ON THE SINGULARITIES OF SIMPLE PLANE CURVES

Tibor Bisztriczky

Let Γ be a differentiable curve in a real projective plane P^2 met by any line in P^2 at a finite number of points. The singular points of Γ are inflections, cusps (cusps of the first kind), and beaks (cusps of the second kind). Let n_1, n_2, and n_3 be the number of these points in Γ respectively. Γ is non-singular if $n(\Gamma) = n_1+n_2+n_3 = 0$. We are interested in the following questions: Under what conditions is Γ non-singular? What is then the minimum value of $n(\Gamma)$ and how is this minimum value related to n_1, n_2, and n_3? We assume that every line in P^2 meets Γ and thus exclude all non-singular conics.

It is known that either every line in P^2 meets Γ with an odd multiplicity or every line in P^2 meets Γ with an even multiplicity. The curves of the former type have been studied by Möbius [3], Kneser [2], and Scherk [6], among others. In [1], we began an investigation of curves of the latter type. Though there exist such non-singular curves with as few as one multiple point, it was shown that the existence of singularities is dependent not only on the number, but also the type, of multiple points in the curve. More precisely, if a curve Γ is *almost-simple* (any closed subarc of Γ with coincident end-points is met by every line in P^2) then $n(\Gamma) \geq 2$ and $n_1+2n_2+n_3 \geq 4$. This investigation is continued in the present paper under the assumption that Γ possesses no multiple points. We claim that $n(\Gamma) \geq 3$ and if $n_2 > 0$, then $n_1+2n_2+n_3 \geq 6$.

We assume that P^2 has the usual topology. Let $p, q, \ldots$ and $L, M, \ldots$ denote the points and lines of P^2 respectively. We denote by $\langle p, L, \ldots \rangle$, the flat of P^2 spanned by $p, L, \ldots$.

Differentiable curves. Let $T \subset P^2$ be an oriented line. For $t_0 \neq t_1$ in T, let $[t_0, t_1]$ denote the oriented closed segment of T with the initial point t_0 and the terminal point t_1. We set $(t_0, t_1) = [t_0, t_1] \setminus \{t_0, t_1\}$, $[t_0, t_1) = (t_0, t_1) \cup \{t_0\}$, and $(t_0, t_1] = (t_0, t_1) \cup \{t_1\}$. Then $T = [t_0, t_1] \cup (t_1, t_0) = [t_0, t_1) \cup [t_1, t_0)$. By a (two-sided) neighborhood of t in T we mean a segment $U(t) = (t_0, t_1)$ containing t. Then $U^-(t) = (t_0, t)$, $U^+(t) = (t, t_1)$, and $U'(t) = (t_0, t) \cup (t, t_1)$ are left, right, and deleted neighborhoods of t in T respectively.

A *curve* Γ in P^2 is a continuous map from T into P^2. A line M is the *tangent* of Γ at t if $M = \lim \langle \Gamma(t), \Gamma(t') \rangle$ as t' tends to t in $T \setminus \{t\}$; in which case we set $M = \Gamma_1(t)$. A curve Γ is (*directly*) *differentiable* if $\Gamma_1(t)$ exists for each $t \in T$ and any line in P^2 meets $\Gamma(T)$ at a finite number of points. Henceforth Γ is differentiable, and for convenience we identify $\Gamma(T)$ with Γ.

Let $\mathfrak{M} \subseteq T$ be a segment. We call $\Gamma|_{\mathfrak{M}}$ a *subarc* of Γ and again identify $\Gamma(\mathfrak{M})$ with $\Gamma|_{\mathfrak{M}}$. As $|L \cap \Gamma(\mathfrak{M})| < \infty$ for any L, we say that $\mathfrak{M}$ has *finite order*. If in addition $n = \sup_{L \subset P^2} |L \cap \Gamma(\mathfrak{M})|$ is finite, we say that $\mathfrak{M}$ is of *order* n. The *order*

Received March 17, 1983. Revision received September 25, 1984.
Michigan Math. J. 32 (1985).

of a point $t \in T$, $\mathrm{ord}(t)$, is the minimum order which an $U(t)$ can possess. Clearly $\mathrm{ord}(t) \geq 2$. A point t is *ordinary* if $\mathrm{ord}(t)=2$; otherwise, t is *singular*. A point t is *elementary* if there exist $U^-(t)$ and $U^+(t)$, both of order two. A subarc of Γ is *ordinary* [*elementary*] if each of its points is ordinary [elementary].

Let $\Gamma(t) \in L \subset P^2$. We say that L *supports* Γ at t if there is an $L' \neq L$ with $\Gamma(t) \notin L'$ and an $U'(t)$ such that $\Gamma(U'(t))$ is contained in one of the open half-spaces of P^2 determined by L' and L. When L does not support Γ at t, we say that L *cuts* Γ at t. Let

$$S(t) = \{L \subset P^2 \mid \Gamma(t) \in L \neq \Gamma_1(t)\}.$$

Then either all $L \in S(t)$ support Γ at t or all $L \in S(t)$ cut Γ at t; cf. [5]. There are then four types of points t in T with respect to Γ: t is *regular* if $L \in S(t)$ [$\Gamma_1(t)$] cuts [supports] Γ at t; t is an *inflection* if $L \in S(t)$ and $\Gamma_1(t)$ both cut Γ at t; t is a *beak* if $L \in S(t)$ and $\Gamma_1(t)$ both support Γ at t; t is a *cusp* if $L \in S(t)$ [$\Gamma_1(t)$] supports [cuts] Γ at t. We note that an ordinary point is regular and that inflections, cusps, and beaks are singular ([4, 3.2.2]).

Since Γ is met by any line at a finite number of points, either every line of P^2 cuts Γ at an odd number of points or every line of P^2 cuts Γ at an even number of points ([4, 7.3.1]). In the case of the former [latter], we say that Γ has *odd* [*even*] *order*. The *index* of Γ, $\mathrm{ind}(\Gamma)$, is the minimum number of points of Γ which can lie on a line of P^2. Thus $\mathrm{ind}(\Gamma)>0$ if Γ has odd order. Finally Γ is *simple* if $\Gamma(t) \neq \Gamma(t')$ for $t \neq t'$ in T.

From here on, we assume that Γ is a simple curve of even order and positive index. We wish to determine the minimum value of $n(\Gamma)$ and hence we may assume that $n(\Gamma)<\infty$. This final restriction then implies that Γ is elementary ([4, 9.2.3]). We refer to [4] for the following properties of Γ.

1. The tangent $\Gamma_1(t)$ depends continuously on $t \in T$.
2. A regular point is ordinary. Hence inflections, cusps, and beaks are the only singular points of Γ.
3. If Γ has even [odd] order, then $n_1+2n_2+n_3$ is even [odd].

Furthermore, since Γ is simple, no subarc of Γ has coincident end-points. Thus Γ is almost-simple and, as mentioned above $n(\Gamma) \geq 2$ and $n_1+2n_2+n_3 \geq 4$. We formally state our main theorem.

