TOPOLOGICAL RESULTS IN COMBINATORICS

James R. Munkres

Let Δ be a finite simplicial complex with vertex set $V=\{x_0,\dots,x_n\}$. Let $X=|\Delta|$ denote its underlying topological space. Let K be a field. Associated with Δ and K is a certain ring $K[\Delta]$, described as follows: Let $S=K[x_0,\dots,x_n]$ be the polynomial ring over K with indeterminates $x_0,\dots,x_n$. Let I_Δ be the ideal of S generated by all monomials $x_{i_0}\cdots x_{i_r}$ such that $i_0<\cdots<i_r$ and the vertices $x_{i_0},\dots,x_{i_r}$ do *not* span a simplex of Δ. Define $K[\Delta]=S/I_\Delta$.

Now when $K[\Delta]$ is considered as a module over the polynomial ring S, it has a finite free resolution

$$0\longrightarrow M_j\longrightarrow\cdots\longrightarrow M_1\longrightarrow M_0\longrightarrow K[\Delta]\longrightarrow 0;$$

this is an exact sequence of S-modules, where each M_i is free. Furthermore, there is a unique such graded resolution which minimizes the rank of each M_i; such a resolution is called *minimal*. Let $b_i=b_i(K[\Delta])$ be the rank of the module M_i in this minimal free resolution. The largest integer i for which $b_i\neq 0$ is called the *homological dimension* (or the *depth*) of $K[\Delta]$, and denoted $h(\Delta)$.

It is known that $n-\dim\Delta\leqslant h(\Delta)\leqslant n+1$, where we recall that $n+1$ is the number of vertices of Δ. If $h(\Delta)=n-\dim\Delta$, then the ring $K[\Delta]$, and by extension the complex Δ, is said to be *Cohen-Macaulay.* If this condition is satisfied and if in addition $b_{h(\Delta)}=1$, then the ring and the complex are said to be *Gorenstein.* These conditions have been extensively studied by M. Hochster [1] and R. Stanley [4].

Hochster conjectured that the Cohen-Macaulay condition is independent of the simplicial structure of Δ, depending only on the underlying topological space. This conjecture has turned out to be correct, and in fact was almost proved by a student of Hochster's, G. Reisner. In his thesis [3], Reisner derived a condition involving the links of simplices in Δ, which he proved equivalent to the condition that Δ be Cohen-Macaulay. It requires only a short additional argument to show his condition equivalent to one which is topologically invariant. See Corollary 3.4 following.

A more general conjecture was suggested to the author by Stanley. The Cohen–Macaulay condition is just the condition that the number $n-h(\Delta)-\dim\Delta$ should vanish. Stanley conjectured that *this number itself* is a topological invariant of $|\Delta|$. Our purpose in this paper is to prove this conjecture. It suffices to prove $n-h(\Delta)$ a topological invariant, since it is well-known that $\dim\Delta$ is.

The proof relies on a theorem of Hochster's, stated in §1, which expresses the numbers b_i in terms of the cohomology (with coefficients in K) of Δ and its subcomplexes. In §2 we use Hochster's theorem to give a proof of our conjecture for complexes whose underlying spaces are the sphere S^N and the ball B^N. This case

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was already known; the proof is included because of its simplicity. All one needs besides Hochster's theorem is the Alexander duality theorem of topology.

We state our general theorem in §3 and prove it in §4 and §5. Section 6 gives an application to partially ordered sets.

1. Hochster's formula. Throughout this paper, all homology and cohomology groups have coefficients in K; the coefficients are suppressed from the notation. As usual, the reduced singular cohomology groups of Δ are denoted $\tilde{H}^i(\Delta)$; these are the derived groups of the cochain complex

$$\longleftarrow \mathrm{Hom}(C_i(\Delta),K) \longleftarrow \cdots \longleftarrow \mathrm{Hom}(C_0(\Delta),K) \xleftarrow{\bar{\epsilon}} K \longleftarrow 0.$$

(Here $C_i(\Delta)$ denotes the group of simplicial i-chains of Δ. The map $\bar{\epsilon}$ is the dual of the standard augmentation map, and is surjective.) We shall allow Δ to be empty; in this case the cochain complex has only one non-trivial term; namely, the group K in dimension -1. Thus $\tilde{H}^i(\varnothing)$ vanishes if $i \neq -1$, and it equals K if $i=-1$.

If W is a subset of the vertex set V of Δ, we let $\#W$ denote the cardinality of W, and we let Δ_W denote the full subcomplex of spanned by W. (That is, Δ_W consists of all simplices of Δ whose vertices belong to W.)

With this notation, we can state Hochster's formula as follows:

THEOREM 1.1 [1]. *Consider the direct sum of vector spaces* $B_i(\Delta)=\sum_{W\subset V}\tilde{H}^{\#W-i-1}(\Delta_W)$. *Then* $b_i(K[\Delta])=\dim B_i(\Delta)$.

Note that the summation extends over all subsets W of V, including $W=\varnothing$.

We now apply this theorem to reformulate Stanley's conjecture. Let $A_j(\Delta)=B_{n-j}(\Delta)$; let $\alpha(\Delta)$ be the smallest index j for which $A_j(\Delta)\neq 0$. Then by Hochster's formula, $\alpha(\Delta)=n-h(\Delta)$. The Cohen-Macaulay condition is the statement that $\alpha(\Delta)=\dim\Delta$; and the Gorenstein condition is the additional statement that $\dim A_{\alpha(\Delta)}=1$. We shall prove $\alpha(\Delta)$ is a topological invariant of $|\Delta|$; this is Stanley's conjecture.

We obtain an alternate expression for $A_j(\Delta)$ as follows: Given $W\subset V$, let $T=V-W$ be the complementary vertex set to W. Since $|\Delta_W|$ is a strong deformation retract of the space $|\Delta|-|\Delta_T|$, and since $\#T+\#W=n+1$, we can write A_j in the form

$$(*) \qquad A_j(\Delta)=\sum_{T\subset V}\tilde{H}^{j-\#T}(|\Delta|-|\Delta_T|).$$

This is the expression we shall use to calculate $\alpha(\Delta)$.

2. The sphere and the ball. In this section we prove the conjecture in two special cases, that of the N-sphere and the N-ball.

THEOREM 2.1. *If* Δ *is a triangulation of* S^N, *then* Δ *is Gorenstein. That is,* $\alpha(\Delta)=N$ *and* $\dim A_N=1$.

Proof. Let $X=S^N$. Then $\tilde{H}^i(X)=0$ for $i<N$ and $\tilde{H}^N(X)\cong K$. In view of $(*)$, it suffices to prove that for $j\leqslant N$ and $T\neq\varnothing$, $\tilde{H}^{j-\#T}(X-|\Delta_T|)=0$.

