ON A HARDY AND LITTLEWOOD IMBEDDING THEOREM

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Introduction. For f in the class $H^p(\text{disc})$, $0<p<\infty$, Hardy and Littlewood [8, Theorem 31] showed that

$$\left\{\int_0^1 \left[(1-\rho)^{1/p-1/r}\left(\int_0^{2\pi} |f(\rho e^{i\theta})|^r\, d\theta\right)^{1/r}\right]^q (1-\rho)^{-1}\, d\rho\right\}^{1/q} \leqslant c\|f\|_{H^p},$$

where $0<p<r\leqslant\infty$, $p\leqslant q\leqslant\infty$. They used this inequality in their discussion of fractional integrals and convolutions of power series. The case $0<p<1=r=q$ has also been used by Duren, Romberg and Shields [2] to identify the bounded linear functionals on $H^p(\text{disc})$. Recently Flett [5] observed that the inequality gives easy proofs of a number of interesting results, and simplified its proof.

The purpose of this note is to present a simple proof of a general version of this inequality and to discuss some of its applications in various settings. We begin by introducing a maximal function. Let (X,μ) and (T,ν) be measure spaces with positive measures $d\mu$ and $d\nu$ respectively. Assume that to each $(x,t)\in X\times T$ we associate a μ-measurable set $B(x,t)\subseteq X$ so that the family $\mathcal{B}=\{B(x,t)\}$ verifies three conditions, namely

(i) $x\in B(x,t)$ for each $t\in T$;
(ii) if $y\in B(x,t)$, then $x\in B(y,t)$; and
(iii) $0\lneqq\mu(B(x,t))\lneqq\infty$.

For functions f defined on $X\times T$ and $x\in X$ we set

$$M_{\mathcal{B}}f(x)=\sup_{t\in T}\ \sup_{y\in B(x,t)} |f(y,t)|.$$

We begin by observing the following:

PROPOSITION. *Suppose $M_{\mathcal{B}}f\in L^p(X)$, $0<p<\infty$. Then*

$$|f(x,t)|\leqslant \min\left(M_{\mathcal{B}}f(x), \left(\frac{1}{\mu(B(x,t))}\int_{B(x,t)} M_{\mathcal{B}}f(y)^p\, d\mu\right)^{1/p}\right).$$

Proof. It is immediate. From (i) it follows that $|f(x,t)|\leqslant M_{\mathcal{B}}f(x)$, and from (ii) that $|f(x,t)|\leqslant \inf_{y\in B(x,t)} M_{\mathcal{B}}f(y)$, which in view of (iii) gives the desired conclusion at once. □

We can now prove our first imbedding result.

THEOREM 1. *Suppose that $M_{\mathcal{B}}f\in L^p(X)$, $0<p<\infty$, and that q and α verify $p\leqslant q<\infty$ and $-1+q/p\leqslant\alpha<q/p$. Furthermore, assume that the non-negative function $k(x,t)$ verifies*

Received April 22, 1983. Revision received October 31, 1983.
Research supported in part by grants from the NFR and NSF.
Michigan Math. J. 31 (1984).

$$\int_{\{t\in T:\,\mu(B(x,t))^{-1}>s\}} k(x,t)\,d\nu \leqslant s^{-\alpha}\psi(x), \quad s>0,$$

where $\psi\in L^r(X)$, $1/r=1+\alpha-q/p$. *Then there is a constant* $c=c(\alpha,p,q)$ *such that*

$$\left(\iint_{X\times T} |f(x,t)|^q k(x,t)\,d\mu\,d\nu\right)^{1/q} \leqslant c\|\psi\|_{L^r(X)}^{1/q}\,\|M_{\mathfrak{B}}f\|_{L^p(X)}.$$

Proof. With no loss of generality we may assume that $\|M_{\mathfrak{B}}f\|_{L^p(X)}=1$. The proof relies now on the Proposition. Indeed, by Fubini's theorem and the assumption on k we have

$$\begin{aligned}\int_T |f(x,t)|^q k(x,t)\,d\nu &= q\int_0^\infty s^{q-1}\int_{\{t:\,|f(x,t)|>s\}} k(x,t)\,d\nu\,ds\\ &\leqslant q\int_0^{M_{\mathfrak{B}}f(x)} s^{q-1}\int_{\{t:\,\mu(B(x,t))^{-1/p}>s\}} k(x,t)\,d\nu\,ds\\ &\leqslant q\psi(x)\int_0^{M_{\mathfrak{B}}f(x)} s^{q-1}s^{-\alpha p}\,ds=\frac{q}{q-\alpha p}\psi(x)M_{\mathfrak{B}}f(x)^{q-\alpha p}.\end{aligned}$$