4. THEOREM. *Let* Γ *be a simple elementary curve of even order and positive index. Then*

(i) $n(\Gamma) \geq 3$; *and*

(ii) *if* $n_2>0$, *then* $n_1+2n_2+n_3 \geq 6$.

By way of preparation, we list some properties of ordinary subarcs and present a characterization of singular points based upon the orientation of T and the continuity of tangents. All results without a reference are well-known or immediate.

Let (s_1, s_2) be of order two.

5. Then $\Gamma_1(s) \cap \Gamma[s_1,s_2] = \{\Gamma(s)\}$ for $s \in (s_1,s_2)$ and there is a line $L^* \subset P^2$ such that $L^* \cap \Gamma[s_1,s_2] = \varnothing$. Let $H(\Gamma[s_1,s_2])$ be the convex hull of $\Gamma[s_1,s_2]$ (in $P^2 \setminus L^*$). Then $\mathfrak{R} = H(\Gamma[s_1,s_2])$ is the closed region in P^2 bounded by $\Gamma[s_1,s_2]$ and $\langle \Gamma(s_1), \Gamma(s_2) \rangle$ such that $|\Gamma_1(s) \cap \mathfrak{R}| = 1$ for $s \in (s_1,s_2)$.

6. Let N be an oriented line such that $N \cap \Gamma(s_1, s_2) = \varnothing$. Let $\Gamma_1(s)$ meet N at the point $\varphi(s)$, $s \in (s_1, s_2)$. If φ is not onto, then φ is strictly monotone ([4, 3.3.1]).

We rephrase **6** by stating that $\Gamma_1(s)$, $s \in (s_1, s_2)$, meets N in a strictly monotone manner. We generalize this monotone intersection property.

Let $\mathfrak{M}$ and $\mathfrak{M}'$ be disjoint segments of T such that each tangent of $\Gamma(\mathfrak{M})$ (Γ at $m \in \mathfrak{M}$) meets $\Gamma(\mathfrak{M}')$. If $|\Gamma_1(m) \cap \Gamma(\mathfrak{M}')| = 1$ for each $m \in \mathfrak{M}$, then $\Gamma_1(m)$ cuts $\Gamma(\mathfrak{M}')$ at the point of contact and we write $\mathfrak{M} \to \mathfrak{M}'$. If for each $m' \in \mathfrak{M}'$ there is exactly one $m \in \mathfrak{M}$ such that $\Gamma(m') \in \Gamma_1(m)$ and if moreover $\Gamma_1(m)$ cuts Γ at m', then we write $\mathfrak{M} \leftarrow \mathfrak{M}'$. If both $\mathfrak{M} \to \mathfrak{M}'$ and $\mathfrak{M} \leftarrow \mathfrak{M}'$, then clearly $\Gamma_1(m)$, $m \in \mathfrak{M}$, meets $\Gamma(\mathfrak{M}')$ in a strictly monotone manner and we write $\mathfrak{M} \leftrightarrow \mathfrak{M}'$.

7. Let $\Gamma_1(s)$ cut Γ at $t \neq s$.

(i) There exist $U(s)$ and $U(t)$ such that either $U^+(s) \leftrightarrow U^+(t)$ or $U^+(s) \leftrightarrow U^-(t)$ and either $U^-(s) \leftrightarrow U^+(t)$ or $U^-(s) \leftrightarrow U^-(t)$.

(ii) There exist $U(s)$ and $U(t)$ such that $U(s) \leftrightarrow U(t)$ if and only if s is an ordinary or a cusp point ([1, 3.4]).

Let $\Gamma(t) \in \Gamma_1(s)$ and assume $t \neq s$ when $\Gamma_1(s)$ supports Γ at t. Then t is *s-negative* [*s-positive*] if there exist $U^+(s)$ and $U^-(t)$ [$U^+(t)$] such that $U^+(s) \leftrightarrow U^-(t)$ [$U^+(s) \leftrightarrow U^+(t)$]. If $\Gamma_1(s)$ cuts Γ at $t \neq s$, then t is s-positive or s-negative but not both. If $\Gamma_1(s)$ supports Γ at t then t is either both s-positive and s-negative or neither.

8. If $s \in T$ is an inflection or a cusp then s is s-negative. If s is an ordinary [beak] point then s is neither s-positive nor s-negative [s-negative or neither s-positive nor s-negative] ([1, 3.7]).

9. Let t be s-negative [s-positive]. Then there exist $U^+(s)$ and $U^-(t)$ [$U^+(t)$] such that $U^+(s) \leftrightarrow U^-(t)$ [$U^+(s) \leftrightarrow U^+(t)$] and each $t' \in U^-(t)$ [$U^+(t)$] is s'-negative [s'-positive]; $\Gamma(t') \in \Gamma_1(s')$ and $s' \in U^+(s)$ ([1, 3.5]).

Let (t_1, t_2) be ordinary in **10** through **15**.

10. There exist $s_1 < s_2$ (s_1 preceding s_2) in $[t_1, t_2]$ such that (s_1, s_2) is of order two and $\Gamma[t_1, t_2] \subset H(\Gamma[s_1, s_2])$. We call $\Gamma[s_1, s_2]$ the *convex cover* of $\Gamma[t_1, t_2]$ ([1, 3.15]).

11. If $\langle \Gamma(t_1), \Gamma(t_2) \rangle \cap \Gamma(t_1, t_2) = \varnothing$, then (t_1, t_2) is of order two.

12. If $\Gamma_1(t_1) \cap \Gamma(t_1, t_2] = \{\Gamma(t_2)\}$, then t_2 is not t_1-negative ([1, 3.10]).

13. For any $t \in (t_1, t_2)$, $\Gamma_1(t) \cap \Gamma[t_1, t) = \varnothing$ or $\Gamma_1(t) \cap \Gamma(t, t_2] = \varnothing$ ([1, 3.12]).

14. Let $\Gamma_1(s) \cap \Gamma[t_1, t_2] = \{\Gamma(t_1), \Gamma(t_2)\}$, $\Gamma_1(s)$ cut Γ at t_1 and t_2, and $s \notin [t_1, t_2]$. Then t_1 and t_2 are both s-negative or both s-positive if and only if $\Gamma(s) \in H(\Gamma[t_1, t_2])$.

15. If L is a limit of lines, none of which meet $\Gamma[t_2, t_1]$, then L cuts Γ in at most two points and these points lie in $[t_1, t_2]$ ([1, 3.17]).

16. If a line L supports Γ at a point, then $|L \cap \Gamma| \geq 3$.

17. LEMMA. *Let (t_1, t_2) be ordinary. Let $s \in (t_1, t_2)$ such that*

(i) *$\Gamma_1(s) = \Gamma_1(t) = L$ for some $t \in (s, t_2)$, or*

(ii) *$\Gamma_1(s)$ cuts Γ at exactly t' and t'', $t' < t''$ in $[t_1, t_2]$, and $\Gamma(s) \notin H(\Gamma[t', t''])$.*

Then $\Gamma_1(s)$ cuts $\Gamma[t_2, t_1]$ in at least two distinct points.

