STRUCTURE SPACES OF RINGS AND BANACH ALGEBRAS

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1. Introduction. This paper originated in a re-examination of a 1949 result of Kaplansky [8, Theorem 8.1]. Let B be a Banach algebra, $\mathfrak{P}(B)$ be the structure space of B (space of primitive ideals of B) and $\mathfrak{J}(B)$ be the set of isolated points of $\mathfrak{P}(B)$. Kaplansky's theorem asserts that for a B^*-algebra B, $\mathfrak{J}(B)=\mathfrak{P}(B)$ if and only if B is the $B^*(\infty)$-sum of topologically simple B^*-algebras. Alternatively one may say that $\mathfrak{J}(B)=\mathfrak{P}(B)$ if and only if B is the direct topological sum of its minimal closed ideals. We are using the notions of [9] and by an ideal we shall always mean a two-sided ideal unless otherwise specified.

We took as our initial task a description of $\mathfrak{J}(B)$ for a B^*-algebra B. We found that the elements of $\mathfrak{J}(B)$ are precisely the annihilators in B of the minimal closed ideals of B. Inasmuch as the arguments for this result were largely algebraic we were led into a consideration of $\mathfrak{J}(A)$ for any semi-simple topological ring A. For a description of $\mathfrak{J}(A)$ the notion of minimal closed ideal is inadequate.

A basic notion here is that of a purely primitive ring (or algebra). A ring R is purely primitive if (0) is the only primitive ideal of R. Easy examples show that primitive rings need not be purely primitive.

For the semi-simple ring A, the elements of $\mathfrak{J}(A)$ are precisely the annihilators of those non-zero ideals of A which are purely primitive rings. For the B^*-algebra B, $\mathfrak{J}(B)$ is in one-to-one correspondence with the minimal closed ideals. The corresponding fact for A is that $\mathfrak{J}(A)$ is in one-to-one correspondence with the non-zero ideals of A which are maximal with respect to the property of being purely primitive rings. Such ideals are closed in A if the primitive ideals of A are closed in A or if $\mathfrak{P}(A)$ is a T_1-space.

With this information in hand some results are obtained for rings A with discrete structure space and also for those where $\mathfrak{J}(A)$ is dense in $\mathfrak{P}(A)$. The class of semi-simple Banach algebras for which $\mathfrak{J}(A)$ is dense in $\mathfrak{P}(A)$ is fairly natural—a B^*-algebra has this property if and only if every non-zero closed ideal of B contains a minimal closed ideal of B. A semi-simple Banach algebra A has this property if and only if every non-zero closed ideal of A contains a non-zero ideal of A which is a purely primitive Banach algebra. Clearly any semi-simple topologically simple Banach algebra is purely primitive. The converse is false and we now turn to that question.

2. Purely primitive rings. Let R be a primitive topological ring (or algebra). We say that R is *purely primitive* if (0) is the only primitive ideal of R. This is a key concept of our program. Suppose that R has minimal one-sided ideals and

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that S is the socle of R. Then the primitive ring R is purely primitive if and only if R is a modular annihilator ring in the sense of [10, p. 39]. This follows since R is a modular annihilator ring if and only if R/S is a radical ring [10, p. 38] and, moreover, S is contained in every non-zero ideal of R. Also R is topologically simple if and only if S is dense in R. Clearly R is purely primitive if it is topologically simple. The converse fails.

PROPOSITION 2.1. *A primitive topological ring R with minimal one-sided ideals is purely primitive and not topologically simple if and only if R is a modular annihilator ring and its socle is not dense.*

This enables us to give examples of Banach algebras which are purely primitive and not topologically simple. Consider a Banach space E and the Banach algebras $K(E)$ of compact linear operators on E and $F(E)$, the uniform limits of finite rank linear operators. Alexander [1] showed that there is a closed linear subspace of E of l_p, $2<p<\infty$ for which $F(E)\neq K(E)$. Now $K(E)$ is a primitive modular annihilator algebra by [2, p. 66]. Then $K(E)$ is purely primitive and not topologically simple by the preceding proposition.