If $\alpha=-1+q/p$, then $q-\alpha p=p$, $r=\infty$ and the assertion follows by integrating the above inequality with respect to x. If on the other hand $\alpha>-1+q/p$, then after integrating we apply Hölder's inequality with indices r and its conjugate, and again obtain the conclusion. This proves the theorem. □

In case the spaces $H^p_{\mathfrak{B}}=\{f\colon \|M_{\mathfrak{B}}f\|_{L^p(X)}<\infty\}$ can be interpolated by the real method, and $\mu(B(x,t))=v(t)$ and $k(x,t)=k(t)$ are independent of x, the conclusion of Theorem 1 can be considerably strengthened to give the inequality of Hardy and Littlewood. More precisely, we have:

THEOREM 2. *Suppose the spaces* $H^p_{\mathfrak{B}}(X)$ *interpolate by the real method, that* $M_{\mathfrak{B}}f\in L^p(X)$, $0<p<\infty$, *that* $0<p<r\leqslant\infty$, $p\leqslant q\leqslant\infty$, *and that*

$$\int_{\{t:\,v(t)^{-1}>s\}} k(t)\,d\nu\leqslant As^{-1}, \quad s>0.$$

Then there is a constant $c=c(A,p,q,r)$ *such that*

$$\left\{\int_T\left(v(t)^{(1/p-1/r)}\left(\int_X |f(x,t)|^r\,d\mu\right)^{1/r}\right)^q\frac{k(t)}{v(t)}\,d\nu\right\}^{1/q}\leqslant c\|M_{\mathfrak{B}}f\|_{L^p(X)}.$$

Proof. Let

$$T_r f(t)=\left(\frac{1}{v(t)}\int_X |f(x,t)|^r\,d\mu\right)^{1/r}.$$

By the Proposition we see that

$$|f(x,t)|\leqslant(\|M_{\mathfrak{B}}f\|_{L^p(X)}/v(t)^{1/p})^{1-p/r}M_{\mathfrak{B}}f(x)^{p/r},$$

and consequently

$$T_r f(t) \leqslant \|M_{\mathfrak{B}} f\|_{L^p(X)} / v(t)^{1/p}.$$

This proves the Theorem when $q=\infty$. In case $q<\infty$, from the above inequality it follows that

$$\{t: T_r f(t) > s\} \subseteq \{t: v(t)^{-1} > (s/\|M_{\mathfrak{B}} f\|_{L^p(X)})^p\}$$

and

$$\int_{\{t: T_r f(t) > s\}} k(t)\, d\nu \leqslant A(\|M_{\mathfrak{B}} f\|_{L^p(X)}/s)^p.$$

This last inequality asserts that T_r maps $H^p_{\mathfrak{B}}(X)$ weakly into $L^p(k(t)\,d\nu)$ for $0<p\leqslant r$. By interpolation, T_r is also of strong type from $H^p_{\mathfrak{B}}(X)$ into $L^p(k(t)\,d\nu)$ for $0<p<r$. Therefore

$$(1) \qquad \left(\int_T \left(\frac{1}{v(t)} \int_X |f(x,t)|^r\, d\mu\right)^{p/r} k(t)\, d\nu\right)^{1/p} \leqslant c\|M_{\mathfrak{B}} f\|_{L^p(X)}.$$

