2 > \lambda_0$ such that $\omega(\lambda_2) < e^{-\gamma}$ and set $U = [z^{\lambda_2}]$. There is at least one row of $\mathfrak{M}$ with *at most*

$$\frac{U}{\log(P(z)^D)} e^{\gamma}\omega(\lambda_2)(1+O(e^{-cD})) \text{ primes.}$$

We set $l_1 = \{\log(2P(z)^D)\}^{\lambda_0}$ and divide this row into $K_1 = [U/l_1]$ subintervals of equal length. An easy computation similar to the one above shows that at least one of these subintervals contains an interval $(x, x+\Phi(x)]$ with *at most* $(\Phi(x)/\log x)(e^{\gamma}\omega(\lambda_2)+\epsilon)$ primes. This concludes the proof. □

ACKNOWLEDGMENT. The author wishes to thank Professors H. L. Montgomery and R. C. Vaughan for pointing out to him the proof of Lemma 4.

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