Other examples can be obtained as follows. There are classical examples of Banach spaces E for which (1) not every weakly compact linear operator on E is compact and (2) the product of any two weakly compact linear operators is compact. See [5] and [6]. For such a space let $W(E)$ be the set of all weakly compact linear operators on E. Inasmuch as $W(E)/K(E)$ is a radical algebra, it follows from Proposition 2.1 that $W(E)$ is a purely primitive Banach algebra which is not topologically simple. It is readily seen that there are no commutative examples.

3. Basic theory. We turn to a consideration of a semi-simple topological ring A (of course the topology could be the discrete topology). For a subset W of A let $L(W)$ ($R(W)$) denote the left (right) annihilator of W in A. Let K be an ideal in A. By [3, p. 462], $L(K)=R(K)$. We set $K^a=L(K)=R(K)$. As in [11, p. 308] we say that the ideal K is *dual* if $K=K^{aa}$ and note that K^a is always dual. A dual ideal $I\neq(0)$ is called *minimally dual* if no non-zero ideal J of A, $J\subset I$, $J\neq I$ can be dual.

LEMMA 3.1. *If P is a prime ideal in A and $P^a\neq(0)$, then P^a is minimally dual and P is dual.*

Proof. In fact, if $K\neq(0)$ is an ideal of A, $K\subset P^a$, then $K^a=P$. First observe that $K\not\subset P$ and so $K^a\subset P$. But $K^a\supset P^{aa}$. Thus $P^{aa}\subset K^a\subset P\subset P^{aa}$.

LEMMA 3.2. *The intersection of any number of dual ideals of A is a dual ideal.*
Proof. For if $\{I_\beta\}$ is a family of dual ideals,

$$\bigcap I_\beta\subset\left(\bigcap I_\beta\right)^{aa}\subset\bigcap I_\beta^{aa}=\bigcap I_\beta.$$

LEMMA 3.3. *If $P_1\neq P_2$ are two prime ideals of A then $P_1^a\cap P_2^a=(0)$.*

Proof. We may suppose $P_1^a\neq(0)$ and $P_2^a\neq(0)$. By Lemmas 3.1 and 3.2, either $P_1^a\cap P_2^a=(0)$ or else $P_1^a\cap P_2^a=P_1^a=P_2^a$. But in the latter case, $P_1^{aa}=P_2^{aa}$ so that $P_1=P_2$ by Lemma 3.1, which is contrary to our hypotheses. □

The only prime ideals of concern here are primitive ideals.

LEMMA 3.4. *Let $W \neq (0)$ be an ideal in A where W is a primitive ring. Then there is a unique primitive ideal P of A with $W \subset P^a$. Also $W \subset Q$ for any primitive ideal $Q \neq P$ and $Q^a \neq (0)$.*

Proof. By [7, p. 206, Proposition 2] there exists a primitive ideal P of A such that $P \cap W = (0)$. Therefore $W \subset P^a$. The uniqueness of P is given by Lemma 3.3. If $Q \neq P$ is a primitive ideal and $Q^a \neq (0)$ then $W \cap Q^a = (0)$ so that $W \subset Q^{aa} = Q$. □

LEMMA 3.5. *The direct sum $\sum P^a$ of the ideals $P^a \neq (0)$, where P is a primitive ideal, is the smallest ideal of A which contains every ideal of A which is itself a primitive ring.*

Proof. In view of Lemma 3.4 it is sufficient to see that each of the $P^a \neq (0)$ is a primitive ring. Let π be the natural homomorphism of A onto A/P. Then $\pi(P^a)$ is an isomorphic image of P^a and, being an ideal in the primitive ring A/P, is itself a primitive ring. □

Consider the structure space (space of primitive ideals) $\mathfrak{P}(A)$ in the hull-kernel topology. For a set $\mathfrak{S}$ in $\mathfrak{P}(A)$ let $k(\mathfrak{S})$, the kernel of $\mathfrak{S}$, be the intersection of the primitive ideals P in $\mathfrak{S}$. For each primitive ideal P let $W(P)$ be the kernel of the $Q \in \mathfrak{P}(A)$, $Q \neq P$. Inasmuch as A is semi-simple we always have $W(P) \subset P^a$. Note that P is an isolated point of $\mathfrak{P}(A)$ if and only if $P \not\supset W(P)$. As $W(P) \subset P^a$, P is an isolated point if and only if $W(P) \neq (0)$. Suppose $W(P) \neq (0)$. We see from [7, Proposition 2, p. 206] that $W(P)$ is a purely primitive ring.