This is the strongest inequality in the scale $q \geqslant p$. Indeed, since for $q \geqslant p$

$$(2) \qquad T_r f(t) \leqslant (\|M_{\mathfrak{B}} f\|_{L^p(X)}/v(t)^{1/p})^{1-p/q} T_r f(t)^{p/q},$$

the desired conclusion readily follows by combining (1) and (2). □

A particular instance of Theorem 2 is the Hardy and Littlewood inequality stated in the Introduction. To see this let $X=\{e^{ix}: 0\leqslant x<2\pi\}$, $d\mu=$Lebesgue measure on X, $T=\{t: 0\leqslant t\leqslant 1\}$, $d\nu=(1-t)^{-1}dt$. Corresponding to $B(x,t)=\{e^{i\theta}: |x-\theta|<1-t\}$ we obtain that $M_{\mathfrak{B}} f(x)=\sup_{0<t<1} \sup_{e^{i\theta}\in B(x,t)} |f(te^{i\theta})|$ is basically the non-tangential maximal function associated to f in $H^p(\text{disc})$. An easy computation shows that $v(t)=k(t)=(1-t)$ satisfy the hypothesis of Theorem 2. Moreover, since $\|M_{\mathfrak{B}} f\|_{L^p(X)} \leqslant c\|f\|_{H^p(\text{disc})}$, as is well-known, and (by a result of C. Fefferman, Rivière and Sagher [3]) the $H^p(\text{disc})$ spaces interpolate by the real method, the conclusion of Theorem 2 obtains. This is precisely the Hardy and Littlewood inequality.

Further applications, in the Euclidean setting, correspond to the choice $X=\mathbf{R}^n$, $d\mu=$Lebesgue measure on $\mathbf{R}^n$, $T=\{t: 0<t<\infty\}$, $d\nu=dt/t$ and $B(x,t)=\{y\in\mathbf{R}^n: |x-y|<t\}$. Then Theorem 1 applies to $k(x,t)=t^{n(q/p-1)}$ and $v(t)=c_n t^n$, $\alpha=-1+q/p$. The corresponding statement includes Lemma 5 of Fefferman and Stein [4]. If instead we choose a parabolic metric ρ corresponding to a matrix P with trace $P=\gamma$ and put $B(x,t)=\{y\in\mathbf{R}^n: \rho(x-y)<t\}$, $v(t)=c_n t^\gamma$, $k(t)=t^{\gamma(q/p-1)}$, $\alpha=-1+q/p$, then the result reduces to Theorem 2.6 of Calderón and Torchinsky [1]. In both cases the function $M_{\mathfrak{B}} f$ is the usual non-tangential maximal function associated to f.

When $f(y,t)=f*\phi_t/y)$ is the extension to $\mathbf{R}^{n+1}_+$ of a tempered distribution f by means of convolutions with the dilations $\phi_t(y)=t^{-n}\phi(y/t)$ (or $\phi_t(y)=t^{-\gamma}\phi(t^{-P}y)$ in the parabolic case) of a Schwartz function with $\int\phi\neq 0$, then $H^p_{\mathfrak{B}}(\mathbf{R}^n)$ is the Hardy space of several real variables of Fefferman and Stein. In this case the spaces interpolate, again by [3], and consequently Theorem 2 applies with $k(t)=v(t)$ to give the n-dimensional Hardy and Littlewood inequality

$$\left\{\int_0^\infty \left(t^{n(1/p-1/r)}\left(\int_{\mathbf{R}^n} |f(x,t)|^r\,dx\right)^{1/r}\right)^q \frac{dt}{t}\right\}^{1/q} \leqslant c\|M_{\mathcal{B}}(f*\phi_t)\|_{L^p(\mathbf{R}^n)}$$

for $0<p<r<\infty$, $p\leqslant q$.

We list now some applications of Theorem 1. The first corresponds to Hardy spaces (of holomorphic functions) in smooth domains in $\mathbf{C}^n$. For simplicity we restrict ourselves to the unit ball

$$B=\{z=(z_1,\ldots,z_n): z_i\in\mathbf{C} \text{ and } |z|^2=z_1\bar{z}_1+\cdots+z_n\bar{z}_n<1\},$$

but our arguments apply to the more general domains considered by Stein [11]. Let $\partial B=\{\rho\in\mathbf{C}^n: |\rho|=1\}$ and

$$B(\rho_0,r)=\{\rho\in\partial B: |1-\langle\rho,\rho_0\rangle|^{1/2}<r\},\quad \rho_0\in\partial B.$$

Here as usual $\langle\rho,\rho_0\rangle=\rho_1(\overline{\rho_0})_1+\cdots+\rho_n(\overline{\rho_0})_n$. Let σ denote the rotation invariant, positive measure on ∂B for which $\sigma(\partial B)=1$. It is well-known that $\sigma(B(\rho_0,r))\approx r^{2n}$, $0<r<\sqrt{2}$.

