QUASICONFORMAL ANALOGUES OF THEOREMS OF KOEBE AND HARDY-LITTLEWOOD

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Dedicated to Professor George Piranian on his seventieth birthday

1. Introduction. Suppose that D is a domain in euclidean n-space $\mathbf{R}^n$. We say that D is *uniform* if there exist positive constants a and b such that each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$l(\gamma) \leq a|x_1 - x_2| \tag{1.1}$$

and

$$\min_{j=1,2} l(\gamma_j) \leq b d(x, \partial D) \tag{1.2}$$

for each $x \in \gamma$; here $l(\gamma)$ denotes the length of γ and γ_1, γ_2 the components of $\gamma \setminus \{x\}$.

Suppose next that D and D' are domains in $\mathbf{R}^n$ and that $f: D \to D'$ is K-quasiconformal with Jacobian J_f. Then $\log J_f$ is integrable over each ball $B \subset D$ and we set

$$(\log J_f)_B = \frac{1}{m(B)} \int_B \log J_f \, dm. \tag{1.4}$$

In particular, for each $x \in D$ we let

$$a_f(x) = \exp\left(\frac{1}{n} (\log J_f)_{B(x)}\right), \tag{1.5}$$

where $B(x) = B(x, d(x, \partial D))$, the open ball with center x and radius equal to the distance $d(x, \partial D)$ from x to ∂D. If $n = 2$ and f is conformal in D, then $\log J_f$ is harmonic,

$$(\log J_f)_{B(x)} = \log J_f(x) = 2 \log |f'(x)|$$

and hence $a_f(x) = |f'(x)|$.

We observed recently in [1] that for certain distortion properties of quasiconformal mappings the function a_f plays a role exactly analogous to that played by $|f'|$ when $n = 2$ and f is conformal. We investigate this analogy further by establishing in this paper quasiconformal versions of the following two well-known results due to Koebe [8, p. 22] and Hardy-Littlewood [6], respectively.

1.6. THEOREM. *Suppose that D and D' are domains in $\mathbf{R}^2$. If $f: D \to D'$ is conformal, then*

Received July 25, 1984.

This research was supported in part by a grant from the U.S. National Science Foundation.

Michigan Math. J. 32 (1985).

$$\frac{1}{4}\frac{d(f(x),\partial D')}{d(x,\partial D)} \le |f'(x)| \le 4\frac{d(f(x),\partial D')}{d(x,\partial D)}$$

for $x \in D$.

1.7. THEOREM. *Suppose that* D *is a uniform domain in* $\mathbf{R}^2$ *and that* α *and* m *are constants with* $0<\alpha\le 1$ *and* $m \ge 0$. *If* f *is analytic in* D *and if*

$$|f'(x)| \le m d(x,\partial D)^{\alpha-1}$$

for $x \in D$, *then* f *has a continuous extension to* $\bar{D}\backslash\{\infty\}$ *and*

$$|f(x_1)-f(x_2)| \le cm|x_1-x_2|^{\alpha}$$

for $x_1, x_2 \in \bar{D}\backslash\{\infty\}$, *where* c *is a constant which depends only on* α *and the constants for* D.

See [4] for the above somewhat more general version of the original result of Hardy and Littlewood.

We shall prove the following quasiconformal analogues of these two results in Sections 2 and 3.

1.8. THEOREM. *Suppose that* D *and* D' *are domains in* $\mathbf{R}^n$. *If* $f: D \to D'$ *is* K-*quasiconformal, then*

$$\frac{1}{c}\frac{d(f(x),\partial D')}{d(x,\partial D)} \le a_f(x) \le c\frac{d(f(x),\partial D')}{d(x,\partial D)}$$

for $x \in D$, *where* c *is a constant which depends only on* K *and* n.

