ON CERTAIN ANALYTIC (NEVANLINNA) FUNCTIONS

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1. Introduction. We study the classes N_1 and N_2 of all analytic functions having the representations

$$(1)\qquad f(z)=\int_{-1}^{1}\frac{d\mu(t)}{z-t}\quad\text{and}\quad \phi(z)=\int_{-1}^{1}\frac{z\,d\mu(t)}{1-tz},$$

where μ is a probability measure. These classes have been the subject of some interesting research during the recent past. Thale [5] showed that the maximal domain of univalence of $N_1(N_2)$ is the open set $|z|>1(|z|<1)$. In two recent notes ([3], [4]) we found the radii of starlikeness and convexity, of order alpha, of N_1 and N_2. We also proved that for each $\phi\in N_2$, $\phi(z)$ and $z\phi'(z)$ are typically-real for $|z|<1$. On the other hand, Goluzin [2] found sharp bounds on the modulus and on the argument of the set TR of all functions $g(z)=z+\cdots$ that are typically-real for $|z|<1$. But Goluzin's extremal functions do *not* belong to our class N_2. Hence it is reasonable for us to try to obtain sharp bounds on Goluzin's functionals $|\phi(z)|$, $\arg\phi(z)$ for $\phi\in N_2$. We do just that plus more. We also find sharp bounds on $|\operatorname{Im}\phi(z)|$, $|\phi'(z)|$ and $\arg\phi'(z)$, for $\phi\in N_2$, $0\le|z|<1$.

2. Bounds on $|\phi(z)|$ and $|\phi'(z)|$. The kernels

$$(2)\qquad l(z,t)\equiv\frac{z}{1-tz},\quad k(z,t)\equiv\frac{1}{z-t}$$

play a leading role, as we shall see.

THEOREM 1. *For each z, $|z|<1$, and for each $\phi\in N_2$, the following inequalities hold:*

$$(3)\qquad |\phi(z)|\le\left|\frac{z}{1\pm z}\right|,\quad |z\pm\tfrac{1}{2}|\le\tfrac{1}{2},$$

$$(4)\qquad |\phi(z)|\le\frac{1}{|\operatorname{Im}(1/z)|},\quad |z\pm\tfrac{1}{2}|\ge\tfrac{1}{2},$$

where for $z\ne0$, equality in (3) *holds only for the appropriate function $\phi(z)\equiv l(z,\pm1)$ and equality in* (4) *holds only for $\phi(z)\equiv l(z,t)$, with $t=\operatorname{Re}(1/z)$.*

Proof. We suppose $z\ne0$. The integral (1) yields

$$|\phi(z)|\le\int_{-1}^{1}|l(z,t)|\,d\mu(t).$$

Hence we study $|l(z,t)|$, $-1\le t\le1$. Let

Received June 6, 1984. Revision received September 17, 1984.
Michigan Math. J. 32 (1985).

$$|l(z,t)|^2=\frac{1}{\tau(z,t)}, \quad \text{where } \tau(z,t)=t^2-2t\,\mathrm{Re}\,\frac{1}{z}+\frac{1}{|z|^2}.$$

Since $\tau(z,t)$ decreases for $-\infty<t\le \mathrm{Re}(1/z)$ and increases for $\mathrm{Re}(1/z)\le t<\infty$, it is clear that $\tau(z,t)$ achieves its minimum (i) at $t=-1$, if $\mathrm{Re}(1/z)\le -1$, (ii) at $t=1$, if $\mathrm{Re}(1/z)\ge 1$, (iii) at $t=\mathrm{Re}(1/z)$, if $-1\le \mathrm{Re}(1/z)\le 1$. The inequalities (3) and (4), and the assertions concerning the sharpness now follow. □

REMARK 1. For $\phi\in N_2$, Dundučenko [1] obtained the inequality $|\phi(z)|\le |z|/(1-|z|)$ which is less precise than (3) and (4).

THEOREM 2. *For fixed* z, $0<|z|<1$, *and for each* $\phi\in N_2$, *the inequalities*

$$|\phi'(z)|\le \frac{1}{|1\pm z|^2}, \quad |z\pm\tfrac{1}{2}|\le\tfrac{1}{2} \tag{5}$$

$$|\phi'(z)|\le \frac{1}{|z\,\mathrm{Im}(1/z)|^2}, \quad |z\pm\tfrac{1}{2}|\ge\tfrac{1}{2} \tag{6}$$

hold, where equality holds in (5) *only for the appropriate function* $\phi(z)\equiv l(z,\pm 1)$ *and equality holds in* (6) *only for* $\phi(z)\equiv l(z,t)$ *with* $t=\mathrm{Re}(1/z)$.