If A is a proper, non-empty subcomplex of a triangulation of $X=S^N$, there are two versions of the Alexander duality isomorphism, namely $\tilde{H}^i(A)\cong \tilde{H}_{N-i-1}(X-A)$ and $\tilde{H}^i(X-A)\cong\tilde{H}_{N-i-1}(A)$, both of which hold with arbitrary coefficients. Applying the second to our situation, we have

$$\tilde{H}^{j-\#T}(X-|\Delta_T|)\cong\tilde{H}_{(N-j)+(\#T-1)}(\Delta_T).$$

Now if $j\leqslant N$ and Δ_T is not a simplex, then

$$(N-j)+(\#T-1)\geqslant \#T-1>\dim\Delta_T,$$

so the right-hand group vanishes. If Δ_T *is* a (non-empty) simplex, then the right-hand group vanishes identically. Thus our statement is proved. □

THEOREM 2.2. *Let Δ be a triangulation of B^N. Then $\alpha(\Delta)=N$, so Δ is Cohen-Macaulay.*

Proof. Let $X=B^N$. We have $\tilde{H}^i(X)=0$ for all i. We need to show that $\tilde{H}^{j-\#T}(X-|\Delta_T|)$ vanishes for $j<N$ and all non-empty T, and is non-trivial for $j=N$ and at least one T.

If Δ is an N-simplex, one checks the statement easily; the only non-trivial group is $\tilde{H}^{-1}(\varnothing)$, which occurs when $j=N$ and T equals the entire vertex set of Δ.

So suppose Δ is not an N-simplex. Then the number of vertices of Δ is greater than $N+1$. If T is the entire vertex set of Δ, then $H^{j-\#T}$ vanishes for $j\leqslant N$, because $j-\#T<-1$. So let T be a proper non-empty subset of the vertex set of Δ. Let $A=|\Delta_T|$. We use the following isomorphism, which can be derived readily from the Alexander duality theorem:

$$\tilde{H}^i(X-A)\cong H_{N-i-1}(A, A\cap \mathcal{B}d\, X).$$

Let us set $i=j-\#T$ in this formula. If $j<N$, then the right-hand group vanishes for dimensional reasons, since $N-j>0$ and $\#T-1\geqslant\dim A$. If $j=N$, and if T is not the vertex set of a simplex of Δ, then it vanishes because $\#T-1>\dim A$. Finally, if $j=N$ and if T spans a k-simplex σ of Δ, then the right-hand group equals $H_k(\sigma, \sigma\cap\mathcal{B}d\, X)$. This group is non-trivial if and only if $\sigma\cap\mathcal{B}d\, X=\mathcal{B}d\,\sigma$. If we choose σ to be a simplex of *smallest* dimension among those that intersect $\operatorname{Int} B^N$, then $|\sigma|$ will be a proper subcomplex of Δ (since Δ is not a simplex), and will satisfy the condition $\sigma\cap\mathcal{B}d\, X=\mathcal{B}d\,\sigma$. Letting T be the vertex set of σ, we have $H^{N-\#T}(X-|\Delta|_T)\neq 0$, as desired. □

The argument just given shows in fact the following:

COROLLARY 2.3. *If Δ is a triangulation of B^N, then* $\dim A_N(\Delta)$ *equals the number of simplices σ of X for which $\sigma\cap\mathcal{B}d\, X=\mathcal{B}d\,\sigma$.* □

Thus Δ is Gorenstein if and only if there is exactly one such simplex σ. In this case, either Δ consists of an N-simplex and its faces, or one can prove by standard topological arguments that Δ equals the join $\sigma*\Sigma$, where σ is a k-simplex and Σ is a homology $N-k-1$ manifold, whose (integral) homology is that of S^{N-k-1}.

3. The main theorem. Here we state the major result of this paper, and derive a corollary.

THEOREM 3.1. *Let Δ be a finite simplicial complex; let $X=|\Delta|$. Let* $N=\dim\Delta$. *Let $\beta(\Delta)$ be the smallest integer j for which at least one of the following groups is non-trivial:*

$$\tilde{H}^j(X),\{H^j(X,X-p)\mid p\in X\}.$$

Then $\alpha(\Delta)=\beta(\Delta)$, so $\alpha(\Delta)$ depends only on the topological space X.

Note that by its definition, $-1\leqslant\beta(\Delta)\leqslant N$. The left-hand inequality holds because all cohomology vanishes in dimensions less than -1. On the other hand, if p is the barycenter of an N-simplex of Δ, then $H^N(X,X-p)\cong K$, whence $\beta(\Delta)\leqslant N$.

Here is an application of this theorem to manifolds:

COROLLARY 3.2. *Let $X=|\Delta|$ be an N-manifold, with or without boundary. Then X is Cohen–Macaulay if and only if $\tilde{H}^i(X)=0$ for $i<N$.*

Proof. The local homology groups of X vanish (even with integer coefficients) in all dimensions less than N. Therefore $\beta(\Delta)=N$ (which is the Cohen–Macaulay condition) if and only if $\tilde{H}^i(X)=0$ for $i<N$. □

Compare this result with that for the sphere given in §2. This corollary has broader applications, of course; it applies for instance to any real projective space P^m if the coefficient field K has characteristic different from 2.

To derive the next corollary, we need a lemma:

LEMMA 3.3. *Let Δ be a complex; let σ be a nonempty simplex of Δ; let p be an interior point of σ. Then*

$$H^j(X,X-p)\cong\tilde{H}^{j-\dim\sigma-1}(\mathcal{L}k\,\sigma).$$

Proof. Recall that $\overline{\mathcal{S}t}\,\sigma$ is the union of all simplices having σ as a face, and $\mathcal{L}k\,\sigma$ is the union of those simplices of $\overline{\mathcal{S}t}\,\sigma$ that are disjoint from σ. The symbol "$*$" denotes the join operation.

If $\mathcal{L}k\,\sigma\neq\varnothing$, we have the following isomorphisms:

$$
\begin{aligned}
H^j(X,X-p) &\cong H^j(\overline{\mathcal{S}t}\,\sigma,(\overline{\mathcal{S}t}\,\sigma)-p) && \text{by excision;}\\
&\cong H^j(\overline{\mathcal{S}t}\,\sigma,(\mathcal{B}d\,\sigma)*(\mathcal{L}k\,\sigma)) && \text{because } (\mathcal{B}d\,\sigma)*(\mathcal{L}k\,\sigma) \text{ is a deformation retract of } \overline{\mathcal{S}t}\,\sigma-p;\\
&\cong \tilde{H}^{j-1}((\mathcal{B}d\,\sigma)*(\mathcal{L}k\,\sigma)) && \text{by the long exact cohomology sequence;}\\
&\cong \tilde{H}^{j-\dim\sigma-1}(\mathcal{L}k\,\sigma) && \text{by the suspension isomorphism.}
\end{aligned}
$$