1.9. THEOREM. *Suppose that* D *is a uniform domain in* $\mathbf{R}^n$ *and that* α *and* m *are constants with* $0<\alpha\le 1$ *and* $m\ge 0$. *If* f *is* K-*quasiconformal in* D *with* $f(D) \subset \mathbf{R}^n$ *and if*

$$a_f(x) \le m d(x,\partial D)^{\alpha-1}$$

for $x \in D$, *then* f *has a continuous extension to* $\bar{D}\backslash\{\infty\}$ *and*

$$|f(x_1)-f(x_2)| \le cm(|x_1-x_2|+d(x_1,\partial D))^{\alpha} \tag{1.10}$$

for $x_1, x_2 \in \bar{D}\backslash\{\infty\}$, *where* c *is a constant which depends only on* K, n, α *and the constants for* D.

Suppose that $n=2$ and f is conformal. Then $a_f(x)=|f'(x)|$ in D and the conclusion of Theorem 1.8 reduces to that in Theorem 1.6 where $c=4$. Next (1.10) in Theorem 1.9 implies that

$$|f(x_1)-f(x_2)| \le cm|x_1-x_2|^{\alpha} \tag{1.11}$$

for $x_1, x_2 \in \partial D\backslash\{\infty\}$ and, in the case where D is unbounded, that

$$|f(x)| = O(|x|^{\alpha}) \tag{1.12}$$

as $x \to \infty$ in D. From these conditions it follows that f satisfies (1.11) for all $x_1, x_2 \in \bar{D}\backslash\{\infty\}$; see for example [3]. Hence Theorem 1.9 reduces to Theorem 1.7 in this special case, apart from the value of the constant c.

The example given in Remark 3.12 shows that the term $d(x_1, \partial D)$ in (1.10) cannot be omitted.

2. Proof of Theorem 1.8. Suppose that D and D' are domains in $\mathbf{R}^n$ and that $f: D \to D'$ is K-quasiconformal. Fix $x_1 \in D$ and set $r_1 = d(x_1, \partial D)$ and $d_1 = d(f(x_1), \partial D')$. We shall show that there exists a constant $c = c(K, n)$ such that

$$\frac{1}{c}\frac{d_1}{r_1} \le a_f(x_1) \le c\frac{d_1}{r_1}. \tag{2.1}$$

Set $r_2 = r_1/2$, choose $x_2 \in D$ so that $|x_2 - x_1| = r_2$ and

$$|f(x_1) - f(x_2)| = \max_{|x_1 - x| = r_2} |f(x_1) - f(x)| = d_2,$$

and set $R = B_1 \setminus \bar{B}_2$ where $B_j = B(x_1, r_j)$ for $j = 1, 2$. Since f is K-quasiconformal and since $f(R)$ separates $f(x_1)$ and $f(x_2)$ from the complement of D',

$$\log 2 = \operatorname{mod} R \le K^{1/(n-1)} \operatorname{mod} f(R) \le K^{1/(n-1)} \operatorname{mod} R_T\left(\frac{d_1}{d_2}\right), \tag{2.2}$$

where $R_T(t)$ is the Teichmüller ring

$$R_T(t) = \mathbf{R}^n \setminus \{x = (s, 0, \dots, 0): -1 \le s \le 0,\ t \le s < \infty\}$$

for $0 < t < \infty$ ([2], [7]). Next, if $d_1 > d_2$ let $R' = B_1' \setminus \bar{B}_2'$, where $B_j' = B(f(x_1), d_j)$ for $j = 1, 2$. Then $f^{-1}(R')$ separates x_1 and x_2 from the complement of D and

$$\log \frac{d_1}{d_2} = \operatorname{mod} R' \le K^{1/(n-1)} \operatorname{mod} f^{-1}(R') \le K^{1/(n-1)} \operatorname{mod} R_T(2), \tag{2.3}$$

an inequality which holds trivially if $d_1 \le d_2$. Combining (2.2) and (2.3) yields

$$\frac{1}{c_1} \le \frac{d_1}{d_2} \le c_1, \tag{2.4}$$

where $c_1 = c_1(K, n)$.