Proof. We note that $\operatorname{ind}(\Gamma) > 0$ and **10** imply that there is an $r \in (t_2, t_1)$ such that $\Gamma(r) \notin \mathcal{R} = H(\Gamma[t_1, t_2])$.

If (i) then $L\cap(\Gamma[t_1,s)\cup\Gamma(t,t_2])=\varnothing$ by **13.** If $L\cap\Gamma(s,t)=\varnothing$ then (s,t) is of order two by **11.** As L supports Γ at both s and t, it is clear that

$$\Gamma[t_1,s)\cup\Gamma(t,t_2]\subset\operatorname{int}H(\Gamma[s,t])\subseteq\operatorname{int}(\mathcal{R}).$$

Since Γ is simple and $\operatorname{bd}(H(\Gamma[s,t]))=\Gamma(s,t)\cup(L\cap H(\Gamma[s,t]))$, we obtain that L cuts Γ at (t_2,r) and (r,t_1). If $L\cap\Gamma(s,t)\neq\varnothing$ then $|L\cap\Gamma|<\infty$ yields that $L\cap\Gamma(s,\bar{t}]=\{\Gamma(\bar{t})\}$ and $L\cap\Gamma[\bar{s},t)=\{\Gamma(\bar{s})\}$ for some $\bar{s}$ and $\bar{t}$ in (s,t). Then $\Gamma[t_1,s)\subset\operatorname{int}H(\Gamma[s,\bar{t}])$, $\Gamma(t,t_2]\subset\operatorname{int}H(\Gamma[\bar{s},t])$, and L still cuts Γ at (t_2,r) and (r,t_1).

If (ii) then either $\{t',t''\}\subset(t_1,s)$ or $\{t',t''\}\subset(s,t_2)$ by **13.** With the proper orientation, we may assume that $\{t',t''\}\subset(s,t_2)$ and that by **14**, t' is s-positive and t'' is s-negative. Since $\Gamma(s)\in H(\Gamma[t',t''])$, it now follows that $\Gamma[t_1,s]\subset H(\Gamma[s,t'])$ and that Γ again enters $H(\Gamma[s,t'])$ at t''. Thus $\Gamma[t'',t_2]\subset H(\Gamma[s,t'])$ and $\Gamma(t'')\in L\cap H(\Gamma[s,t'])$. The result now follows by arguments similar to those in (i).

The main theorem.

PROOF OF THEOREM 4(i). Suppose that $n(\Gamma)=n_1+n_2+n_3=2$. Then $n_1+2n_2+n_3\geq 4$ implies $n_1=n_3=0$, $n_2=2$; that is, Γ possesses exactly two cusps, say t_1 and t_2, as singular points. Then (t_1,t_2) and (t_2,t_1) are ordinary. There exist $s_1<s_2$ in $[t_1,t_2]$ such that $\Gamma[t_1,t_2]\subset\mathcal{R}=H(\Gamma[s_1,s_2])$ by **10.**

As $\operatorname{ind}(\Gamma)>0$, $\Gamma\not\subset\mathcal{R}$ and there exist $v_2<v_1$ in $[t_2,t_1]$ such that $\langle\Gamma(v_1),\Gamma(v_2)\rangle=L=\langle\Gamma(s_1),\Gamma(s_2)\rangle$ and $\Gamma[v_1,v_2]$ is the largest subarc of Γ contained in $\mathcal{R}$. Then **5** readily yields that each of L and $\Gamma_1(s)$, $s\in[s_1,s_2]$, is a limit of lines, none of which meets $\Gamma[v_1,v_2]$. As (v_2,v_1) is ordinary, **15** implies that each of these lines cuts Γ in at most two points and these points lie in $[v_2,v_1]$.

Let $s\in(s_1,s_2)$. Then $\Gamma_1(s)$ meets, and supports, Γ at exactly $s\in[v_1,s_2]$, cuts Γ in at most two points of (v_2,v_1), and by **16** meets Γ in at least two points (say v and v') in (v_2,v_1). If $\Gamma_1(s)$ does not cut Γ at any point of (v_2,v_1) then $\Gamma_1(s)=\Gamma_1(v)=\Gamma_1(v')$, and by **17**(i) $\Gamma_1(s)$ cuts Γ in at least two points of (v_1,v_2), a contradiction. Hence $\Gamma_1(s)$ cuts Γ at least once and this point lies in (v_2,v_1). As Γ is of even order, we may assume that $\Gamma_1(s)$ cuts Γ at v and v'. Suppose that $\Gamma_1(s)$ supports Γ at $\bar{v}\in(v_2,v_1)$. Let $U(\bar{v})$, $U(v)$, and $U(v')$ be mutually disjoint in (v_2,v_1). Then **1** and **7**(i) readily yield that there is an s', arbitrarily close to s in (s_1,s_2), such that $\Gamma_1(s')$ cuts Γ at a point in both $U(v)$ and $U(v')$ and at two points in $U(\bar{v})$, a contradiction. Hence

(1) the tangents of $\Gamma(s_1,s_2)$ meet, and cut, Γ in exactly two points of (v_1,v_2) and in a strictly monotone manner at each point.

Let $\Gamma_1(s_1)$ $[\Gamma_1(s_2)]$ cut Γ at $u<u'$ $[w<w']$ in $[v_2,v_1]$. Let $s\in(s_1,s_2)$ be close to s_1. If $\Gamma_1(s_1)$ cuts Γ at $v(s)$ $[v'(s)]$ close to u $[u']$ then $v(s)<v'(s)$. Let s range between s_1 and s_2. Then $v(s)<v'(s)$ for all s by (1) and so $v(s)$ $[v'(s)]$ ranges between u and w $[u'$ and $w']$. By **9** and **7**(ii), we have

(2) u is s_1-positive if and only if w is s_2-positive, and u' is s_1-positive if and only if w' is s_2-positive.

Case 1: $s_1\neq t_1$ and $s_2\neq t_2$; cf. Figure 1(a).