LEMMA 3.6. *Let $K \neq (0)$ be an ideal of A which is a purely primitive ring. Then there is a unique primitive ideal P of A so that $K \subset W(P)$.*

Proof. Take the primitive ideal P so that $K \subset P^a$ as given by Lemma 3.4. Let $Q \neq P$ be a primitive ideal of A. Suppose that $K \not\subset Q$. Then $K \cap Q$ is a primitive ideal of K and so $K \cap Q = (0)$. Hence $K \subset Q^a$, which is impossible by Lemma 3.3. Thus $K \subset W(P)$. □

We characterize the $W(P) \neq (0)$ intrinsically.

LEMMA 3.7. *The ideals $W(P) \neq (0)$ are the non-zero ideals of A which are purely primitive rings and properly contained in no ideal which is a purely primitive ring.*

The following is a counterpart to Lemma 3.5.

LEMMA 3.8. *The direct sum $\sum W(P)$ of the $W(P) \neq (0)$ is the smallest ideal of A which contains every ideal of A which is a purely primitive ring.*

It should be noted that one can have $W(P) \neq P^a$. For let A be the algebra of all bounded linear operators on l_2. $P = (0)$ is a primitive ideal with $W(P)$ all the compact linear operators on l_2 and $P^a = A$.

This direct sum $N = \sum W(P)$, $P \in \mathfrak{J}(A)$, plays an important role in our theory. (We take $N = (0)$ if $\mathfrak{J}(A)$ is empty.)

THEOREM 3.9. (1) *The ideals of A which are in $\mathfrak{J}(A)$ are precisely the annihilators of those non-zero ideals of A which are purely primitive rings.* (2) *The closure of $\mathfrak{J}(A)$ in $\mathfrak{P}(A)$ is the hull of N^a.*

Proof. As noted above, $P \in \mathfrak{J}(A)$ if and only if $W(P) \neq (0)$. Let $K \neq (0)$ be any ideal of A which is a purely primitive ring. By Lemma 3.6, $K \subset W(P)$ for some $P \in \mathfrak{J}(A)$. Also $K^a = P$ by the calculation of Lemma 3.1. Conversely, if $P \in \mathfrak{J}(A)$, then $W(P) \neq (0)$ and $P = (W(P))^a$ where $W(P)$ is a purely primitive ring.

Since the sum $N = \sum W(P)$, $P \in \mathfrak{J}(A)$, is a direct sum we have

$$N^a = \bigcap \{(W(P))^a : P \in \mathfrak{J}(A)\}.$$

Inasmuch as $(W(P))^a = P$ for these P, we get $N^a = k(\mathfrak{J}(A))$. □

Clearly $\mathfrak{J}(A)$ is in one-to-one correspondence with the set of ideals $W(P) \neq (0)$. The ideals $W(P)$ are closed if the primitive ideals of A are all closed ideals. If these primitive ideals need not be closed the matter is obscure. One case is easy.

LEMMA 3.10. *Suppose that $P_0 \in \mathfrak{P}(A)$ and $P_0^a \neq (0)$. If $\{P_0\}$ is a closed point of $\mathfrak{P}(A)$, then $W(P_0) = P_0^a$.*

Proof. We have $W(P_0) \subset P_0^a$. If $P \in \mathfrak{P}(A)$, $P \neq P_0$, then P cannot contain P_0 by hypothesis. Therefore $P \supset P_0^a$, so $P_0^a \subset W(P_0)$. □

For a set $\mathfrak{S}$ in $\mathfrak{P}(A)$ let $\mathrm{Int}(\mathfrak{S})$ denote the interior of $\mathfrak{S}$. For $P \in \mathfrak{P}(A)$ let $[P]$ denote the closure of the point P.

LEMMA 3.11. *For P, Q in $\mathfrak{P}(A)$, $Q \in \mathrm{Int}([P])$ if and only if $Q \not\supset P^a$.*

Proof. Consider any $Q \in \mathfrak{P}(A)$ where $Q \notin [P]$. As $Q \not\supset P$ we have $Q \supset P^a$. Thus $P^a \subset K$ where K is the kernel of the complement of $[P]$. On the other hand $P = k([P])$ so that, by semi-simplicity, $K \cap P = (0)$ or $K \subset P^a$. Hence $K = P^a$.