If $\mathcal{L}k\,\sigma=\varnothing$, a similar argument gives the isomorphisms

$$\begin{aligned} H^j(X, X-p) &\cong H^j(\sigma, \sigma - p) \cong H^j(\sigma, \mathcal{B}d\ \sigma) \\ &\cong \tilde{H}^{j-1}(\mathcal{B}d\ \sigma) \cong \begin{cases} K & \text{if } j = \dim \sigma \\ 0 & \text{if } j \neq \dim \sigma \end{cases} \\ &\cong \tilde{H}^{j-\dim\sigma-1}(\varnothing). \end{aligned}$$ □

COROLLARY 3.4. *Let Δ be a complex; let $X=|\Delta|$; let $N=\dim \Delta$. Then the following conditions on Δ are equivalent:*

(i) *Δ is Cohen–Macaulay.*
(ii) *$\tilde{H}^j(X)=0=H^j(X,X-p)$ for $j<N$ and all $p\in X$.*
(iii) *(Reisner's condition). $\tilde{H}^k(\mathcal{L}k\ \sigma)=0$ for $k<\dim(\mathcal{L}k\ \sigma)$ and all $\sigma\in\Delta$, including $\sigma=\varnothing$. (We make the convention that $\mathcal{L}k\ \sigma=\Delta$ if $\sigma=\varnothing$.)*

Proof. Step 1. A complex Δ is said to be *pure* if it consists entirely of N-simplices and their faces. We show that each of (ii) and (iii) implies that Δ is pure.

Suppose (ii) holds. If σ is a k-simplex of Δ that is a proper face of no simplex of Δ, then letting p be an interior point of σ, one has $H^k(X,X-p)\cong K$. It follows that $k=N$.

Similarly, (iii) implies that Δ is pure: Suppose that Δ is a complex for which (iii) holds, which is not pure. Then $N>0$, since any 0-dimensional complex is pure. Let L be the collection of all N-simplices of Δ and their faces; let $Y=|\Delta|-|L|$, and let J be the subcomplex of Δ whose underlying space is $\bar{Y}$. Then $\dim J<N$. Now $|J|\cap|L|$ is non-empty, since condition (iii) applied to the case $\sigma=\varnothing$ tells us that $|\Delta|$ is connected. Let t be a simplex of highest dimension in $J\cap L$. Then t is a face of an N-simplex σ of L, and it is a face of a simplex s whose interior is contained in Y. Then

$$\dim t<\dim s<N=\dim \sigma.$$

The link of t in Δ equals the union of $\mathcal{L}k(t,J)$ and $\mathcal{L}k(t,L)$. Both these links are non-empty; furthermore, they are disjoint because t is a simplex of *highest* dimension in $J\cap L$. Thus $\mathcal{L}k(t,\Delta)$ is not connected. Hence $\tilde{H}^0(\mathcal{L}k(t,\Delta))\neq 0$. This contradicts condition (iii), for $\mathcal{L}k(t,\Delta)$ has dimension at least 1, since t is a face of σ of dimension at most $N-2$.

Step 2. We show that (ii) and (iii) are equivalent. If either holds, then Δ is pure, so that

$$\dim(\mathcal{L}k\ \sigma)+\dim \sigma=N-1.$$

Then the condition that $\tilde{H}^k(\mathcal{L}k\ \sigma)=0$ for σ non-empty and $k<\dim(\mathcal{L}k\ \sigma)$ the same as the condition $\tilde{H}^{j-\dim\sigma-1}(\mathcal{L}k\ \sigma)=0$ for σ non-empty and $j<N$. The latter is equivalent to the condition $H^j(X,X-p)=0$ for $j<N$ and all p, by the preceding lemma. Similarly, the condition that $\tilde{H}^k(\mathcal{L}k\ \sigma)=0$ for $k<\dim(\mathcal{L}k\ \sigma)$ when $\sigma=\varnothing$ is, by our convention, exactly the condition that $\tilde{H}^k(X)=0$ for $k<N$.

Step 3. The corollary follows, since Theorem 3.1 implies that (i) and (ii) are equivalent. □

This corollary also follows readily from Reisner's theorem. Instead of using Theorem 3.1 in Step 3, one can use Reisner's theorem, which states that (i) and (iii) are equivalent.

4. The cohomology of $(X, X-A)$. In order to prove Theorem 3.1, we need to study the cohomology of the space $X-A$, where $A=|\Delta_T|$ and Δ_T is a full subcomplex of X. As a first step in this direction, we study the relative cohomology groups of the pair $(X, X-A)$, under suitable hypotheses on the local cohomology groups of X. We shall in this section and the next abuse notation and make no distinction between a complex and its underlying space, when no confusion will result.

DEFINITION. Let X' denote the first barycentric subdivision of X; and let $X^*(A)$ denote the subcomplex of X' consisting of all (closed) simplices of X' that do not intersect A.

Now $X^*(A)$ is a full subcomplex of X'; therefore it is a strong deformation retract of its open star in X', which set equals $X-A$. Therefore in computing cohomology, we may replace $X-A$ by $X^*(A)$ whenever convenient; this we shall do freely in what follows.

We now introduce the notion of the *dual block complex* associated to X. The reader may recognize this complex as the device used when X is a manifold to give the classical proof of Poincaré duality.

DEFINITION. Let X be a complex of dimension N. Order the vertices of each simplex of the first barycentric subdivision X' in order of *increasing* dimension of the simplices of X of which they are the barycenters. Then, given a simplex σ of X, let $D(\sigma)$ denote the collection of all simplices of X' whose *initial* vertex is $\hat{\sigma}$, the barycenter of σ, along with all faces of such simplices. We call $D(\sigma)$ the *block dual to* σ; and we call the collection $\mathcal{X}$ of all the blocks $D(\sigma)$ the *dual block complex* of X.

Given σ, let $\dot{D}(\sigma)$ denote the collection of those simplices of $D(\sigma)$ that do not have $\hat{\sigma}$ as a vertex. Then $D(\sigma)=\sigma * \dot{D}(\sigma)$, where $*$ denotes "join". If σ has dimension m, we will call $D(\sigma)$ a "dual $(N-m)$-block", even though as a complex it may have dimension less than $N-m$ (if Δ is not pure). We call $N-m$ the *formal dimension* of the block $D(\sigma)$.

Note that $\dot{D}(\sigma)$ is the union of all blocks $D(\Sigma)$ for which Σ has σ as a face. Each of these blocks has formal dimension less than that of $D(\sigma)$; such a block will be called a (proper) *face* of $D(\sigma)$. Note also that the intersection $D(\sigma_1)\cap D(\sigma_2)$ of two blocks is a face of each of them: If the vertices of σ_1 and σ_2 together span the simplex Σ of X, then this intersection equals $D(\Sigma)$; if they do not span a simplex of X, the intersection is empty.

We define a *block subcomplex* $\mathcal{A}$ of $\mathcal{X}$ to be a collection of blocks of $\mathcal{X}$ such that for each block belonging to $\mathcal{A}$, all of its faces belong to $\mathcal{A}$ as well. The notation $|\mathcal{A}|$ denotes the space which is the union of the blocks of $\mathcal{A}$.