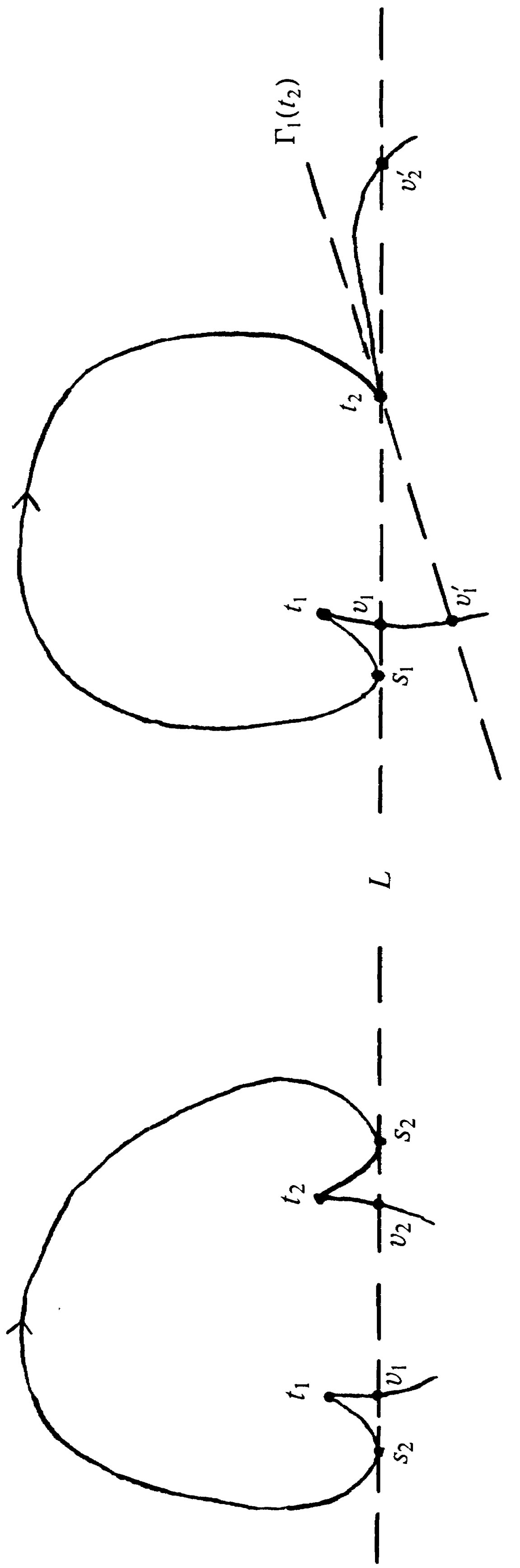

Figure 1.

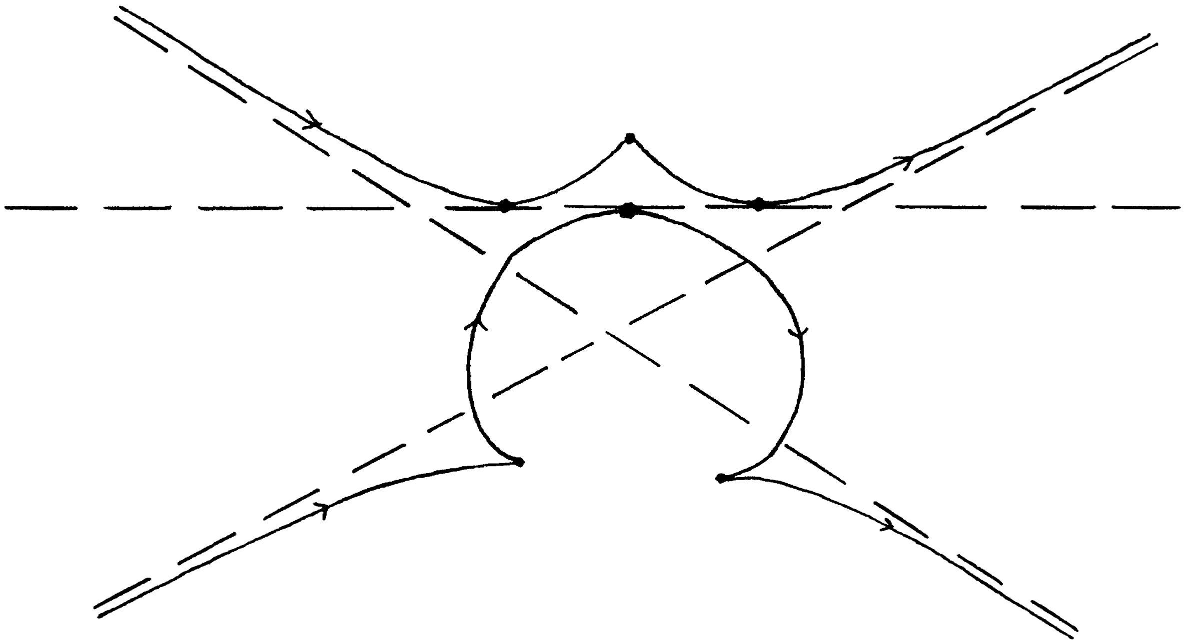

Figure 2.

Then L supports Γ at s_i, $L=\Gamma_1(s_i)$, $\Gamma(t_i)\in \text{int}(\mathcal{R})$, $t_i \neq v_i$, and L cuts Γ at v_i; $i=1,2$.

Since $\Gamma_1(s_1)$ cuts Γ at v_1, there exist $U^+(s_1)\subset(s_1,s_2)$ and $U(v_1)\subset(v_2,t_1)$ such that either $U^+(s_1)\leftrightarrow U^+(v_1)$ or $U^+(s_1)\leftrightarrow U^-(v_1)$ by **7**(i). As

$$\Gamma(U^+(v_1))\subset\Gamma(v_1,t_1)\subset \text{int}(\mathcal{R}),$$

no tangent of $\Gamma(U^+(s_1))$ meets $\Gamma(U^+(v_1))$ by **5**. Hence $U^+(s_1)\leftrightarrow U^-(v_1)$ and v_1 is s_1-negative. Similarly we obtain that v_2 is s_1-positive, v_2 is s_2-negative, and v_1 is s_2-positive. Then $u=v_2=w$ and $u'=v_1=w'$ contradicts (2). Thus $s_1=t_1$ or $s_2=t_2$.

Case 2: $s_1\neq t_1$ and $s_2=t_2$; cf. Figure 1(b).

Then $L=\Gamma_1(s_1)$ supports Γ at s_1, $t_1\neq v_1$, $\Gamma_1(s_1)$ cuts Γ at v_1 and as in Case 1, v_1 is s_1-negative. Since t_2 is a cusp, $t_2=s_2$ readily yields that $t_2=v_2$. We note that $\Gamma_1(t_2)$ cuts Γ at t_2 and by **8**, t_2 is t_2-negative.

Since Γ is of even order, $\Gamma_1(s_1)$ cuts Γ at exactly one point $v_2'\in[v_2,v_1)$ and $\Gamma_1(t_2)$ cuts Γ at exactly one point $v_1'\in(v_2,v_1]$. As (t_2,v_1') is ordinary, $\Gamma_1(t_2)$ does not support Γ at any point of (t_2,v_1') by **13**. Hence $\Gamma_1(t_2)\cap\Gamma(t_2,v_1')=\varnothing$ and by **12**, v_1' is t_2-positive. Then $u=v_2'$, $u'=v_1$, $w=t_2$, and $w'=v_1'$ contradicts (2).

The preceding is symmetric in s_1,t_1 and s_2,t_2. Thus $s_1=t_1$ and $s_2=t_2$.

Case 3: $s_1=t_1$ and $s_2=t_2$.

Then L does not support Γ at any point of $(t_1,t_2)\cup(t_2,t_1)$ by **13** and $L\cap\Gamma=\{\Gamma(t_1),\Gamma(t_2)\}$. By **16**, L cuts Γ at t_1 and t_2. As t_i is a cusp, $L=\Gamma_1(t_i)$, $t_i=v_i$ and t_i is t_i-negative; $i=1,2$. By **12**, t_2 is t_1-positive and t_1 is t_2-positive. Hence $u=t_2=w$ and $u'=t_1=w'$ contradicts (2).