From this we see that $Q \not\supset P^a$ if and only if Q is in the complement of the closure of the complement of $[P]$ which is $\mathrm{Int}([P])$. □

LEMMA 3.12. *$P_0^a \neq (0)$ if and only if $P_0 \in \mathrm{Int}([P_0])$. If $P_0^a \neq (0)$ the mapping $Q \to Q \cap P_0^a$ is a homeomorphism of $\mathrm{Int}([P_0])$ onto $\mathfrak{P}(P_0^a)$.*

Proof. Let $P = Q = P_0$ in Lemma 3.11, where $P \not\supset P^a$ if and only if $P^a \neq (0)$. The homeomorphism follows from [7, p. 206]. □

If the point P is closed in $\mathfrak{P}(A)$, P is an isolated point if and only if $P^a \neq (0)$. As the example following Lemma 3.8 shows, an isolated point need not be closed in $\mathfrak{P}(A)$.

While $P^a \neq (0)$ for every isolated point P, we have no example of $P \in \mathfrak{P}(A)$ where $P^a \neq (0)$ and P is not an isolated point. But at least for the closure of $\mathfrak{J}(A)$ we can be definite.

THEOREM 3.13. *For P in the closure of $\mathfrak{J}(A)$, $P \in \mathfrak{J}(A)$ if $P^a \neq (0)$.*

Proof. Take P in the closure of $\mathfrak{J}(A)$, $P^a \neq (0)$. By Lemma 3.12, $P \in \operatorname{Int}([P])$. Hence there exists $Q \in \mathfrak{J}(A)$ with $Q \in \operatorname{Int}([P])$. Since $Q \in \mathfrak{J}(A)$, $Q^a \neq (0)$. As $Q \in [P]$, we have $P \subset Q$. It follows from [11, Lemma 2.1] that $P = Q$. □

We now give characterizations for $\mathfrak{J}(A) = \mathfrak{P}(A)$ and for $\mathfrak{J}(A)$ to be dense in $\mathfrak{P}(A)$.

THEOREM 3.14. *The following are equivalent.*
(1) $\mathfrak{J}(A) = \mathfrak{P}(A)$.
(2) *$P^a \neq (0)$ for all $P \in \mathfrak{P}(A)$.*
(3) *A/N is a radical ring.*

Proof. As noted earlier $P^a \neq (0)$ for all $P \in \mathfrak{J}(A)$, so (1) implies (2). Suppose (2) and that $P_1 \neq P_2$ are in $\mathfrak{P}(A)$. It follows from [11, Lemma 2.1] that $P_1 \not\subset P_2$. Hence $\{P_1\}$ is a closed point of $\mathfrak{P}(A)$, so $P_1^a = W(P_1)$ by Lemma 3.10. Clearly no $Q \in \mathfrak{P}(A)$ contains every P^a and therefore does not contain the ideal N. Hence A/N is a radical ring.

Suppose (3) and let $Q \in \mathfrak{P}(A)$. Since Q fails to contain N, there is $P \in \mathfrak{P}(A)$ so that $Q \not\supset W(P)$. Since $W(P)$ is a purely primitive ring we have $W(P) \cap Q = (0)$ and $Q \subset (W(P))^a = P$ by Lemma 3.1. Since $Q^a \neq (0)$ we have $Q = P$ by [11, Lemma 2.1]. Therefore $W(Q) \neq (0)$ so that $Q \in \mathfrak{J}(A)$. □

THEOREM 3.15. *The following are equivalent.*
(1) *$\mathfrak{J}(A)$ is dense in $\mathfrak{P}(A)$.*
(2) $N^a = (0)$.
(3) *Every closed ideal $K \neq (0)$ of A contains a non-zero ideal of A which is a purely primitive ring.*

Proof. By semi-simplicity a subset $\mathfrak{S}$ of $\mathfrak{P}(A)$ is dense in $\mathfrak{P}(A)$ if and only if $k(\mathfrak{S}) = (0)$. By Theorem 3.9, $N^a = k(\mathfrak{J}(A))$. Hence (1) and (2) are equivalent.