For example, consider the collection $\mathcal{X}^m$ of all blocks of $\mathcal{X}$ having formal dimension at most m. Then $\mathcal{X}^m$ is a block subcomplex of $\mathcal{X}$. We call it the *m-skeleton* of $\mathcal{X}$.

For another example, let A be a subcomplex of X, and let $\mathcal{A}$ be the collection of all blocks $D(\sigma)$ for which σ is *not* in A. Then $\mathcal{A}$ is a block subcomplex of $\mathcal{X}$. Its underlying topological space is just the space $|X^*(A)|$ defined earlier.

We need the following lemma relating the local cohomology groups of X with the cohomology of the dual blocks:

LEMMA 4.1. *Let σ be a k-simplex of X; let $D(\sigma)$ be its dual block. If p is an interior point of σ, then for all j,*

$$H^j(X,X-p)\cong H^{j-k}(D(\sigma),\dot{D}(\sigma)).$$

Proof. Note first that

$$\begin{aligned}\overline{\mathcal{S}t}(\hat{\sigma},X')&=\sigma*\dot{D}(\sigma)=p*(\mathcal{B}d\,\sigma)*\dot{D}(\sigma)\\&=p*\mathcal{L}k(\hat{\sigma},X').\end{aligned}$$

It follows that $\mathcal{L}k(\hat{\sigma},X')$ is a strong deformation retract of $\overline{\mathcal{S}t}(\hat{\sigma},X')-p$. We then consider the isomorphisms:

$$\begin{aligned}H^j(X,X-p)&\cong H^j(\overline{\mathcal{S}t}(\hat{\sigma},X'),\overline{\mathcal{S}t}(\hat{\sigma},X')-p)\\&\cong H^j(\overline{\mathcal{S}t}(\hat{\sigma},X'),\mathcal{L}k(\hat{\sigma},X'))\cong\tilde{H}^{j-1}(\mathcal{L}k(\hat{\sigma},X'))\\&\cong\tilde{H}^{j-1}((\mathcal{B}d\,\sigma)*\dot{D}(\sigma))\cong\tilde{H}^{j-k-1}(\dot{D}(\sigma))\\&\cong H^{j-k}(D(\sigma),\dot{D}(\sigma)).\end{aligned}$$

If σ is a *principal* simplex of X (that is, if σ is a proper face of no simplex of X), then $D(\sigma)$ is the single point $\hat{\sigma}$ and $\dot{D}(\sigma)$ is empty. Nevertheless, the isomorphism stated in our lemma still holds; both groups are isomorphic to K if $j=k$, and vanish otherwise. □

We use the following fact, whose proof is elementary:

LEMMA 4.2. *Let $\mathcal{X}$ be the dual block complex of X; let $\mathcal{A}$ be a block subcomplex of $\mathcal{X}$. The group $H^i(|\mathcal{X}^m\cup\mathcal{A}|,|\mathcal{X}^{m-1}\cup\mathcal{A}|)$ is isomorphic to the direct sum $\sum H^i(D(\sigma),D(\sigma))$, where the sum extends over all blocks $D(\sigma)$ of formal dimension m that do not lie in $\mathcal{A}$.*

We now compute the cohomology of $(X,X^*(A))$, under suitable hypotheses on the local cohomology of X.

LEMMA 4.3. *Let X be a simplicial complex of dimension N; let A be a proper non-empty subcomplex. Let γ be an integer such that $0\leqslant\gamma\leqslant N$; assume that the local cohomology groups $H^i(X,X-p)$ vanish for $i<\gamma$ and all $p\in|A|$. Then if $k=\dim A$,*

(i) *$H^j(X,X^*(A))=0$ for $j<\gamma-k$.*

(ii) *$H^{\gamma-k}(X,X^*(A))$ is isomorphic to the kernel of the map*

$$H^{\gamma-k+1}(Y_{N-k+1},Y_{N-k})\xleftarrow{\ \delta^*\ }H^{\gamma-k}(Y_{N-k},Y_{N-k-1}).$$

Here $\mathcal{X}$ denotes the dual block complex of X, $\mathcal{A}$ denotes the subcomplex of $\mathcal{X}$ whose underlying space is $|X^*(A)|$, and $Y_i = |\mathcal{X}^i \cup \mathcal{A}|$.

Proof. Let $\lambda = N - \gamma$.

Step 1. We prove that $H^i(Y_{j+\lambda+1}, Y_{j+\lambda}) = 0$ for $i \leqslant j$. This group is isomorphic to the direct sum $\sum H^i(D(\sigma), \dot{D}(\sigma))$, where $D(\sigma)$ ranges over all blocks of formal dimension $j+\lambda+1$ that are not in $\mathcal{A}$; i.e., as σ ranges over all simplices of A of dimension $N-(j+\lambda+1) = \gamma-(j+1)$. By Lemma 4.1, $H^i(D(\sigma), \dot{D}(\sigma)) \cong H^{i+\dim \sigma}(X, X-p)$, where p is an interior point of σ. Since $i + \dim \sigma = \gamma - (j-i) - 1$, this group vanishes for $i \leqslant j$.

Step 2. We show that $H^i(X, Y_{j+\lambda}) = 0$ for $i \leqslant j$. For this purpose, we begin by considering the long exact sequence of the triple $(Y_{j+\lambda+2}, Y_{j+\lambda+1}, Y_{j+\lambda})$. Applying Step 1 to appropriate terms of this sequence, we see that $H^i(Y_{j+\lambda+2}, Y_{j+\lambda}) = 0$ for $i \leqslant j$. One continues similarly to show that $H^i(Y_{j+\lambda+m}, Y_{j+\lambda}) = 0$ for $i \leqslant j$ and all $m > 0$. If m is sufficiently large, $Y_{j+\lambda+m} = X$, and our result follows.

Step 3. We prove part (i) of the lemma. Every simplex σ of A has dimension at most k, so every block $D(\sigma)$ of $\mathcal{X}$ not in $\mathcal{A}$ has formal dimension at least $N-k$. Therefore $\mathcal{X}^i \subset \mathcal{A}$ if $i < N-k$, so that $Y_i = |\mathcal{A}| = X^*(A)$.