Since $f(B_2) \subset B(f(x_1), d_2)$,

$$\begin{cases} (\log J_f)_{B_2} \le \log\left(\dfrac{m(f(B_2))}{m(B_2)}\right) \le n \log \dfrac{d_2}{r_2} \\ \qquad \le n\left(\log \dfrac{2c_1 d_1}{r_1}\right) \end{cases} \tag{2.5}$$

by Jensen's inequality and (2.4). Next, by the n-dimensional version of Lemma 5.10 in [1],

$$|(\log J_f)_{B_1} - (\log J_f)_{B_2}| \le \frac{e}{2}\left(\log \frac{m(B_1)}{m(B_2)} + 1\right) \|\log J_f\|_*, \tag{2.6}$$

where $\|\log J_f\|_*$ denotes the BMO norm of $\log J_f$ in D, a quantity which is bounded above by a constant $c_2 = c_2(K, n)$ [10]. Thus by (2.5) and (2.6),

$$(\log J_f)_{B_1} < n\left(\log \frac{2c_1 d_1}{r_1} + 2c_2\right)$$

and

$$a_f(x_1) = \exp\left(\frac{1}{n}(\log J_f)_{B_1}\right) < \frac{2c_1 d_1}{r_1} \exp(2c_2). \tag{2.7}$$

Finally, if we apply the n-dimensional form of Lemma 5.15 in [1] to f in B_1, we obtain

$$\begin{aligned} d_2 &= |f(x_1) - f(x_2)| \\ &\leq c_3 a_f(x_1) d(x_1, \partial B_1)^a |x_2 - x_1|^{1-a} \\ &\leq 2^a c_3 a_f(x_1) r_1, \end{aligned}$$

where $a = (e/2)\|\log J_f\|_* < 2c_2$ and $c_3 = c_3(K, n) \geq 1$. Thus

$$\frac{d_1}{r_1} \leq c_1 c_3 2^{2c_2} a_f(x_1) \tag{2.8}$$

and (2.1) follows from (2.7) and (2.8) with

$$c = 2c_1 c_3 e^{2c_2}.$$

Theorem 1.8 yields the following analogue for the operator a_f of the familiar composition rule for derivatives,

$$|(g\circ f)'| = |g'\circ f|\,|f'|.$$

2.9. COROLLARY. *Suppose that D, D', D'' are domains in $\mathbf{R}^n$. If $f: D \to D'$ and $g: D' \to D''$ are K_1- and K_2-quasiconformal mappings, then*

$$\frac{1}{c} a_g(f(x)) a_f(x) \leq a_{g\circ f}(x) \leq c a_g(f(x)) a_f(x) \tag{2.10}$$

for $x \in D$, where c is a constant which depends only on K_1, K_2 and n.

Proof. For example, by Theorem 1.8 there exist constants c_1, c_2, c_3 depending only on K_1, K_2 and n such that

$$a_f(x) \geq \frac{1}{c_1} \frac{d(f(x), \partial D')}{d(x, \partial D)},$$

$$a_g(f(x)) \geq \frac{1}{c_2} \frac{d(g\circ f(x), \partial D'')}{d(f(x), \partial D')},$$

$$a_{g\circ f}(x) \leq c_3 \frac{d(g\circ f(x), \partial D'')}{d(x, \partial D)}$$

for $x \in D$, and we obtain the right-hand side of (2.10) with $c = c_1 c_2 c_3$.

3. Proof of Theorem 1.9. We require the following two preliminary results.

3.1. LEMMA. *Suppose that $1 \leq b < \infty$ and that $\{u_j\}_0^\infty$ is a sequence of nonnegative numbers such that*

$$\sum_{j=k}^{\infty} u_j \le b u_k \tag{3.2}$$

for $k=0,1,\ldots$. *Then*

$$\sum_{j=0}^{\infty} u_j^\alpha \le c u_0^\alpha \tag{3.3}$$

for each $0<\alpha\le 1$, *where* c *is a constant which depends only on* b *and* α.