Assume (2). Let $K \neq (0)$ be a closed ideal of A. There exists $P \in \mathfrak{J}(A)$ so that $KW(P) \neq (0)$. Inasmuch as any ideal of a purely primitive ring is purely primitive by [7, p. 206, Proposition 2], we see that (2) implies (3).

Assume (3). If $N^a \neq (0)$ then the closed ideal N^a contains an ideal $J \neq (0)$ of A which is a purely primitive ring. By Lemma 3.6, $J \subset W(P)$ for some $P \in \mathfrak{P}(A)$. But then $J \subset N$, which is impossible. □

COROLLARY 3.16. *Let K be an ideal of A. If $\mathfrak{J}(A)$ is dense in $\mathfrak{P}(A)$, then $\mathfrak{J}(K)$ is dense in $\mathfrak{P}(K)$.*

Proof. Consider any non-zero ideal L of K. That L contains a non-zero ideal J of A is seen by [7, p. 65]. We apply Theorem 3.15 to A in the discrete topology to see that J contains a non-zero ideal of A which is a purely primitive ring. Then, by Theorem 3.15 again, $\mathfrak{J}(K)$ is dense in $\mathfrak{P}(K)$. □

We apply our results to B^*-algebras. Note that a B^*-algebra B is purely primitive if and only if it is topologically simple. For if $K \neq (0)$ is a closed ideal and

$K \neq B$ then, as B/K is semi-simple, there is a primitive ideal of B containing K. Here the ideals $W(P)$ are all closed. Thus (see Lemma 3.6), the ideals $W(P) \neq (0)$, $P \in \mathfrak{P}(B)$, are the minimal closed ideals of B.

COROLLARY 3.17. *For a B^*-algebra B, the elements of $\mathfrak{J}(B)$ are the annihilators of the minimal closed ideals of B. Also $\mathfrak{J}(B)$ is dense in $\mathfrak{P}(B)$ if and only if every closed ideal $K \neq (0)$ of B contains a minimal closed ideal of B.*

4. Discrete structure spaces. Let A be a semi-simple topological ring or algebra. The hypothesis that no non-zero continuous homomorphic image of A is a radical ring occurs in Theorem 4.1 and elsewhere below. This hypothesis is equivalent to the statement that every proper closed ideal of A is contained in a primitive ideal of A. It is satisfied by B^*-algebras and by any A for which no proper closed ideal contains every non-zero idempotent. Let B be a semi-simple Banach algebra which has pointwise bounded left approximate units (see [4, p. 50] for definitions). It follows from [4, Proposition 11.6] that B also satisfies the above hypothesis. In particular this holds for $L(G)$ where G is a locally compact group. In §2 above we gave examples of semi-simple Banach algebras which fail to satisfy the hypothesis.

THEOREM 4.1. *All the primitive ideals of A are closed in A and A is the direct topological sum of its minimal closed ideals if and only if A has discrete structure space, and no non-zero continuous homomorphic image of A is a radical ring.*

Proof. Suppose all $P \in \mathfrak{P}(A)$ are closed and A is the direct topological sum of its minimal closed ideals. Let $P \in \mathfrak{P}(A)$. There is a minimal closed ideal K such that $P \not\supset K$. Then $K \subset P^a$ and A has discrete structure space by Theorem 3.14. Consider a proper closed ideal J of A. If A/J is a radical ring, then no primitive ideal of A contains J. For any $P \in \mathfrak{P}(A)$, we have $J^a \subset P$. Therefore $J^a = (0)$ so that $J \cap K = K$ for every minimal closed ideal K. This makes $J = A$ which is impossible. Thus A/J is not a radical ring.

Suppose $\mathfrak{P}(A) = \mathfrak{J}(A)$ and the homomorphic image requirement. Inasmuch as $P^a \neq (0)$ for all $P \in \mathfrak{P}(A)$ we see that each $P \in \mathfrak{P}(A)$ is dual and therefore closed by Lemma 3.1. Let R be a proper closed ideal of A. Then there exists a primitive ideal Q containing R. Inasmuch as $Q^a \neq (0)$ we have also $R^a \neq (0)$. That A is the direct topological sum of its minimal closed ideals now follows from [11, Theorem 2.6]. □

Let $e \neq 0$ be an idempotent in A. We say that e is a *purely primitive idempotent* if eAe is a purely primitive ring.