For $\alpha>1$, consider the approach region

$$D_\alpha(\rho)=\{z\in\mathbf{C}^n: |1-\langle z,\rho\rangle|<\alpha(1-|z|^2)\},\quad \rho\in\partial B,$$

and let $M_\alpha F(\rho)=\sup_{z\in D_\alpha(\rho)}|F(z)|$. As above consider those F's with $M_\alpha F$ in $L^p(\partial B,d\sigma)$, $0<p<\infty$. As in the proof of the Proposition it is readily seen that if $z\in B$, $|z|=r$, and $\rho=z/|z|\in\partial B$, then

$$|F(z)|\leqslant\min(M_\alpha F(\rho),\ \|M_\alpha F\|_{L^p(\partial B)}/\sigma(B(\rho,\delta(z))^{1/p})$$

where $\delta(z)=c_\alpha(1-|z|^2)^{1/2}=c_\alpha(1-r^2)^{1/2}$ (we may take $c_\alpha=\sqrt{\alpha-1}$). Thus we obtain

$$\left(\int_{\partial B}\int_0^1 |F(r\rho)|^q(1-r^2)^{n(q/p-1)-1}r\,dr\,d\sigma\right)^{1/q}\leqslant c_\alpha\|M_\alpha F\|_{L^p(\partial B)}$$

for $0<p<q<\infty$.

If F is analytic in B, the estimate holds for the Hardy spaces $H^p(B)$. In analogy to the case of the upper half-space discussed above, $H^p(B)$ gives rise to the consideration of the Hardy spaces in hermitian hyperbolic space, once we recall the geometrical interpretation (due to Pyatecki–Shapiro) and identify ∂B with H_n, the Heisenberg group of order n. We will not pursue this matter here.

We pass to discuss an example in which the shape of the balls $B(x,t)$ change from those corresponding to $|x|$ into those corresponding to $\rho(x)$. In order to be able to include this case suppose that (i) and (ii) in the Introduction are replaced by: there is a measurable function $\emptyset: T\to T$ such that

(i′) $x\in B(x,\emptyset(t))$ for each t, and

(ii′) if $y\in B(x,t)$ then $x\in B(y,\emptyset(t))$.

Then, as in the proof of the Proposition, we have

$$|f(x,\emptyset(t))|\leqslant\min(M_{\mathcal{B}}f(x),\ \|M_{\mathcal{B}}f\|_{L^p(X)}/\mu(B(x,t))^{1/p}),$$

and if $k(x,t)$ satisfies the hypothesis of Theorem 1 we have that

$$\left(\iint_{X\times T}|f(x,\emptyset(t))|^q k(x,t)\,d\mu\,d\nu\right)^{1/q}\leqslant c\|\psi\|_{L^r(X)}^{1/q}\,\|M_{\mathcal{B}}f\|_{L^p(X)}.$$

The example we have in mind is this. Let $X=\mathbf{R}^n$, $d\mu=$ Lebesgue measure, $T=\{t: 0<t\leqslant 1\}$, $d\nu=dt/t$. Assume $\beta: \mathbf{R}^+\to[0,2]$ is a smooth, nondecreasing function which $=1$ in $[0,1]$, and $=2$ in $[2,\infty)$. Put $B(x,t)=\{y\in\mathbf{R}^n: |x-y|<t^{\beta(|x|)}\}$ and let $\emptyset(t)=t^{1/2}$, $0<t\leqslant 1$. The reader can verify that (i′) and (ii′) hold, and for $k(x,t)=t^{n\beta(|x|)(q/p-1)}$ the conclusion of Theorem 1 reads

$$\left(\int_0^1\int_{\mathbf{R}^n}|f(x,t)|^q t^{2n\beta(|x|)(q/p-1)}\,dx\,\frac{dt}{t}\right)^{1/q}\leqslant c\|M_{\mathcal{B}}f\|_{L^p(\mathbf{R}^n)}.$$

Notice that in this instance k is a function of x as well as of t.