Proof. Set

$$s_k = \sum_{j=k}^{\infty} u_j$$

for $k=0,1,\ldots$. Then $u_k = s_k - s_{k+1}$ and (3.2) is equivalent to

$$s_{k+1} \le \left(\frac{b-1}{b}\right) s_k \tag{3.4}$$

for $k=0,1,\ldots$. Inequality (3.4) then implies that

$$\sum_{j=0}^{\infty} u_j^\alpha \le \sum_{j=0}^{\infty} s_j^\alpha \le s_0^\alpha \sum_{j=0}^{\infty} \left(\frac{b-1}{b}\right)^{j\alpha}$$

$$= s_0^\alpha \left(1 - \left(\frac{b-1}{b}\right)^\alpha\right)^{-1} \le c u_0^\alpha,$$

where

$$c = b^{2\alpha}(b^\alpha - (b-1)^\alpha)^{-1}.$$

3.5. LEMMA. *Suppose that* D *is a uniform domain in* $\mathbf{R}^n$ *and that* α, r *and* m *are constants with* $0<\alpha\le 1$, $0<r<1$ *and* $m\ge 0$. *If* $g: D \to \mathbf{R}^n$ *is an open mapping and if*

$$|g(x_1) - g(x_2)| \le m|x_1 - x_2|^\alpha \tag{3.6}$$

for $x_1, x_2 \in D$ *with* $|x_1 - x_2| = r d(x_1, \partial D)$, *then* g *has a continuous extension to* $\bar{D} \backslash \{\infty\}$ *and*

$$|g(x_1) - g(x_2)| \le c m (|x_1 - x_2| + d(x_1, \partial D))^\alpha \tag{3.7}$$

for $x_1, x_2 \in \bar{D} \backslash \{\infty\}$, *where* c *is a constant which depends only on* α, r *and the constants for* D.

Proof. Fix $x_1, x_2 \in D$. Because D is uniform, we can find a rectifiable arc γ joining x_1 and x_2 in D which satisfies (1.1) and (1.2) with constants a and b which depend only on D. Let x_0 denote the midpoint of γ. Since $d(\gamma, \partial D) > 0$, we can choose points $y_0, y_1, \ldots, y_l \in \gamma$ with the following properties:

$$\begin{cases} y_0 = x_0, \\ y_{j+1} \text{ lies in the component of } \gamma \backslash \{y_j\} \text{ which contains } x_1, \\ |y_{j+1} - y_j| = r d(y_j, \partial D), \\ |x_1 - y_l| \le r d(y_l, \partial D). \end{cases} \tag{3.8}$$

We now use Lemma 3.1 to show that

$$|g(y_l)-g(x_0)| \le c_1 m |x_1-x_2|^\alpha, \tag{3.9}$$

where $c_1 = c_1(r, \alpha, a, b)$; obviously we may assume that $l \ge 1$. Set

$$u_j = \begin{cases} |y_{j+1}-y_j| & \text{if } 0 \le j \le l-1, \\ 0 & \text{if } l \le j < \infty. \end{cases}$$

If $0 \le k \le l-1$, then

$$\sum_{j=k}^{\infty} u_j = \sum_{j=k}^{l-1} |y_{j+1}-y_j| \le l(\gamma_1) \le l(\gamma_2),$$

where γ_i is the component of $\gamma \setminus \{y_k\}$ which contains x_i, while

$$l(\gamma_1) \le bd(y_k, \partial D) = \frac{b}{r}|y_{k+1}-y_k| = \frac{b}{r} u_k$$

by (1.2) and (3.8). Hence

$$\sum_{j=k}^{\infty} u_j \le \frac{b}{r} u_k. \tag{3.10}$$

Inequality (3.10) is trivially true if $k \ge l$ and hence we can apply (3.8), (3.6), (3.3), and (1.1) to obtain

$$\begin{aligned} |g(y_l)-g(x_0)| &\le \sum_{j=0}^{l-1} m|y_{j+1}-y_j|^\alpha \\ &\le c_0 m |y_1-y_0|^\alpha \\ &\le c_1 m |x_1-x_2|^\alpha, \end{aligned}$$

where $c_0 = c_0(b/r, \alpha)$ and $c_1 = (a/2)^\alpha c_0$.