LEMMA 4.2. *If e is a purely primitive idempotent, $e \neq 0$, then e lies in exactly one P^a, $P \in \mathfrak{P}(A)$. If $j \neq 0$ is an idempotent in P^a, where P is a closed point of $\mathfrak{P}(A)$, then j is a purely primitive idempotent.*

Proof. Consider a purely primitive idempotent $e \neq 0$. Let Q be any primitive ideal of A. If $Q \supset eAe$ then $e \in Q$. Otherwise $Q \cap eAe$ is a primitive ideal of eAe by [7, Proposition 4, p. 206]. In that case $Q \cap eAe = (0)$ so that $eQe = (0)$. Then by semi-simplicity $eQ = (0)$ or $e \in Q^a$. Hence either $e \in Q$ or $e \in Q^a$.

Now e cannot lie in every primitive ideal so there exists $P \in \mathfrak{P}(A)$ where $e \notin P$. Then $e \in P^a$. The uniqueness of P follows from Lemma 3.3.

Now let $j \neq 0$ be an idempotent in P^a where P is a closed point of $\mathfrak{P}(A)$. By [7, Proposition 4, p. 206], (0) is a closed point in $\mathfrak{P}(jAj)$. Hence (0) is the only primitive ideal of jAj. □

THEOREM 4.3. *Suppose that A has discrete structure space. Then the idempotents of A are the finite sums of mutually orthogonal purely primitive idempotents of A.*

Proof. Theorem 3.14 shows that A/N is a radical ring, so that every idempotent of A must lie in N. For such an idempotent $p \neq 0$ we can write $p = x_1 + \cdots + x_n$ where each $x_k \in P_k^a$, $P_k \in \mathfrak{P}(A)$, and $P_i^a \cap P_j^a = (0)$ if $i \neq j$ (see Lemma 3.3). It follows that each x_k is an idempotent and so is a purely primitive idempotent by Lemma 4.2. □

It follows readily that if A has discrete structure space and has an identity, then A is the direct sum of a finite number of simple rings with identity.

THEOREM 4.4. *Suppose that A has discrete structure space and is strongly semi-simple. Then every primitive ideal is a modular maximal ideal.*

Proof. Suppose P is a primitive ideal which is not a modular maximal ideal. For each modular maximal ideal M, $P^a \cap M^a = (0)$ by Lemma 3.3. Let K be the direct sum of the M^a as M runs over the set of modular maximal ideals. Now $M = M^{aa}$ for each M so that K^a is the intersection of the modular maximal ideals or $K^a = (0)$. But then $P^a = (0)$, which is absurd. □

5. $\mathfrak{I}(A)$ dense in $\mathfrak{P}(A)$. We treat semi-simple topological rings A for which $\mathfrak{I}(A)$ is dense in $\mathfrak{P}(A)$. We saw earlier, in Corollary 3.16, that if K is an ideal of A then $\mathfrak{I}(K)$ is dense in $\mathfrak{P}(K)$. We now examine the quotient ring A/K.

THEOREM 5.1. *Let K be an ideal in A, $K = K^{aa}$. Suppose $\mathfrak{I}(A)$ is dense in $\mathfrak{P}(A)$. Then A/K is semi-simple and $\mathfrak{I}(A/K)$ is dense in $\mathfrak{P}(A/K)$.*

Proof. Let $\mathscr{L} = \{P \in \mathfrak{I}(A) : P \supset K\}$ and $\mathscr{L}'$ be the complement of $\mathscr{L}$ in $\mathfrak{I}(A)$. If $Q \in \mathscr{L}'$ then $K \not\subset Q$, so that $K^a \subset Q$ and therefore $Q^a \subset K$ as K is dual. By Lemma 3.3 the algebraic sum $\sum Q^a$, for $Q \in \mathscr{L}'$, is a direct sum. Then

$$\sum Q^a \subset K \subset k(\mathscr{L})$$

where $k(\mathscr{L})$, the kernel of $\mathscr{L}$, is the intersection of the $P \in \mathscr{L}$. Now by Lemma 3.1, each $Q \in \mathscr{L}'$ is dual. Taking annihilators of the ideals in the formula displayed above we obtain $k(\mathscr{L}') \supset K^a \supset [k(\mathscr{L})]^a$.