It is also possible to iterate Theorem 1, and thus obtain results for families $\mathcal{B}$ of the form $\mathcal{B}_1\times\cdots\times\mathcal{B}_n$. For simplicity assume that $\mathcal{B}=\mathcal{B}_1\times\mathcal{B}_2$, where $\mathcal{B}_i=\{B_i(x_i,t_i)\}_{(x_i,t_i)\in X_i\times T_i}$, $i=1,2$, satisfies (i)–(iii) above. Now for $(x_1,x_2)\in X_1\times X_2$ let

$$M_{\mathcal{B}}f(x_1,x_2)=\sup_{t_i\in T_i}\ \sup_{y_i\in B_i(x_i,t_i)}|f(y_1,t_1,y_2,t_2)|.$$

We then have:

THEOREM 3. *Suppose that* $M_{\mathcal{B}}f\in L^p(X_1\times X_2)$, *and that for* $q\geqslant p>0$

$$\int_{\{t_i:\,\mu_i(B_i(x_i,t_i))^{-1}>s\}}k_i(x_i,t_i)\,d\nu_i\leqslant cs^{-(q/p-1)},\quad s>0,$$

for $i=1,2$. *Then there is a constant* c *such that*

$$\left\{\iint_{X_2\times T_2}\iint_{X_1\times T_1}|f(x_1,t_1,x_2,t_2)|^q k_1(x_1,t_1)\,k_2(x_2,t_2)\,d\mu_1\,d\nu_1\,d\mu_2\,d\nu_2\right\}^{1/q}$$
$$\leqslant c\|M_{\mathcal{B}}f\|_{L^p(X_1\times X_2)}.$$

Proof. We iterate Theorem 1. Indeed, with

$$M_{\mathcal{B}_1}f(x_1,x_2,t_2)=\sup_{t_1\in T_1}\ \sup_{y_1\in B_1(x_1,t_1)}|f(y_1,t_1,x_2,t_2)|,$$

from Theorem 1 it readily follows that

$$\iint_{X_1\times T_1}|f(x_1,t_1,x_2,t_2)|^q k_1(x_1,t_1)\,d\mu_1\,d\nu_1\leqslant c\left(\int_{X_1}M_{\mathcal{B}_1}f(x_1,x_2,t_2)^p\,d\mu_1\right)^{q/p}$$
$$\equiv g(x_2,t_2)^q,$$

say. To complete the proof it suffices to apply Theorem 1 to $g(x_2,t_2)$ once we observe that

$$\sup_{t_2\in T_2}\ \sup_{y_2\in B_2(x_2,t_2)}g(y_2,t_2)\leqslant\left(\int_{X_1}M_{\mathcal{B}}f(x_1,x_2)^p\,d\mu_1\right)^{1/p}. \qquad\square$$

Let us consider the particular instance of the bi-half space. Let $X_1=\mathbf{R}^{n_1}$, $d\mu_1=$ Lebesgue measure, $X_2=\mathbf{R}^{n_2}$, $d\mu_2=$ Lebesgue measure, $T_1=[0,\infty)$, $d\nu_1=dt_1/t_1$,

$T_2=[0,\infty)$, $d\nu_2=dt_2/t_2$. Put

$$B(x_1,t_1,x_2,t_2)=\{y\in \mathbf{R}^{n_1}\times \mathbf{R}^{n_2}: |x_1-y_1|<t_1,\ |x_2-y_2|<t_2\},$$

and $\mu_i(B_i(x_i,t_i))=c_{n_i}t^{n_i}$, $i=1,2$. Then Theorem 3 holds for $k_1(x_1,t_1)k_2(x_2,t_2)=t_1^{n_1(q/p-1)}t_2^{n_2(q/p-1)}$. Applications of this result depend on the maximal function characterization of the Hardy spaces in the poly half-space due to Gundy and Stein [7] and Merryfield [10]. Let ϕ be a Schwartz function in $\mathbf{R}^{n_1+n_2}$ with $\int\phi\neq 0$. For $x=(x_1,x_2)\in\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$ and $t=(t_1,t_2)\in\mathbf{R}^+\times\mathbf{R}^+$ put

$$\phi_t(x)=t_1^{-n_1}t_2^{-n_2}\phi(x_1/t_1,x_2/t_2).$$

We say that a tempered distribution f is in H^p if $M_{\mathfrak{B}}(f*\phi_t)\in L^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})$, $0<p<\infty$, and set

$$\|f\|_{H^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})}=\|M_{\mathfrak{B}}(f*\phi_t)\|_{L^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})}.$$