Thus $n(\Gamma)>2$. In Figure 2, we present a simple Γ with positive index, even order and $n(\Gamma)=n_2=3$.

We note that each cusp in Figure 2 can be replaced by a combination of two non-cusp singularities in such a manner that the resultant curve is still simple with positive index and even order. Thus there exist such curves with exactly 4, 5, or 6 singular points.

It is also easy to check that a beak of a simple curve with positive index and even order can be replaced by an inflection in such a manner that the resultant curve is still simple with positive index, even order, and an unchanged number of singularities. Hence to simplify the following arguments, we assume that a singular point is either a cusp or an inflection.

PROOF OF THEOREM 4(ii). Since $n(\Gamma)\geq 3$ and $n_1+2n_2+n_3$ is even, $n_1+2n_2+n_3\geq 6$ if either $n_2\geq 2$ or $n(\Gamma)>3$ and $n_2=1$. Suppose that $n(\Gamma)=3$ and $n_2=1$. Let t_0, t_1 and t_2 to be the singular points of Γ; t_0 a cusp and t_1 and t_2 inflections. We assume that $t_0\notin(t_1,t_2)$. Hence (t_0,t_1), (t_1,t_2), and (t_2,t_0) are all ordinary. We assume that

(3) none of $\Gamma[t_0,t_1]$, $\Gamma[t_1,t_2]$, or $\Gamma[t_2,t_0]$ is contained in the convex hull of another.

Otherwise we can consider Γ as a curve with at most two singular points. Then arguing as in the proof of **4**(i) or [1, 3.1], we obtain a contradiction.

Since (t_1,t_2) is ordinary, there exist $s_1<s_2$ in $[t_1,t_2]$ such that $\Gamma[t_1,t_2]\subset\mathcal{R}=H(\Gamma[s_1,s_2])$ and $v_2<v_1$ in $[t_2,t_1]$ such that $\langle\Gamma(v_1),\Gamma(v_2)\rangle=L=\langle\Gamma(s_1),\Gamma(s_2)\rangle$ and $\Gamma[v_1,v_2]$ is the largest subarc of Γ contained in $\mathcal{R}$. We note that (s_1,s_2) is of order two by **10** and $t_0\in(v_2,v_1)$ by (3). As t_1 and t_2 are both inflections, it is immediate that L cuts Γ at v_1 and v_2. We also note that if $L\neq\Gamma_1(v_i)$, then $\Gamma_1(v_i)$ meets, and cuts, Γ at exactly one point of (s_1,s_2); $i=1,2$.

Case 1: $\Gamma_1(t)\cap\mathcal{R}\neq\varnothing$ for any $t\in(v_2,v_1)$; cf. Figure 3.

We show that (v_2,t_0) and (t_0,v_1) are both of order two, there is an $s^*\in(s_1,s_2)$ such that $\Gamma(t_0)\in\Gamma_1(s^*)\neq\Gamma_1(t_0)$ and any line through $\Gamma(s^*)$ meets Γ in at most one point of, say, $[v_2,t_0]$. Then $\Gamma_1(s^*)$ meets, and supports, Γ at exactly s^* and t_0 in $[v_1,t_0]$ by **5**. Since (t_0,v_1) is of order two, any line through $\Gamma(t_0)$ meets, and cuts, Γ in at most one point of (t_0,v_1). As Γ is of even order, this implies that $\Gamma_1(s^*)\cap\Gamma(t_0,v_1)=\varnothing$ and $|\Gamma_1(s^*)\cap\Gamma|=2$. This is a contradiction by **16**.

We first note that $|\Gamma_1(t)\cap\mathcal{R}|\neq 1$ for any $t\in(v_2,v_1)$ and $L\cap\Gamma(v_2,v_1)=\varnothing$. If $|\Gamma_1(t)\cap\mathcal{R}|=1$ for some $t\in(v_2,v_1)$ then there is a $t'\in(v_2,v_1)$, arbitrarily close to t, such that $\Gamma_1(t')\cap\mathcal{R}=\varnothing$ by **1**. If $L\cap\Gamma(v_2,v_1)\neq\varnothing$ then L meets Γ at $(v_2,t_0]$ or $[t_0,v_1)$, say $(v_2,t_0]$, and there is a $t\in(v_2,t_0]$ such that $L\cap\Gamma(v_2,t]=\{\Gamma(t)\}$. Then Γ simple and **11** imply respectively that $\mathcal{R}\cap\Gamma(v_2,t)=\varnothing$ and (v_2,t) is of order two. As $\mathcal{R}$ is bounded, it is immediate that there is a $t'\in(v_2,t_0)$ such that $\Gamma_1(t')\cap\mathcal{R}=\varnothing$.

We next claim that for $t\in(t_2,v_1)$, $\Gamma_1(t)\cap\Gamma(s_1,s_2)\neq\varnothing$, and $\Gamma_1(t)$ cuts Γ at each point of contact in (s_1,s_2). Since $\Gamma(t_2,v_2)\subset\operatorname{int}(\mathcal{R})$, the claim is immediate if $t\in(t_2,v_2)$ or $t=v_2\neq t_2$. Let $t\in(v_2,v_1)$. Since (s_1,s_2) is of order two, $\Gamma_1(t)\neq L$ and $|\Gamma_1(t)\cap\mathcal{R}|\neq 1$; the claim follows by **5**. Suppose $\Gamma(s)\in\Gamma_1(t)$ for $s\in(s_1,s_2)$ and $t\in(s_2,v_1)$. Since s_1 is s_2-positive and $[s_2,t_2)$ is ordinary, **9**, **7**(ii), and **14** applied to $\Gamma[s_1,s_2]$ yield that s is t-positive for $t\in(s_2,t_2)$. Since $\Gamma(t_2)$ is an inflection, $\Gamma_1(t)$ changes direction at $t=t_2$. Thus now the preceding argument yields

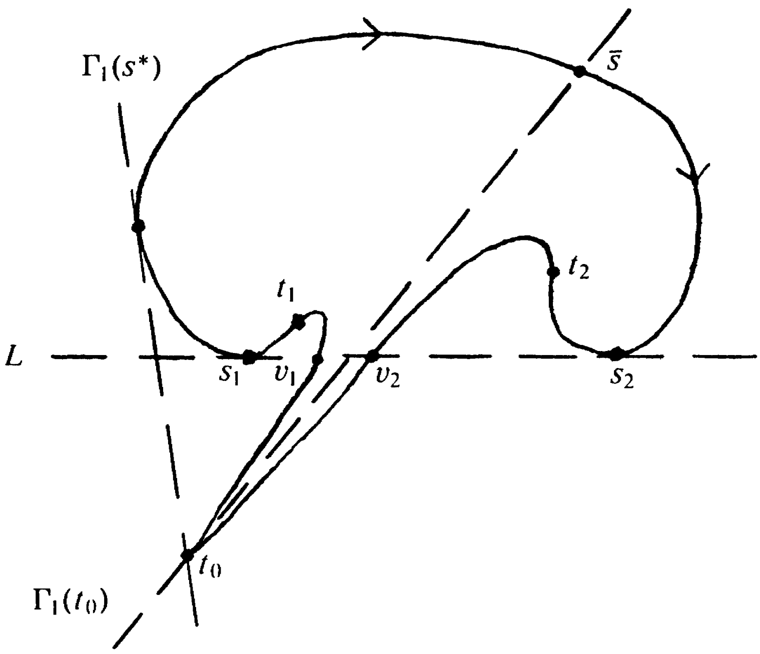

Figure 3.