Proof. From (1) and (2) we obtain

$$|\phi'(z)|\le \int_{-1}^{1}|l'_z(z,t)|\,d\mu(t)=\frac{1}{|z|^2}\int_{-1}^{1}|l(z,t)|^2\,d\mu(t),$$

and this last allows us to use the reasoning in the proof of Theorem 1 to obtain our result. □

REMARK 2. Our result is a better one than Dundučenko's $|\phi'(z)|\le 1/(1-|z|^2)$ [1].

3. Bounds on $|\mathrm{Im}\,\phi(z)|$ and $\sup_{|z|=r}|\mathrm{Im}\,\phi(z)|$.

THEOREM 3. *For fixed* z, $0<|z|<1$, *and for each* $\phi\in N_2$, *the following hold:*

$$|\mathrm{Im}\,\phi(z)|\le\left|\frac{\mathrm{Im}\,z}{(1\pm z)^2}\right|, \quad |z\pm\tfrac{1}{2}|\le\tfrac{1}{2}, \tag{7}$$

$$|\mathrm{Im}\,\phi(z)|\le\left|\frac{z^2}{\mathrm{Im}\,z}\right|, \quad |z\pm\tfrac{1}{2}|\ge\tfrac{1}{2}, \tag{8}$$

with equality in (7) *only for the appropriate* $\phi(z)\equiv l(z,\pm1)$, *and equality in* (8) *only for* $\phi(z)\equiv l(z,t)$, $t=\mathrm{Re}(1/z)$, *if* $\mathrm{Im}\,z\ne 0$.

Proof. Again, from (1) and (2) we find

$$|\mathrm{Im}\,\phi(z)|\le\int_{-1}^{1}|\mathrm{Im}\,l(z,t)|\,d\mu=\left|\frac{\mathrm{Im}\,z}{z^2}\right|\int_{-1}^{1}|l(z,t)|^2\,d\mu. \tag{9}$$

Once again the reasoning in the proof of Theorem 1 yields the desired results. □

THEOREM 4. *Let r be fixed,* $0<r<1$; *then for each* $\phi \in N_2$, *the inequality*

$$|\operatorname{Im} \phi(z)| \leq \frac{r}{1-r^2}, \quad |z|=r,$$

holds, with equality only for the functions $\phi(z) \equiv l(z, \pm 1)$, *for the appropriate*

$$z = \pm r\left(\frac{2r}{1+r^2} \pm i\frac{1-r^2}{1+r^2}\right).$$

Proof. As z varies on the circle $|z|=r$, $0<r<1$, subject to $|z \pm \frac{1}{2}| \geq \frac{1}{2}$, we see that the right-hand side of (8) increases as $|\operatorname{Im} z|$ decreases so that the maximum of $|\operatorname{Im} \phi(z)|$ occurs where $|z|=r$ meets $|z \pm \frac{1}{2}| = \frac{1}{2}$, that is, at the points $z = \pm r \exp(\pm i \arccos r)$, $0 < \arccos r < \pi/2$. This yields the maximum of $|\operatorname{Im} \phi(z)|$ on $|z|=r$, subject to $|z \pm \frac{1}{2}| \geq \frac{1}{2}$, is $r/\sqrt{1-r^2}$. But if we use (7), then elementary calculations show that the maximum of $|\operatorname{Im} z|/|1 \pm z|^2$, subject to the constraints $|z \pm \frac{1}{2}| \leq \frac{1}{2}$, is $r/(1-r^2)$, and this occurs only at the points

$$z^* = r\left(\frac{2r}{1+r^2} \pm i\frac{1-r^2}{1+r^2}\right) \quad \text{for } \phi(z) \equiv l(z, +1)$$

and at the points $-z^*$ for $l(z, -1)$. Our theorem now follows from the inequality $r/\sqrt{1-r^2} \leq r/(1-r^2)$. □

4. Bounds on $\arg \phi(z)$ **and** $\arg \phi'(z)$**.** In this section, if $z \neq 0$, then $\arg z$ satisfies $-\pi < \arg z \leq \pi$.