In particular, if $j < \gamma - k$, then $j + \lambda < N - k$ and $Y_{j+\lambda} = |X^*(A)|$, so that

$$H^j(X, X^*(A)) = H^j(X, Y_{j+\lambda}) = 0.$$

Step 4. We construct, using the long exact sequence of a triple, the following commutative diagram of exact sequences, from which part (ii) of the theorem follows easily:

$$\begin{array}{ccccccc}
0 & & & & & & \\
\downarrow & & & & & & \\
H^{\gamma-k+1}(X, Y_{N-k}) & \xleftarrow{\delta^*} & H^{\gamma-k}(Y_{N-k}, |\mathcal{A}|) & \longleftarrow & H^{\gamma-k}(X, |\mathcal{A}|) & \longleftarrow & 0 \\
\downarrow & & \| & & & & \\
H^{\gamma-k+1}(Y_{N-k+1}, Y_{N-k}) & \xleftarrow{\delta^*} & H^{\gamma-k}(Y_{N-k}, Y_{N-k-1}). & & & &
\end{array}$$

The zero at the top follows from the equation $H^{\gamma-k+1}(X, Y_{N-k+1}) = 0$; the zero at the right, from the equation $H^{\gamma-k}(X, Y_{N-k}) = 0$; both are a consequence of Step 2. The equality in the middle follows from Step 3. □

As a particular application of part (ii) of the preceding lemma, we prove the following:

LEMMA 4.4. *Let X be a complex that does not consist of a simplex and its faces. Assume that the local cohomology groups $H^i(X, X-p)$ vanish for $i < \gamma$ and all $p \in X$.*

If $H^\gamma(X, X-p)$ is non-trivial for at least one $p \in X$, then there is a non-empty subcomplex A of X, consisting of a k-simplex σ and its faces, such that $H^{\gamma-k}(X, X^*(A))$ is non-trivial. Conversely, if there is such a subcomplex A of X, then $H^\gamma(X, X-p)$ is non-trivial for each p interior to σ.

Proof. Let A be a non-empty subcomplex of X consisting of a k-simplex and its faces. Let N denote the dimension of X. By hypothesis, A is not all of X, so the preceding lemma applies; the group $H^{\gamma-k}(X,X^*(A))$ is isomorphic to the kernel of the map

$$H^{\gamma-k+1}(Y_{N-k+1},Y_{N-k}) \xleftarrow{\delta^*} H^{\gamma-k}(Y_{N-k},Y_{N-k-1}). \tag{4.1}$$

Consider the collection of those blocks of $\mathfrak{X}$ that do *not* belong to $\mathfrak{A}$. This collection consists of the single block $D(\sigma)$ dual to σ (of formal dimension $N-k$), along with blocks $D(s)$ (of larger formal dimension) that are dual to proper faces s of σ. Therefore by Lemma 4.2, the right-hand group of formula (4.1) is isomorphic to the group $H^{\gamma-k}(D(\sigma),\dot{D}(\sigma))$.

Similarly, the left-hand group of (4.1) is isomorphic to the direct sum $\sum H^{\gamma-k+1}(D(s),\dot{D}(s))$, where the sum extends over all $(k-1)$-faces s of σ. [If σ has dimension 0, there are no such faces and the group in question vanishes.] Thus $H^{\gamma-k}(X,X^*(A))$ is isomorphic to the kernel of a certain homomorphism

$$\sum_{s\in\mathfrak{Bd}\,\sigma} H^{\gamma-k+1}(D(s),\dot{D}(s)) \longleftarrow H^{\gamma-k}(D(\sigma),\dot{D}(\sigma)).$$

Applying Lemma 4.1, we can rewrite this as a homomorphism

$$\sum_{s\in\mathfrak{Bd}\,\sigma} H^{\gamma}(X,X-p_s) \xleftarrow{\phi} H^{\gamma}(X,X-p_\sigma), \tag{4.2}$$

where p_τ is interior to the simplex τ, for each τ. Therefore $H^{\gamma-k}(X,X^*(A))$ is non-trivial if and only if $\ker\phi\neq 0$.

In order that $\ker\phi\neq 0$, it is necessary that the group $H^{\gamma}(X,X-p_\sigma)$ be non-trivial. This proves the converse of the theorem.

To prove the theorem itself, suppose that $H^{\gamma}(X,X-p)$ is non-trivial for at least one $p\in X$. Let σ be a simplex *of lowest dimension* such that $H^{\gamma}(X,X-p)$ is non-trivial at an interior point p of σ. In that case, the right-hand group in formula (4.2) is non-trivial, while the left-hand group is trivial because s has smaller dimension than σ. If we take A to be the complex of dimension k consisting of σ and its faces, we have $H^{\gamma-k}(X,X^*(A))\neq 0$, as desired. □

5. Proof of the main theorem. In this section we prove Theorem 3.1. In fact, we prove the following slightly more precise result:

THEOREM 5.1. *Let Δ be a complex of dimension N; let $X=|\Delta|$. Assume the notation of Theorem 3.1. Then*

(i) $A_j(\Delta)=0$ *for* $j<\beta(\Delta)$.
(ii) $A_j(\Delta)\neq 0$ *for* $j=\beta(\Delta)$.
(iii) *In order that* $\tilde{H}^{\beta(\Delta)-\#T}(X-|\Delta_T|)$ *be non-trivial, for non-empty T, it is necessary (but not sufficient) that:*
   (a) *T span a simplex σ of Δ, and*
   (b) $H^{\beta(\Delta)}(X,X-p)$ *be non-trivial for at least one (and hence every) p interior to σ.*

Proof. Step 1. We prove the theorem first in the case where Δ consists of an N-simplex and its faces. In this case, $\beta(\Delta)=N$ because $|\Delta|$ is an N-cell. We have already noted (in proving Theorem 2.2) that the only case in which $\tilde{H}^{j-\#T}(X-|\Delta_T|)$ is non-trivial is when $j=N$ and T is the vertex set of Δ. Thus (i), (ii), and (iii) hold in this case.

Henceforth we assume that Δ does not consist of an N-simplex and its faces.

Step 2. Let T be a set of vertices of X; let $A=|\Delta_T|$. We show that the equation $\tilde{H}^{j-\#T}(X-A)=0$ holds for $j<\beta(\Delta)$ in general, and for $j\leqslant\beta(\Delta)$ if A is non-empty and does not equal a simplex of X. Parts (i) and (iii)(a) of the theorem follow immediately.

If T is empty, then $\tilde{H}^{j-\#T}(X-A)$ equals $\tilde{H}^{j}(X)$, which vanishes for $j<\beta(\Delta)$ by hypothesis. If T is the entire vertex set of Δ, then $\#T>N+1$ (since Δ does not consist of a single simplex plus its faces). The condition $j\leqslant\beta(\Delta)$ implies that $j\leqslant N$, whence $\tilde{H}^{j-\#T}(X-A)$ vanishes because $j-\#T<-1$.