Next, because g is open,

$$\begin{aligned} |g(x_1)-g(y_l)| &\le \sup\{|g(x)-g(y_l)| : |x-y_l| = rd(y_l, \partial D)\} \\ &\le m(rd(y_l, \partial D))^\alpha \\ &\le c_2 md(x_1, \partial D)^\alpha, \end{aligned}$$

where $c_2 = r^\alpha(1-r)^{-\alpha}$, and with (3.9) we have

$$|g(x_1)-g(x_0)| \le (c_1+c_2)m(|x_1-x_2|+d(x_1, \partial D))^\alpha.$$

By the same argument with x_2 in place of x_1,

$$|g(x_2)-g(x_0)| \le (c_1+c_2)m(|x_1-x_2|+d(x_2, \partial D))^\alpha,$$

and since

$$d(x_2, \partial D) \le |x_1-x_2|+d(x_1, \partial D),$$

we obtain (3.7) with $c = 3(c_1+c_2)$ for $x_1, x_2 \in D$.

Next if $x_0 \in \partial D \setminus \{\infty\}$ and if $\{x_j\}$ is a sequence of points in D which converge to x_0, then

$$|g(x_j)-g(x_k)| \le cm(|x_j-x_k|+|x_j-x_0|)^\alpha \to 0$$

as $j, k \to \infty$ by (3.7). Hence g has a finite limit at x_0 and a continuous extension to $\bar{D}\backslash\{\infty\}$ which satisfies (3.7). This completes the proof of Lemma 3.5.

3.11. PROOF OF THEOREM 1.9. Choose $x_1, x_2 \in D$ with

$$|x_1-x_2| = \tfrac{1}{2} d(x_1, \partial D).$$

Then by the n-dimensional version of Lemma 5.15 in [1] applied to $B(x_1, d(x_1, \partial D))$,

$$\begin{aligned} |f(x_1)-f(x_2)| &\le c_1 a_f(x_1) d(x_1, \partial D)^a |x_1-x_2|^{1-a} \\ &= 2^a c_1 a_f(x_1)\, |x_1-x_2| \end{aligned}$$

where $a=(e/2)\|\log J_f\|_* \le c_2$ and $c_j = c_j(K, n)$ for $j=1,2$. Next by hypothesis,

$$a_f(x_1) \le md(x_1, \partial D)^{\alpha-1} \le m|x_1-x_2|^{\alpha-1}$$

and hence

$$|f(x_1)-f(x_2)| \le c_1 2^{c_2} m |x_1-x_2|^\alpha.$$

The desired conclusion now follows from Lemma 3.5 with $r=\frac{1}{2}$.

3.12. REMARK. The mapping

$$f(x) = |x|^{a-1}x, \quad a = K^{1/(1-n)},$$

is K-quasiconformal with $a_f(x)$ bounded in the unit ball D. Hence f satisfies the hypotheses of Theorem 1.9 with $\alpha=1$. Since

$$|f(x)-f(0)| = |x-0|^a,$$

we see that when $K>1$, i.e. when $K^{1/(1-n)} < \alpha$, the conclusion that

$$|f(x_1)-f(x_2)| \le cm(|x_1-x_2|+d(x_1, \partial D))^\alpha$$

in Theorem 1.9 cannot be replaced by the stronger assertion that

$$|f(x_1)-f(x_2)| \le cm|x_1-x_2|^\alpha.$$

On the other hand, the following two alternative quasiconformal analogues of Theorem 1.7 yield sharper estimates for $|f(x_1)-f(x_2)|$ in the special cases where $\alpha \le K^{1/(1-n)}$ or D is bounded.