THEOREM 5. *For each* z, $0<|z|<1$, *and for each* $\phi \in N_2$, *we have*

$$\arg \frac{z}{1 \pm z} \leq \arg \phi(z) \leq \arg \frac{z}{1 \mp z}, \quad \operatorname{Im} z \gtrless 0. \tag{10}$$

Equality holds only for the appropriate choice among the functions $\phi(z) \equiv l(z, \pm 1)$.

Proof. The inequalities certainly hold for $\operatorname{Im} z = 0$. Hence we assume $\operatorname{Im} z > 0$. It follows from (1) and (2) that each $\phi \in N_2$ is the limit of a weighted sum of complex numbers, with positive weights, each of which lies in the upper half plane, i.e., $0 < \arg z/(1-tz) < \pi$ for $-1 \leq t \leq 1$. Moreover, it is geometrically clear that

$$\arg(1-z) = [\arg(1-tz)]_{t=1} \leq \arg(1-tz) \leq [\arg(1-tz)]_{t=-1} = \arg(1+z)$$

holds for $-1 \leq t \leq 1$ and that

$$\min_{-1 \leq t \leq 1} \arg l(z, t) \leq \arg \phi(z) \leq \max_{-1 \leq t \leq 1} \arg l(z, t)$$

also holds. Hence we have

$$\arg \frac{z}{1+z} \equiv \arg l(z, -1) \leq \arg \phi(z) \leq \arg l(z, 1) \equiv \arg \frac{z}{1-z},$$

which we have established for all $0<|z|<1$, $\operatorname{Im} z > 0$. A similar reasoning yields the analogous result for $0<|z|<1$ with $\operatorname{Im} z < 0$. This completes our proof. □

COROLLARY 5.1. *With the hypotheses of Theorem 5, we obtain*

$$\arg \frac{1}{1\pm z} \le \arg \frac{\phi(z)}{z} \le \arg \frac{1}{1\mp z}, \quad \operatorname{Im} z \gtrless 0,$$

where, for $\operatorname{Im} z \ne 0$, *equality holds only for one of the functions* $\phi(z) \equiv l(z, \pm 1)$.

REMARK 3. Dundučenko established the less precise inequality [1]

$$\left|\arg \frac{\phi(z)}{z}\right| \le \arcsin|z|, \quad 0 \le |z| < 1, \quad \phi \in N_2.$$

THEOREM 6. *For each* z, $|z|<1$, *and for each* $\phi \in N_2$, *we have the inequalities*

$$\arg \frac{1}{(1\pm z)^2} \le \arg \phi'(z) \le \arg \frac{1}{(1\mp z)^2}, \quad \operatorname{Im} z \gtrless 0, \tag{11}$$

where, for $\operatorname{Im} z \ne 0$, *equality holds only for the proper choice among* $\phi(z) \equiv l(z, \pm 1)$.

Proof. From (1) and (2) we obtain

$$\phi'(z) = \int_{-1}^{1} l_z'(z, t)\, d\mu \equiv \int_{-1}^{1} \left(\frac{l(z, t)}{z}\right)^2 d\mu(t).$$

Once again, the same reasoning as in the proof of Theorem 5 yields (11). □

COROLLARY 6.1. *For* $\phi \in N_2$, *for* $|z|<1$, *we have* $|\arg \phi'(z)| \le 2 \arcsin|z|$, *with equality only for the functions* $\phi(z) \equiv l(z, \pm 1)$ *if* $z \ne 0$.

REMARK 4. For $\phi \in N_2$, Dundučenko [1] showed that

$$|\arg \phi'(z)| \le \arccos(1 - 2|z|^2)$$

holds for $|z| \le 1/\sqrt{2}$. If one chooses the range $0 \le \arccos(1-2|z|^2) \le \pi$, then Dundučenko's result (which is the same as ours in Corollary 6.1) holds for all z, $|z|<1$.