Finally, let T be a non-empty proper subset of the vertex set of Δ. Consider the exact sequence

$$\cdots\longleftarrow H^{i+1}(X,X-A)\longleftarrow \tilde{H}^{i}(X-A)\longleftarrow \tilde{H}^{i}(X)\longleftarrow\cdots. \tag{5.1}$$

If $i<\beta(\Delta)$, the right-hand group vanishes, by hypothesis. If $i+1<\beta(\Delta)-\dim A$, part (i) of Lemma 4.3 tells us that the left-hand group vanishes. (Here we substitute $X^*(A)$ for $X-A$, as we are allowed to do.) Therefore the middle group of formula (5.1) vanishes for $i+1<\beta(\Delta)-\dim A$. That is, the equation

$$\tilde{H}^{j-\#T}(X-A)=0 \tag{5.2}$$

holds for $j<\beta(\Delta)+[\#T-\dim A-1]$. Now in general, $\#T\geqslant(\dim A)+1$; therefore (5.2) holds in general if $j<\beta(\Delta)$. If A does not consist of a simplex plus its faces, then $\#T>(\dim A)+1$; formula (5.2) holds for $j\leqslant\beta(\Delta)$ in this case.

Step 3. Now we prove part (ii) of the theorem; that is, we show that $\tilde{H}^{\beta(\Delta)-\#T}(X-A)$ is non-trivial for some vertex set T, where $A=|\Delta_T|$. If $\tilde{H}^{\beta(\Delta)}(X)\neq 0$, we can take T to be the empty set and we are finished. So we assume $\tilde{H}^{\beta(\Delta)}(X)=0$. In this case, $\beta(\Delta)$ must equal the smallest integer j for which at least one of the groups $H^{j}(X,X-p)$, for $p\in X$, is non-trivial. Lemma 4.4 then applies, with $\gamma=\beta(\Delta)$. We conclude that there is a non-empty set T of vertices, such that the subcomplex A spanned by T consists of a k-simplex and its faces, and such that $H^{\gamma-k}(X,X-A)$ is non-trivial (substituting $X-A$ for $X^*(A)$ as usual). Consider the exact sequence

$$\tilde{H}^{\gamma-k}(X)\longleftarrow H^{\gamma-k}(X,X-A)\longleftarrow \tilde{H}^{\gamma-k-1}(X-A)\longleftarrow \tilde{H}^{\gamma-k-1}(X). \tag{5.3}$$

The end groups vanish because $\gamma=\beta(\Delta)$ and we have assumed $\tilde{H}^{i}(X)=0$ for $i\leqslant\beta(\Delta)$. We conclude that the group $H^{\gamma-k-1}(X-A)=H^{\beta(\Delta)-\#T}(X-A)$ is non-trivial.

Step 4. Finally, we prove part (iii)(b) of the theorem. Let T be any non-empty vertex set that spans a subcomplex A consisting of a k-simplex σ and its faces. Consider the exact sequence (5.3), with $\gamma=\beta(\Delta)$. The right-hand group vanishes by the hypotheses of the theorem. Therefore, the assumption that the second group from the right is non-trivial implies that the third group from the right is non-trivial. By Lemma 4.4, this in turn implies that $H^{\beta(\Delta)}(X,X-p)$ is non-trivial for each p interior to σ. □

REMARK. The astute reader, who has noted the use of Alexander duality in §2, and the use of techniques similar to those used in the proof of Poincaré duality in §4, may wonder if there is some more general version of duality underlying our results. There is. It can be stated as follows:

Let Δ be a finite simplicial complex of dimension N; let $X=|\Delta|$. Let G be a fixed coefficient group for homology and cohomology. Assume $H_i(X,X-p)=0$ for $i<\gamma$ and all p. Let A be a non-empty subcomplex of Δ. Then for $i\geqslant \dim A$,

$$H_{\gamma-i}(X,X-A)\cong \check{H}^i(A;\mathcal{L}_\gamma).$$

Here $\mathcal{L}_\gamma$ is the presheaf of local homology groups of X in dimension γ, given by $\mathcal{L}_\gamma(U)=H_\gamma(X,X-U)$, and $\check{H}$ denotes Čech cohomology.

Note that this theorem resembles the Lefschetz duality theorem for manifolds, except for the restricted range of dimensions. If it happens that $\gamma=N$, then this restriction can be removed; the isomorphism holds for all i. In the case where X is a manifold, one obtains the usual Lefschetz duality theorem; the coefficient sheaf is simple if X is orientable, and twisted otherwise.

One can prove this duality theorem directly, following the pattern of §4. Alternatively, one can derive it as a consequence of the powerful "Zeeman spectral sequence" using the interpretation due to C. McCrory. See [2: 3.3] for details.

Note that Theorem 5.1 is an immediate consequence of this result. Letting $G=K$, one has

$$H_{\gamma-i}(X,X^*(A))\cong \check{H}^i(A;\mathcal{L}_\gamma).$$

Now if $i>\dim A$, the right-hand group is trivial for dimensional reasons. Thus (i) of Lemma 4.3 holds; cohomology is interchangeable with homology because one is using field coefficients.

On the other hand, suppose A is a simplex σ of lowest dimension k such that $H_\gamma(X,X-p)\neq 0$ at some interior point p of σ. Then the coefficient group attached to the simplex σ is non-trivial, while the coefficient groups attached to the proper faces of σ are trivial. It follows that there is a non-trivial cocycle of A in dimension k; it does not cobound because all the $k-1$ cochain groups of A (with coefficients in $\mathcal{L}_\gamma$) vanish!

6. An application to partially ordered sets. Let P be a finite poset (partially ordered set). Let $\Delta(P)$ denote the complex whose vertices are the elements of P and whose simplices are the chains (totally ordered subsets) of P. The condition that the complex $\Delta(P)$ be pure of dimension N is equivalent to the condition that every maximal chain in P have $N+1$ elements. In this case, it is easy to see that if x is the kth element of one maximal chain in P, then x is the kth element of every other maximal chain in P containing x. Let us define the *rank* $\rho(x)$ of this element x to be the number $k-1$. Thus $\rho(x)=0$ if and only if x is a minimal element of P, and $\rho(x)=N$ if and only if x is a maximal element of P. The function ρ

induces a partition of P into $N+1$ non-empty sets: $\rho^{-1}(0), \rho^{-1}(1), \ldots, \rho^{-1}(N)$. We will call these sets the *level sets* of the partially ordered set P.

Suppose we consider the partially ordered set Q obtained from P by deleting all elements of one of these level sets, say $Q = P - \rho^{-1}(i)$. It is easy to see that $\Delta(Q)$ is pure of dimension $N-1$, and that the collection of level sets of Q is precisely the same as the collection of level sets of P, except that the set $\rho^{-1}(i)$ is deleted from the collection.

In general, any complex that is Cohen-Macaulay must be pure. (See Step 1 of Corollary 3.4.) In particular, if $\Delta(P)$ is Cohen-Macaulay, it is pure and the preceding discussion applies.

Stanley conjectured that if $\Delta(P)$ is Cohen-Macaulay and Q is formed by deleting one level from P, then $\Delta(Q)$ is Cohen-Macaulay as well; or more generally that if P is a partially ordered set with $\Delta(P)$ pure, $\alpha(\Delta(Q)) \geqslant \alpha(\Delta(P)) - 1$. The second conjecture implies the first: If $\Delta(P)$ is Cohen-Macaulay, then $\alpha(\Delta(P)) = N$. The second conjecture implies that $\alpha(\Delta(Q)) \geqslant N-1$; since $\dim \Delta(Q) = N-1$, this means that equality holds and $\Delta(Q)$ is Cohen-Macaulay.