that s is t-negative for $t \in (t_2, v_2]$. As (v_2, v_1) contains neither breaks nor cusps, s is t-negative for $t \in (v_2, v_1)$. By $\mathcal{R} \cap \Gamma(v_2, v_1) = \varnothing$ and **14** applied to $\Gamma[s_1, s_2]$, each tangent of $\Gamma(v_2, v_1)$ meets, and cuts, Γ at exactly one point of (s_1, s_2). Since $\mathcal{R}$ is bounded, the hypothesis clearly yields that

(4) the tangents of $\Gamma(v_2, v_1)$ meet $\Gamma(s_1, s_2)$ in a strictly monotone manner and each point of $\Gamma(s_1, s_2)$ lies on at most one tangent of $\Gamma(v_2, v_1)$.

Let L be oriented. Then $L \cap \Gamma(v_2, v_1) = \varnothing$ and (4) readily imply that the tangents of $\Gamma(v_2, v_1)$ meet L in a strictly monotone manner. In particular, each tangent of $\Gamma(v_2, v_1)$ meets the open segment $(L \cap \mathcal{R}) \setminus \{\Gamma(v_1), \Gamma(v_2)\}$. It is now easy to check that $\langle \Gamma(v_2), \Gamma(t_0) \rangle \cap \Gamma(v_2, t_0) = \varnothing$ and $\langle \Gamma(v_1), \Gamma(t_0) \rangle \cap \Gamma(t_0, v_1) = \varnothing$. Hence by **11**, *$(v_2, t_0)$ and (t_0, v_1) are both of order two.*

Let $\Gamma_1(t_0) \cap \Gamma(s_1, s_2) = \{\Gamma(\bar{s})\}$. As $\bar{s}$ is t_0-negative; **1**, **9** and (4) imply that

(5) no point of $\Gamma(s_1, \bar{s})$ $[\Gamma(\bar{s}, s_2)]$ lies on a tangent of $\Gamma[v_2, t_0]$ $[\Gamma[t_0, v_1]]$.

Let $s \in (s_1, s_2)$. Since $L \cap \Gamma(v_2, v_1) = \varnothing$ and $L = \langle \Gamma(v_1), \Gamma(v_2) \rangle$, there is a line L_s such that $L_s \cap \Gamma[v_2, v_1] = \varnothing$ and L_s separates $\mathcal{R}$ into two disjoint regions, one of which contains $\Gamma(s)$ and the other $\Gamma(v_1)$ and $\Gamma(v_2)$. As $\mathcal{R} \cap \Gamma(v_2, v_1) = \varnothing$, this implies that $\Gamma(s) \notin H(\Gamma[v_2, t_0]) \cup H(\Gamma[t_0, v_1])$. Now (5) readily yields that

(6) any line through a point of $\Gamma(s_1, \bar{s})$ $[\Gamma(\bar{s}, s_2)]$ meets $\Gamma[v_2, t_0]$ $[\Gamma[t_0, v_1]]$ in at most one point.

As $\Gamma(t_0) \notin \mathcal{R}$ and (v_2, t_0) is of order two, there is an $s^* \in (s_1, s_2)$ such that $\Gamma(t_0) \in \Gamma_1(s^*)$. Since $\Gamma_1(t_0)$ meets, and cuts, Γ at exactly $\bar{s} \in (s_1, s_2)$ and $\Gamma_1(s^*)$ supports Γ at s^*, we obtain that $\Gamma_1(s^*) \neq \Gamma_1(t_0)$, $s^* \neq \bar{s}$ and thus $s^* \in (s_1, \bar{s})$, say. Then by (6), *any line through $\Gamma(s^*)$ meets $\Gamma[v_2, t_0]$ in at most one point.*

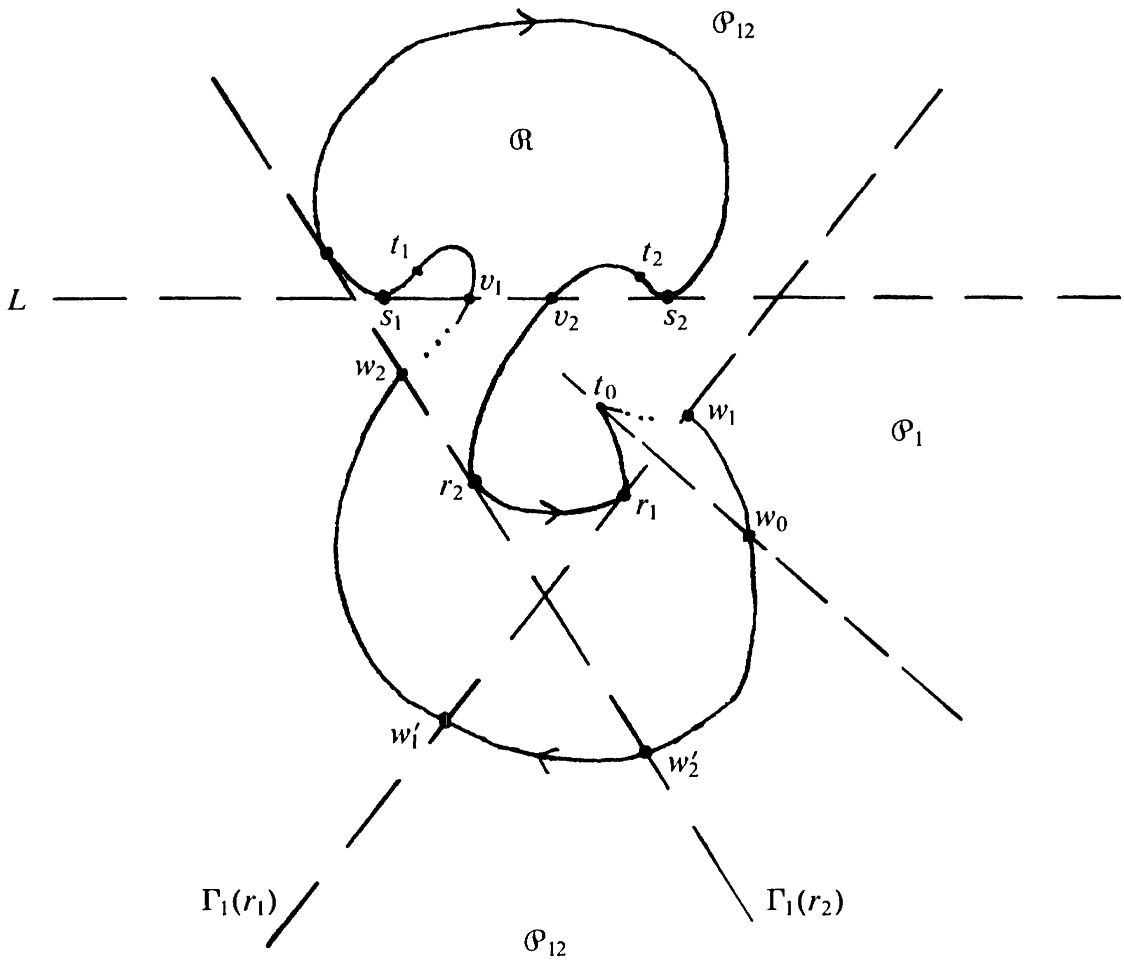

Figure 4.