We then have:

LEMMA 1. *Suppose that $q=1$ in Theorem 3. Then if $k_i(x_i,t_i)=k_i(t_i)$, $i=1,2$, satisfies the hypothesis of the theorem, for $\xi=(\xi_1,\xi_2)\in\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$ and ϕ a Schwartz function we have*

$$\left(\int|\hat{\phi}(t_1\xi_1,t_2\xi_2)|k_1(t_1)k_2(t_2)\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)|\hat{f}(\xi_1,\xi_2)|\leqslant c\|f\|_{H^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})}.$$

Proof. Since

$$|(f*\phi_t)\hat{\ }(\xi_1,\xi_2)|\leqslant\int_{\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}}|(f*\phi_t)(x_1,x_2)|\,dx_1\,dx_2,$$

the result follows at once from Theorem 3. □

COROLLARY. *Let $f\in H^p$, $0<p<1$. Then*

$$|\hat{f}(\xi_1,\xi_2)|\leqslant c|\xi_1|^{n_1(1/p-1)}|\xi_2|^{n_2(1/p-1)}\|f\|_{H^p}.$$

Proof. Choose $\phi(x_1,x_2)=\eta(|x_1|)\eta(|x_2|)$, where η is a Schwartz function in $\mathbf{R}^1$ with $\hat{\eta}(\rho)=1$ for $\frac{1}{2}<|\rho|<2$. Apply Lemma 1 to $k_1(t_1)k_2(t_2)=t_1^{n_1(1/p-1)}t_2^{n_2(1/p-1)}$. □

Results of this nature, as well as duality, are discussed in the context of the polydisc by Frazier [6]. As for the duals we have the following representation. For a multi-index $\alpha=(\alpha_1,\alpha_2)$ let

$$\Lambda_\alpha(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})=\{\text{tempered distributions } f\text{: } |f*\phi_t(x)|\leqslant ct_1^{\alpha_1}t_2^{\alpha_2}\},$$

where $\phi(x_1,x_2)=\eta_1(x_1)\eta_2(x_2)$ with η_i Schwartz functions with $\operatorname{supp}\hat{\eta}_i\subset\{\frac{1}{4}<|\rho|<4\}$ and $\hat{\eta}_i(\rho)=1$ for $\frac{1}{2}<|\rho|<2$, $i=1,2$. In the spirit of Duren, Romberg and Shields [2], and Frazier [6] and Madych [9], we have:

DUALITY. Let $0<p<1$ and $\alpha_i=n_i(1/p-1)$, $i=1,2$. Then

$$(H^p)^*\approx\Lambda_\alpha(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}).$$

We would like to thank K. Merryfield for clarifying comments concerning this result, and to note another characterization of $(H^p)^*$ due to him (personal

communication). For simplicity we state his result when $n_1=n_2=1$. Assume $1/p\neq$ integer and let $1/p-2<N<1/p-1$. Then the linear functionals in H^p can be represented by functions $g(x_1,x_2)$ such that (i)

$$\left(\frac{\partial}{\partial x_1}\right)^k g(0,x_2)=\left(\frac{\partial}{\partial x_2}\right)^k g(x_1,0)=0, \quad 0\leqslant k\leqslant N$$

and (ii) if

$$h(x_1,x_2)=\left(\frac{\partial}{\partial x_1}\right)^N\left(\frac{\partial}{\partial x_2}\right)^N g(x_1,x_2),$$

and $H(x_1,x_2,h_1,h_2)=h(x_1+h_1,x_2+h_2)-h(x_1+h_1,x_2)-h(x_1,x_2+h_2)$, then

$$\sup_{x_1,x_2}|H(x_1,x_2,h_1,h_2)|\leqslant c|h_1 h_2|^{(1/p-1)-N}.$$

ACKNOWLEDGEMENT. We would like to thank the referee for valuable suggestions concerning this paper.

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