We shall verify this general conjecture, as an application of our theorem. (It can also be proved by algebraic means, as Stanley has shown.) We remark that there is no converse; there is an example of a partially ordered set P such that each of the complexes $\Delta(Q)$ is Cohen-Macaulay, but $\Delta(P)$ is not.

DEFINITION. Let Δ be a complex; let $X = |\Delta|$. Define $\gamma(\Delta)$ to be the smallest integer j for which at least one of the following groups does not vanish:

$$H^j(X, X-p), \quad \text{for } p \in X.$$

If $\beta(\Delta)$ is the number defined in Theorem 3.1, then $\beta(\Delta) \leqslant \gamma(\Delta)$. If X is empty, the local homology groups of X are not defined; in this case, we make the convention that $\beta(\varnothing) = \gamma(\varnothing) = -1$.

LEMMA 6.1. *Let Δ be a non-empty complex of dimension N. Let β_0 be an integer. The following are equivalent:*

(i) $\gamma(\Delta) \geqslant \beta_0$.

(ii) $\beta(\mathcal{L}k(v, \Delta)) \geqslant \beta_0 - 1$ *for each vertex v of Δ.*

(iii) $\beta(\mathcal{L}k(\sigma, \Delta)) \geqslant \beta_0 - \dim \sigma - 1$ *for each non-empty σ in Δ.*

Proof. We note first that if $\sigma = v * s$, where v is a vertex of σ, then $\mathcal{L}k(s, \mathcal{L}k(v, \Delta)) = \mathcal{L}k(\sigma, \Delta)$.

(i)$\Rightarrow$(ii). Let v be a vertex of Δ. If $\mathcal{L}k(v, \Delta) = \varnothing$, then $H^0(X, X-v) = H^0(v) \neq 0$, so $\beta_0 \leqslant 0$. Then $\beta(\mathcal{L}k(v, \Delta)) \geqslant \beta_0 - 1$ automatically.

So suppose $\mathcal{L}k(v, \Delta) \neq \varnothing$. In view of (i) and Lemma 3.3, the cohomology groups of $\mathcal{L}k(v, \Delta)$ vanish in dimensions less than $\beta_0 - 1$. To show that its *local* cohomology groups also vanish in these dimensions, it will suffice, again in view of Lemma 3.3, to show that for each non-empty simplex s of $\mathcal{L}k(v, \Delta)$ and each $i < \beta_0 - 1$,

$$\tilde{H}^{i - \dim s - 1}(\mathcal{L}k(s, \mathcal{L}k(v, \Delta))) = 0.$$

If we let σ denote the simplex $v*s$, then this condition is the same as the requirement that for $i<\beta_0-1$,

$$\tilde{H}^{i-\dim\sigma}(\mathcal{Lk}(\sigma,\Delta))=0;$$

and the latter holds in view of (i) and Lemma 3.3.

(ii)$\Rightarrow$(iii). We proceed by induction on $\dim\Delta$. Condition (ii) implies that $\beta_0\leqslant\dim\Delta$. If $\dim\Delta=0$, conditions (ii) and (iii) are identical. If $\dim\Delta=1$, the proof is easy: If v is a vertex, then $\beta(\mathcal{Lk}(v,\Delta))\geqslant\beta_0-1$ by (ii); while if σ is a 1-simplex, $\mathcal{Lk}(\sigma,\Delta)=\varnothing$ and $\beta(\mathcal{Lk}(\sigma,\Delta))=-1$. The latter is at least $\beta_0-\dim\sigma-1$, since $\beta_0-\dim\sigma\leqslant\dim\Delta-\dim\sigma=0$.

Suppose now that the implication holds in dimensions less than N, for all β_0. Let Δ have dimension N; let σ be a non-empty simplex of Δ. If $\dim\sigma=0$, then (iii) holds at once. If $\dim\sigma>0$, let us write $\sigma=v*s$, where v is a vertex of σ. By hypothesis, $\beta(\mathcal{Lk}(v,\Delta))\geqslant\beta_0-1$, so that in particular, $\gamma(\mathcal{Lk}(v,\Delta))\geqslant\beta_0-1$. Thus (i) holds for the non-empty complex $\mathcal{Lk}(v,\Delta)$ if β_0 is replaced by β_0-1. Now (i)$\Rightarrow$(ii) for all non-empty complexes and (ii)$\Rightarrow$(iii) for complexes of dimension less than N and all β_0 (by the induction hypothesis). Since $\mathcal{Lk}(v,\Delta)$ has dimension less than N, we conclude that for the simplex s of $\mathcal{Lk}(v,\Delta)$, we have

$$\beta(\mathcal{Lk}(s,\mathcal{Lk}(v,\Delta)))\geqslant(\beta_0-1)-\dim s-1.$$

Thus $\beta(\mathcal{Lk}(\sigma,\Delta))\geqslant\beta_0-1-\dim\sigma$, as desired.

(iii)$\Rightarrow$(i). This is an immediate consequence of Lemma 3.3. □

COROLLARY 6.2. *Let Δ be a complex; let $\dim\Delta=N$; let $X=|\Delta|$. Assume $H_j(X,X-p)=0$ for $j<N$ and all $p\in X$. If σ is a non-empty simplex of Δ, then $\mathcal{Lk}(\sigma,\Delta)$ is Cohen–Macaulay.* □

COROLLARY 6.3. *Let Δ be a non-empty complex. Then*

$$\gamma(\Delta)-1=\min_v\{\beta(\mathcal{Lk}(v,\Delta))\},$$

as v ranges over all vertices of Δ. □

We now prove the following, of which our conjecture is a special case (see Corollary 6.6):

THEOREM 6.4. *Let Δ be a complex of dimension N. Let T be a subset of the vertex set of Δ such that each principal simplex of Δ has a vertex in T, and no simplex of Δ has more than one vertex in T. Let $X=|\Delta|$ and $Y=|\Delta|-\mathcal{St}\,T$. Then Y has dimension $N-1$ and the following hold:*

(i) $\beta(Y)\geqslant\beta(X)-1$,

(ii) $\gamma(Y)\geqslant\gamma(X)-1$.

Proof. Let $(\mathrm{i})_N$ denote the statement that (i) holds whenever X has dimension N, and let $(\mathrm{ii})_N$ denote the corresponding version of (ii). If $N=0$, then $\beta(X)=\gamma(X)=0$. Because Y is empty, $\beta(Y)=\gamma(Y)=-1$, so $(\mathrm{i})_0$ and $(\mathrm{ii})_0$ hold.

If $N=1$, then $\beta(X)$ equals 0 or 1, and so does $\gamma(X)$. Because Y has dimension 0, $\beta(Y)=\gamma(Y)=0$. Thus $(\mathrm{i})_1$ and $(\mathrm{ii})_1$ hold. Henceforth we assume $N\geqslant2$.