Case 2: $\Gamma_1(t)\cap\mathcal{R}=\varnothing$ for some $t\in(v_2,v_1)$.

By **1**, there exist $r_2<r_1$ in (v_2,v_1) such that $t_0\notin(r_2,r_1)$ and $\Gamma_1(r)\cap\mathcal{R}=\varnothing$ for $r\in(r_2,r_1)$. Without loss of generality, we may assume that $v_2<r_2<r_1\leq t_0$ in $[t_2,t_0]$ and $\Gamma_1(t)\cap\mathcal{R}\neq\varnothing$ for each $t\in(v_2,r_2)$. As in Case 1, we obtain $L\cap\Gamma(v_2,r_2]=\varnothing$ and (v_2,r_2) is of order two. Then $\Gamma_1(r_2)\cap\Gamma[v_2,r_2)=\varnothing$ and from **1**, $|\Gamma_1(r_2)\cap\mathcal{R}|=1$. We now show (cf. Figure 4) that (r_2,r_1) is of order two and there exist $w_1<w_2$ in $[t_0,t_1]$ such that each tangent of $\Gamma(r_2,r_1)$ meets, and cuts, Γ at exactly two points of (w_1,w_2), and w is r-negative if $\Gamma_1(r)$ cuts Γ at w, $w\in[w_1,w_2]$ and $r\in[r_2,r_1]$. This will then imply that $\Gamma_1(t_0)$ cuts Γ at a point w_0 in $(t_0,t_1]$ such that $\Gamma_1(t_0)\cap\Gamma(t_0,w_0)=\varnothing$ and w_0 is t_0-negative. Since (t_0,w_0) is ordinary, this is a contradiction by **12**.

Suppose that $\Gamma_1(r_2)$ meets Γ at a point $r'\neq r_2$ in $[t_2,t_0]$. Then $\Gamma[t_2,v_2]\subset\operatorname{int}(\mathcal{R})$ and $\Gamma_1(r_2)\cap(\operatorname{int}(\mathcal{R})\cup\Gamma[v_2,r_2))=\varnothing$ imply $r'\in[r_2,t_0]$. We may assume that $\Gamma_1(r_2)\cap\Gamma(r_2,r')=\varnothing$ and hence (r_2,r') is of order two by **11**. Then $|\Gamma_1(r_2)\cap\mathcal{R}|=1$ and $\Gamma_1(r)\cap\mathcal{R}=\varnothing$ for r arbitrarily close to r_2 in (r_2,r') imply that $\mathcal{R}\subseteq H(\Gamma[r_2,r'])\subseteq H(\Gamma[t_2,t_0])$; a contradiction by (3).

Suppose that $\Gamma_1(r_1)$ meets Γ at a point $r' \neq r_1$ in $[t_2, t_0]$. Then $\Gamma_1(r_1) \cap \operatorname{int}(\mathcal{R}) = \varnothing$ implies that $r' \in [v_2, t_0]$. If $r' \in (r_1, t_0)$, then $r_1 \neq t_0$ and $\Gamma_1(r_1)$ supports Γ at r_1. We assume that $\Gamma_1(r_1) \cap \Gamma(r_1, r') = \varnothing$ and hence (r_1, r') is of order two. As Γ is simple and $\Gamma_1(r_1)$ supports Γ at r_1, **13** implies that $\Gamma_1(r_1) \cap \Gamma[t_2, r_1) = \varnothing$ and $\Gamma(r_2) \in \Gamma[t_2, r_1) \subset \operatorname{int}(H(\Gamma[r_1, r']))$. But then $\Gamma_1(r_2) \cap \Gamma[r_1, r'] \neq \varnothing$, a contradiction by the preceding. If $r' \in [r_1, r_2)$, we assume that (r', r_1) is of order two. As every point $p \notin H(\Gamma[r', r_1])$ lies on a tangent of $\Gamma[r', r_1]$, $\mathcal{R} \subseteq H(\Gamma[r', r_1])$ by definition, a contradiction by (3). Finally, though less obvious, it is easy to check that $r' \in [v_2, r_2)$ also implies that $\mathcal{R} \subset H(\Gamma[r', r_1])$.

Since $\Gamma_1(r_i) \cap \Gamma[t_2, t_0] = \{\Gamma(r_i)\}$ for $i = 1, 2$, neither $\Gamma(r_1)$ nor $\Gamma(r_2)$ is contained in $\operatorname{int}(H(\Gamma[t_2, t_0]))$. Hence $\Gamma(r_1)$ and $\Gamma(r_2)$ are contained in the convex cover of $\Gamma[t_2, t_0]$ and by **5**, (r_2, r_1) *is of order two.*

As $\Gamma_1(r_1) \neq \Gamma_1(r_2)$, let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the open half-spaces of P^2 determined by $\Gamma_1(r_2)$ and $\Gamma_1(r_1)$; $\Gamma(r_2, r_1) \subset \mathcal{P}_1$. As each point of $\mathcal{P}_2$ lies on a tangent of $\Gamma(r_2, r_1)$, $\mathcal{R} \subset \mathcal{P}_1$. Then $|\Gamma_1(r_i) \cap \Gamma[t_2, t_0]| = 1$ for $i = 1, 2$ implies that $\Gamma[t_2, r_2) \cup \Gamma(r_1, t_0) \subset \mathcal{P}_1$. We also note that $\Gamma(r_1, r_2)$ separates $\mathcal{P}_1$ into two disjoint regions $\mathcal{P}_{11}$ and $\mathcal{P}_{12}$ such that each point of say $\mathcal{P}_{11}$ $[\mathcal{P}_{12}]$ lies on two tangents [on no tangent] of $\Gamma(r_2, r_1)$. Therefore $\Gamma[t_1, t_0] \subset \bar{\mathcal{P}}_{12}$, the closure of $\mathcal{P}_{12}$ in P^2.