Since $\mathfrak{I}(A)$ is dense and $\mathfrak{I}(A) = \mathscr{L} \cup \mathscr{L}'$ we see that $k(\mathscr{L})k(\mathscr{L}') = (0)$ or $[k(\mathscr{L})]^a \supset k(\mathscr{L}')$. Therefore $k(\mathscr{L}') = K^a = [k(\mathscr{L})]^a$. Then

$$K = K^{aa} = [k(\mathscr{L})]^{aa} \supset k(\mathscr{L}) \supset K.$$

Therefore $K = k(\mathscr{L})$ and K is the intersection of the primitive ideals of A containing K. This shows that A/K is semi-simple.

The mapping $\tau: P \to P+K$ is a homeomorphism of $\{P \in \mathfrak{P}(A): P \supset K\}$ onto $\mathfrak{P}(A/K)$ by [7, Proposition 1, p. 205] and $\mathscr{L}$ is in the domain of definition of τ. Inasmuch as $k(\mathscr{L}) = K$, we see $\tau(\mathscr{L})$ is dense in $\mathfrak{P}(A/K)$. Since the point P of $\mathscr{L}$ is an open set in $\mathfrak{P}(A)$ we see that $\tau(P)$ is an isolated point in $\mathfrak{P}(A/K)$. Therefore $\mathfrak{J}(A/K)$ is dense in $\mathfrak{P}(A/K)$. □

That the property $\mathfrak{J}(A)$ dense in $\mathfrak{P}(A)$ is preserved under the process of taking direct sums follows from the next result.

THEOREM 5.2. *Let R be a semi-simple topological ring which is equal to the topological sum of a given family $\{K_\lambda, \lambda \in \Lambda\}$ of its ideals. If $\mathfrak{J}(K_\lambda)$ is dense in $\mathfrak{P}(K_\lambda)$ for each $\lambda \in \Lambda$ then $\mathfrak{J}(R)$ is dense in $\mathfrak{P}(R)$.*

Proof. We consider the ideal N of R which is the smallest ideal of R containing every ideal of R which is a purely primitive ring (see Lemma 3.8). By Theorem 3.15 it is sufficient to show that $N^a = (0)$.

Let $x_0 \in N^a$ and let V be a non-zero ideal of K_λ which is a purely primitive ring. Now V need not be an ideal of A, but $K_\lambda V K_\lambda$ is a non-zero ideal of A and is a purely primitive ring. Then $x_0 K_\lambda V K_\lambda = (0)$. As K_λ is semi-simple, $x_0 K_\lambda V = (0)$. Therefore $x_0 K_\lambda$ annihilates every ideal of K_λ which is a purely primitive ring. By applying Theorem 3.15 to K_λ we have $x_0 K_\lambda = (0)$. As A is the topological sum of the ideals K_λ and A is semi-simple, $x_0 = 0$. □

THEOREM 5.3. *Suppose that* (1) *$\mathfrak{J}(A)$ is dense in $\mathfrak{P}(A)$,* (2) *each $P \in \mathfrak{J}(A)$ is a closed point of $\mathfrak{P}(A)$, and* (3) *no non-zero continuous homomorphic image of A is a radical ring. Then every closed ideal $K \neq (0)$ of A contains a minimal closed ideal of A.*

Proof. By (1) and Theorem 3.15, K contains the ideal $K \cap W(P) \neq (0)$ for some $P \in \mathfrak{J}(A)$. We show that P is a maximal-closed ideal of A. For if J is a closed ideal of A and $P \subset J \subset A$, $P \neq J$, $J \neq A$, by (3) there must be a primitive ideal of A containing J which is contrary to (2). Then [11, Lemma 2.3] shows that $L = (P^a)^2$ must be a minimal closed ideal of A. We claim that $K \cap L \neq (0)$ so that $K \supset L$. For otherwise we would have $K \subset L^a$. Now $L^a = P$ by Lemma 3.1. But then $K \cap W(P) \subset P \cap P^a$, which is impossible. □

A well-known example [9, p. 82] satisfies (1), (2) and (3) with $\mathfrak{J}(A) \neq \mathfrak{P}(A)$.

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