COROLLARY 6.2. *Each* $\phi \in N_2$ *maps each chord*

$$\left[z \mid \operatorname{Im} z = c, \ -\frac{1}{\sqrt{2}} < c < \frac{1}{\sqrt{2}}, \ |z| \leqq \frac{1}{\sqrt{2}}\right]$$

onto curves each of which intersects the lines $\operatorname{Re} w = $ *constant at most once.*

5. Bounds on the mean values of $|\phi(z)|^p$ and $|\phi'(z)|^p$ on the circle $|z| = r$.

THEOREM 7. *For* $\phi \in N_2$, $0 \le r < 1$, *for* $p > 1$, *we have*

$$\int_0^{2\pi} |\phi(re^{i\theta})|^p \, d\theta \le \int_0^{2\pi} |l(re^{i\theta}, \pm 1)|^p \, d\theta.$$

Proof. From (1) and (2) and the Hölder inequality we obtain

$$\begin{aligned}\int_0^{2\pi} |\phi(re^{i\theta})|^p\,d\theta &\le \int_0^{2\pi}\left[\int_{-1}^{1} |l(re^{i\theta},t)|^p\,d\mu\right]d\theta \\ &\le \int_{-1}^{1}\left[\int_0^{2\pi} |l(re^{i\theta},t)|^p\,d\theta\right]d\mu \\ &\le \max_{-1\le t\le 1}\int_0^{2\pi} |l(re^{i\theta},t)|^p\,d\theta.\end{aligned} \tag{12}$$

But

$$\begin{aligned}|l(re^{i\theta},t)|^p &= r^p(1-tre^{i\theta})^{-p/2}(1-tre^{-i\theta})^{-p/2} \\ &= r^p\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{k+n}\binom{-p/2}{k}\binom{-p/2}{n}(tr)^{k+n}e^{i(k-n)\theta},\end{aligned}$$

which, with (12), yields

$$\begin{aligned}\int_0^{2\pi} |\phi(re^{i\theta})|^p\,d\theta &\le \max_{-1\le t\le 1} 2\pi r^p\sum_{k=0}^{\infty}\binom{-p/2}{k}^2(tr)^{2k} \\ &\le 2\pi r^p\sum_0^{\infty}\binom{-p/2}{k}^2 r^{2k} = \int_0^{2\pi}|l(re^{i\theta},\pm1)|^p\,d\theta.\end{aligned}$$

This completes our proof of Theorem 7. □

COROLLARY 7.1. *For $\phi\in N_2$, $0<r<1$, and p a positive integer, we have*

$$\int_0^{2\pi}|\phi(re^{i\theta})|^{2p}\,d\theta \le 2\pi\left(\frac{r^2}{1-r^2}\right)^p\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{p+k-1}{k}\left(\frac{r^2}{1-r^2}\right)^k, \tag{13}$$

with equality only for $\phi(z)\equiv l(z,\pm1)$.

Proof. For $0<r<1$, $\phi\in N_2$ we have

$$\int_0^{2\pi}|\phi(re^{i\theta})|^{2p}\,d\theta \le \int_0^{2\pi}\left(\frac{r}{\sqrt{1-2r\cos\theta+r^2}}\right)^{2p}d\theta. \tag{14}$$

If we set $\zeta=e^{i\theta}$ in (14) and if we use the theory of residues, we obtain

$$\int_0^{2\pi}\frac{d\theta}{(1-2r\cos\theta+r^2)^p} = \frac{2\pi(-1)^p}{r^p(p-1)!}\left(\frac{d^{p-1}}{d\zeta^{p-1}}\left[\frac{\zeta^{p-1}}{(\zeta-(1/r))^p}\right]\right)_{\zeta=r},$$

which with (14) yields (13). □

THEOREM 8. *If $\phi\in N_2$, and if $p\ge 1$, then for each r, $0\le r<1$ we have*

$$\int_0^{2\pi}|\phi'(re^{i\theta})|^p\,d\theta \le \int_0^{2\pi}|l_z'(re^{i\theta},\pm1)|^p\,d\theta.$$

Proof. A proof can be given that uses the same technique that was used in the proof of Theorem 7. □

COROLLARY 8.1. *If* $\phi \in N_2$, $0<r<1$, *and* p *a positive integer, then we have*

$$\int_0^{2\pi} |\phi'(re^{i\theta})|^p \, d\theta \le \frac{2\pi}{(1-r^2)^p} \sum_{k=0}^{p-1} \binom{p-1}{k} \binom{p+k-1}{k} \left(\frac{r^2}{1-r^2}\right)^k,$$

with equality only for $\phi(z) \equiv l(z, \pm 1)$.

Proof. We use the fact that

$$l_z'(z,t) \equiv \left(\frac{l(z,t)}{z}\right)^2,$$

and then use Corollary 7.1. □

7. The class N_1. Since the transformation $z \to 1/z$ yields a (1-1) correspondence between N_1 and N_2, it follows that results analogous to those above can be obtained quite easily.

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