3.13. THEOREM. *Suppose that D is a uniform domain in $\mathbf{R}^n$ and that α and m are constants with $0<\alpha\le K^{1/(1-n)}$ and $m \ge 0$. If f is K-quasiconformal in D with $f(D) \subset \mathbf{R}^n$ and if*

$$a_f(x) \le md(x, \partial D)^{\alpha-1}$$

for $x \in D$, then f has a continuous extension to $\bar{D}\backslash\{\infty\}$ and

$$|f(x_1)-f(x_2)| \le cm|x_1-x_2|^\alpha$$

for $x_1, x_2 \in \bar{D}\backslash\{\infty\}$, where c is a constant which depends only on K, n, α and the constants for D.

Proof. By Theorem 1.8,

$$d(f(x), \partial f(D)) \le c_1 a_f(x) d(x, \partial D) \le c_1 m d(x, \partial D)^\alpha, \quad c_1 = c_1(K, n)$$

for $x \in D$. Then since $\alpha \le K^{1/(1-n)}$, the desired conclusions follow from Theorems 3.4 and 2.24 in [5].

3.14. THEOREM. *Suppose that D is a bounded uniform domain in $\mathbf{R}^n$ and that α and m are constants with $0 < \alpha \le 1$ and $m \ge 0$. If f is K-quasiconformal in D with $f(D) \subset \mathbf{R}^n$ and if*

$$a_f(x) \le m d(x, \partial D)^{\alpha - 1}$$

for $x \in D$, then f has a continuous extension to $\bar{D}$ and

$$|f(x_1) - f(x_2)| \le cm|x_1 - x_2|^\beta, \quad \beta = \min(\alpha, K^{1/(1-n)}) \tag{3.15}$$

for $x_1, x_2 \in \bar{D}$, where c is a constant which depends only on K, n, α and the constants and diameter of D.

Proof. By Theorem 1.9, f has a continuous extension to $\bar{D}$ and

$$|f(x_1) - f(x_2)| \le c_1 m |x_1 - x_2|^\alpha$$

for $x_1 \in \partial D$ and $x_2 \in \bar{D}$, where c_1 depends only on K, n, α and the constants for D. Inequality (3.15) then follows from Theorem 1 in [9] with

$$c = c_1 c_2 \max(1, \operatorname{dia}(D)^{\alpha - K^{1/(1-n)}}), \quad c_2 = c_2(n).$$

3.16. REMARK. Using the example in 3.12 it is not difficult to construct a K-quasiconformal self-mapping f of a half space D for which a_f is bounded and the conclusion (3.15) fails. Thus the hypothesis that D be bounded is necessary in Theorem 3.14.

Finally, Theorem 1.8 allows us to establish the following converse for Theorem 1.9.

3.17. THEOREM. *Suppose that D is a domain in $\mathbf{R}^n$ and that α and m are constants with $0 \le \alpha \le 1$ and $m \ge 0$. If f is K-quasiconformal in D with $f(D) \subset \mathbf{R}^n$ and if*

$$|f(x_1) - f(x_2)| \le m(|x_1 - x_2| + d(x_1, \partial D))^\alpha$$

for all $x_1, x_2 \in D$, then

$$a_f(x) \le c m d(x, \partial D)^{\alpha - 1}$$

for $x \in D$, where c is a constant which depends only on K and n.

Proof. Choose $x_1 \in D$ and $x_2 \in \partial D \setminus \{\infty\}$ such that

$$|x_1 - x_2| = d(x_1, \partial D).$$

Then f has a continuous extension to $\bar{D} \setminus \{\infty\}$,

$$d(f(x_1), \partial f(D)) \le |f(x_1) - f(x_2)| \le m|x_1 - x_2|^\alpha,$$

and

$$a_f(x_1) \leq c \frac{d(f(x_1), \partial f(D))}{d(x_1, \partial D)} \leq cmd(x_1, \partial D)^{\alpha-1}$$

by Theorem 1.8.

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