Step 1. We prove that $(\mathrm{i})_0, \ldots, (\mathrm{i})_{N-1} \Rightarrow (\mathrm{ii})_N$. Let Δ have dimension N.

Let v be a vertex of $\Delta - \mathrm{St}\, T$. Let X_v be the complex $\mathcal{Lk}(v, \Delta)$ and let T_v be the set of those elements of T which are vertices of X_v. Note that $\mathcal{Lk}(v, \Delta - \mathrm{St}\, T) = X_v - \mathrm{St}\, T_v$.

Now X_v has dimension at most $N-1$; and we assert that T_v satisfies the hypotheses of the theorem: If s is a principal simplex of $\mathcal{Lk}(v, \Delta)$, then $\sigma = v * s$ is a principal simplex of X. Therefore at least one of the vertices of σ is in T; since it is not v, it must be one of the vertices of s. Similarly, if t were a simplex of $\mathcal{Lk}(v, \Delta)$ having more than one vertex in T_v, then $v * t$ would be a simplex of Δ having more than one vertex in T, contrary to hypothesis.

We conclude that

$$\beta(X_v - \mathrm{St}\, T_v) \geqslant \beta(X_v) - 1, \quad \text{or} \quad \beta(\mathcal{Lk}(v, \Delta - \mathrm{St}\, T)) \geqslant \beta(\mathcal{Lk}(v, \Delta)) - 1.$$

This inequality holds for each vertex of $\Delta - \mathrm{St}\, T$. Suppose the minimum value of the left side occurs when v is the vertex v_0. Using the preceding inequality and Corollary 6.3, we conclude that

$$\gamma(\Delta) - 1 \leqslant \beta(\mathcal{Lk}(v_0, \Delta)) \leqslant \beta(\mathcal{Lk}(v_0, \Delta - \mathrm{St}\, T)) + 1 = \gamma(\Delta - \mathrm{St}\, T),$$

as desired.

Step 2. Now we show that $(\mathrm{ii})_N \Rightarrow (\mathrm{i})_N$, and the theorem is proved. Let Δ have dimension N. Let $\beta_0 = \beta(\Delta)$. In view of $(\mathrm{ii})_N$, we know that the local cohomology groups of $Y = \Delta - \mathrm{St}\, T$ vanish in dimensions less than $\beta_0 - 1$. It remains to prove the same about the cohomology groups of Y.

Consider T as a 0-dimensional subcomplex of Δ. Then T is a full subcomplex of Δ, since no simplex of Δ has more than one of its vertices in the set T. Therefore we may, in computing cohomology groups, replace $Y = X - \mathrm{St}\, T$ by $X - T$, or by $X^*(T)$, as in §4. Applying Lemma 4.3 to the complex X, we have

$$H^j(X, X^*(T)) = 0 \quad \text{for } j < \beta_0 - \dim T = \beta_0.$$

Consider the exact sequence

$$H^{i+1}(X, X^*(T)) \longleftarrow \tilde{H}^i(X^*(T)) \longleftarrow \tilde{H}^i(X).$$

The left group vanishes for $i+1 < \beta_0$, as just shown; and the right group vanishes for $i < \beta_0$, by hypothesis. We conclude that the group $\tilde{H}^i(X^*(T)) \cong \tilde{H}^i(Y)$ vanishes for $i < \beta_0 - 1$, as desired. Thus $(\mathrm{i})_N$ holds. □

The proof just given implies the following additional fact:

COROLLARY 6.5. *Let X and Y be as in Theorem 6.4. If $\gamma(X) > \beta(X)$, then $\beta(Y) = \beta(X)$.*

Proof. Let $\beta_0 = \beta(X)$, as before. Because $\gamma(X) > \beta(X) = \beta_0$, we must have $\tilde{H}^{\beta_0}(X) \neq 0$. Because $\gamma(X) \geqslant \beta_0 + 1$, we conclude from Lemma 4.3 that $H^j(X, X^*(T)) = 0$ for $j < \beta_0 + 1$. Consider the exact sequence

$$H^{i+1}(X, X^*(T)) \longleftarrow \tilde{H}^i(X^*(T)) \longleftarrow \tilde{H}^i(X) \longleftarrow H^i(X, X^*(T))$$

The left group vanishes for $i+1<\beta_0+1$, as just noted, and so does $\tilde{H}^i(X)$; therefore $\tilde{H}^i(X^*(T))=H^i(Y)=0$ for $i<\beta_0$. The right-hand group vanishes for $i<\beta_0+1$ and in particular for $i<\beta_0$. Then $\tilde{H}^{\beta_0}(X)$ injects into $\tilde{H}^{\beta_0}(X^*(T))$; therefore the latter group is non-trivial. Thus β_0 is the smallest index for which $H^i(Y)\neq 0$. On the other hand, $\gamma(Y)\geqslant\gamma(X)-1\geqslant\beta_0$, so the local cohomology groups of Y vanish in dimensions less than β_0. We conclude that $\beta(Y)=\beta_0$, as desired. □

COROLLARY 6.6. *Let P be a partially ordered set with $\Delta(P)$ pure. Let T be a level set of P; let $Q=P-T$. Then*

$$\beta(\Delta(Q))\geqslant\beta(\Delta(P))-1.$$

In particular, if $\Delta(P)$ is Cohen-Macaulay, so is $\Delta(Q)$. □

This corollary is all one can say in general about the relation between $\beta(\Delta(Q))$ and $\beta(\Delta(P))$. Examples show that in general, $\beta(\Delta(Q))$ may be any number between $\beta(\Delta(P))-1$ and $\dim(\Delta(Q))$.

COROLLARY 6.7. *Let P be a partially ordered set with $\Delta(P)$ pure. If $\gamma(\Delta(P))-\beta(\Delta(P))\geqslant m>0$, then deletion of m levels from P leaves a poset Q for which $\beta(\Delta(Q))=\beta(\Delta(P))$.*

Proof. This follows from Corollary 6.5, once one notes that by Theorem 6.4, deletion of one level from P decreases the value of γ by at most 1. □

EXAMPLE. Let X be a triangulated N-manifold, and let P be its collection of simplices, ordered by inclusion. The complex $\Delta(P)$ is then just the first barycentric subdivision of X, so $\gamma(\Delta(P))=N$. Suppose that $\Delta(P)$ is *not* Cohen-Macaulay. Let $k<N$ be the smallest index for which $\tilde{H}^i(X)$ is non-trivial. Then $\beta(\Delta(P))=k$. In this case, if Q is obtained by deleting no more than $N-k$ levels from P, then $\beta(\Delta(Q))=\beta(\Delta(P))$. In particular, if Q is obtained by deleting precisely $N-k$ levels from P, then $\Delta(Q)$ is Cohen-Macaulay.

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Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139