Since $\bar{\mathcal{P}}_{12}$ is a closed bounded region in P^2, $\operatorname{ind}(\Gamma) > 0$ implies that there exist $w_1 < w_2$ in $[t_0, t_1]$ such that $\Gamma[w_2, w_1]$ is the longest subarc of Γ contained in $\bar{\mathcal{P}}_{12}$. Let $r \in (r_2, r_1)$. Then $\Gamma[w_2, w_1] \subset \bar{\mathcal{P}}_{12}$ implies that $\Gamma_1(r)$ meets, and supports, Γ at exactly $r \in [w_2, w_1]$. Clearly by **1**, $\Gamma_1(r)$ is a limit of lines, none of which meet $\Gamma[w_2, w_1]$. Since (w_1, w_2) $[(r_2, r_1)]$ is ordinary, **15** [**16**] implies that $\Gamma_1(r)$ cuts [meets] Γ in at most [least] two points of (w_1, w_2). We now argue as in the proof of **4**(i) and obtain that *each tangent of* $\Gamma(r_2, r_1)$ *meets, and cuts,* Γ *at exactly two points of* (w_1, w_2).

As $\Gamma[w_2, w_1] \subset \bar{\mathcal{P}}_{12}$, $r_2 < r_1$ in $[t_2, t_0]$ implies that $\Gamma[w_2, r_2] \cup \Gamma[r_1, w_1] \subset \bar{\mathcal{P}}_{12}$. Since Γ is simple, it is easy to check that $\Gamma_1(r_i)$ cuts Γ at w_i, $i = 1, 2$. Then $w_2 \neq r_2$ and **7**(i) imply that there exist $U^+(r_2) \subset (r_2, r_1)$ and $U(w_2) \subset (t_0, t_2)$ such that either $U^+(r_2) \leftrightarrow U^+(w_2)$ or $U^+(r_2) \leftrightarrow U^-(w_2)$. As $\Gamma(U^+(w_2)) \subset \mathcal{P}_{12}$, no tangent of $\Gamma(U^+(r_2))$ meets $\Gamma(U^+(w_2))$. Hence $U^+(r_2) \leftrightarrow U^-(w_2)$ and w_2 is r_2-negative. Similarly, $w_1 \neq r_1$ implies that w_1 is r_1-negative. If $w_1 = r_1$, then $w_1 \in [t_0, t_2]$ and $r_1 \in [t_2, t_0]$ imply that $r_1 = t_0$. As t_0 is a cusp, w_1 is r_1-negative by **8**. Since each tangent of $\Gamma(r_2, r_1)$ cuts Γ in exactly two points of (w_1, w_2), **1** and **7**(i) imply that $\Gamma_1(r_i)$ cuts Γ at exactly two points of $[w_1, w_2]$; $i = 1, 2$. Let $\Gamma_1(r_1)$ $[\Gamma_1(r_2)]$ cut Γ at $w_1' \in (w_1, w_2]$ $[w_2' \in [w_1, w_2)]$. Then arguing as in the proof of **4**(i), $w_1 < w_1'$ and $w_2' < w_2$ in $[w_1, w_2]$; **9**, **7**(ii), and the preceding imply that *w is r-negative if* $\Gamma_1(r)$ *cuts* Γ *at w, $w \in [w_1, w_2]$ and $r \in [r_2, r_1]$.*

Since $\Gamma_1(r_1)$ cuts Γ at exactly w_1 and w_1' in $[t_0, t_1]$, we observe that

$$\Gamma_1(r_1) \cap \Gamma(w_1, w_1') = \varnothing.$$

If $r_1 = w_1$, then $w_1 = t_0$ and we set $w_1' = w_0$.

Let $r_1 \neq w_1$. If there is a $w \in (r_1, w_1)$ such that $\Gamma(w) \in \Gamma_1(r_1)$, then

$$\Gamma_1(r_1) \cap \Gamma[t_2, t_0] = \{\Gamma(r_1)\} \quad \text{implies that} \quad w \in (t_0, w_1) \subset (t_0, t_1).$$

Hence $\Gamma_1(r_1)$ supports Γ at w, $\Gamma_1(r_1)=\Gamma_1(w)$ and by **13**, $\Gamma_1(r_1)\cap\Gamma(r_1,w_1)=\{\Gamma(w)\}$. Since $\Gamma_1(r_1)$ cuts Γ at only w_1 and w_1'; $\Gamma_1(r_1)=\Gamma_1(w)$, $w<w_1<w_1'$ in $(t_0,t_1]$, and **16** imply that $\Gamma(w)\in H(\Gamma[w_1,w_1'])$. Then w_1 and w_1' are either both w-negative or both w-positive by **14**. As w_1 and w_1' are both r_1-negative, and Γ is simple, $\Gamma_1(r_1)\cap\Gamma(r_1,w_1)=\{\Gamma(w)\}$ clearly implies that w_1 and w_1' are also both w-negative. Then $\Gamma_1(w)\cap\Gamma(w,w_1)=\varnothing$ and **12** imply that (w,w_1) is not ordinary, a contradiction.

As $\Gamma_1(r_1)\cap(\Gamma(r_1,w_1)\cup\Gamma(w_1,w_1'))=\varnothing$ and w_1 and w_1' are both r_1-negative, **1** and **9** imply that for r arbitrarily close to r_1 in (r_1,t_0),

(7) $\Gamma_1(r)$ cuts Γ at $w<w'$ in (t_0,t_1) such that w and w' are both r-negative and $\Gamma_1(r)\cap(\Gamma(r,w)\cup\Gamma(w,w'))=\varnothing$.

Arguing as in the preceding, we obtain that (7) is true for all $r\in(r_1,t_0)$. Since t_0 is a cusp, **8** and **9** imply that there exist $U^-(t_0)\subset(r_1,t_0)$ and $U^+(t_0)\subset(t_0,t_1)$ such that $U^-(t_0)\leftrightarrow U^+(t_0)$. Let $r\in(r_1,t_0)$ tend to t_0. Then **1** implies that w tends to t_0 and w' tends to some $w_0\neq t_0$ in (t_0,t_1). Then w_0 is t_0-negative by **9** and $\Gamma_1(t_0)\cap\Gamma(t_0,w_0)=\varnothing$ by **13**. □

Finally, it should be noted that it is not known if there exists a curve of even order and positive index which does not possess cusps and possesses less than six beaks and inflection points. We conjecture that such a curve does not exist and thus the condition $n_2>0$ is not necessary in **4**(ii).

REFERENCES

1. T. Bisztriczky, *On the singularities of almost-simple plane curves,* Pacific J. Math. 109 (1983), 257–273.
2. A. Kneser, *Einige allgemeine Sätze über die einfachsten Gestalten ebenen Kurven,* Math. Ann. 41 (1889), 201–226.
3. F. A. Möbius, *Über die Grundformen der Linien dritter Ordnung,* Ges. Werke II, Leipzig, 1886.
4. R. Park, *Topics in direct differential geometry,* Canad. J. Math. 24 (1972), 98–148.
5. P. Scherk, *Über differenzierbare Kurven and Bögen I. Zum Begriff der Charakteristik.* Časopis pěst. mat. a fys. 66 (1937), 165–171.
6. ———, *On differentiable arcs and curves IV,* Ann. of Math. 46 (1945), 68–82.

Department of Mathematics
University of Calgary
Calgary, Canada