ON THE H^p CLASSES OF DERIVATIVES OF FUNCTIONS ORTHOGONAL TO INVARIANT SUBSPACES

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Let Δ be the unit disk in the complex plane and let H^p denote the usual class of functions analytic on Δ. An analytic function φ which maps Δ into Δ is called an inner function if

$$\lim_{r\to 1^-} |\varphi(re^{i\theta})|=1 \text{ a.e. } [d\theta].$$

Thus

$$\varphi(z)=\prod_{k=1}^{\infty}\frac{\bar{a}_k}{|a_k|}\,\frac{a_k-z}{1-\bar{a}_k z}\cdot\exp\left(-\int_T \frac{\zeta+z}{\zeta-z}\,d\mu(\zeta)\right)$$

where $\Sigma\, 1-|a_k|<\infty$, and μ is a positive measure singular with respect to $d\theta$.

In [9] Protas, in [3] and [4] Ahern and Clark, and in [1] Ahern study the derivative φ' of an inner function φ. In [1] Ahern obtained the following characterization of when $\varphi'\in H^p$ for $1/2<p<1$.

THEOREM. *If φ is inner and $1/2<p<1$, the following are equivalent:*

(a) $\varphi'\in H^p$.

(b) *There is an α, $|\alpha|<1$ such that $\Sigma\,(1-|a_k(\alpha)|)^{1-p}<\infty$, where $\{a_k(\alpha)\}$ are the zeros of $\varphi_\alpha=(\alpha-\varphi)/(1-\bar{\alpha}\varphi)$.*

(c) $\Sigma\,(1-|a_k(\alpha)|)^{1-p}<\infty$ *for all $\alpha\in\Delta\backslash E$, where E is a set of capacity* 0.

Thus, if $\{a_k\}$ is the zero set of φ, $\Sigma\,(1-|a_k|)^{1-p}<\infty$ implies that $\varphi'\in H^p$. Although the converse need not hold, $\Sigma\,(1-|a_k(\alpha)|)^{1-p}$ will converge for nearly all $\alpha\in\Delta$, if $\varphi'\in H^p$.

In this paper we consider inner functions and the functions f belonging to $H^2\ominus\varphi H^2$. We show that $\varphi'\in H^p$ if and only if $f'\in H^\beta$ for all $f\in(\varphi H^2)^\perp$, for appropriate choice of β. We also obtain a characterization of when $\varphi'\in H^p$ in terms of the "level set" $\{z\colon|\varphi(z)|<\epsilon\}$, for small ϵ. In case φ is an interpolating Blaschke product, it turns out that for $1/2<p<1$, $\varphi'\in H^p$ if and only if $\Sigma\,(1-|a_k|)^{1-p}<\infty$, where $\{a_k\}$ is the zero set of φ.

This paper is divided into three sections. In the first we prove our results for interpolating Blaschke products. In the second we call into play the "Carleson curve" surrounding the level set $\{|\varphi(z)|<\epsilon\}$, where φ is an arbitrary inner function. This enables us to "approximate" φ' by B', where B is an interpolating Blaschke product, and we generalize the results of Section 1 to arbitrary inner functions. The third section indicates further theorems the methods at hand make available.

We assume the standard results in the literature about H^p spaces, interpolation

Received March 4, 1983.
Michigan Math. J. 30 (1983).

sequences, Carleson measures, and the non-Euclidean metric. As references we give [6], [7], or [10].

We will follow the convention of Ahern and Clark in [3] and write $f'(\zeta)=\lim_{r\to 1} f'(r\zeta)$ if the limit exists, and $|f'(\zeta)|=\infty$ if the limit does not exist. The theorems on the angular derivative of Carathéodory imply that if φ is inner and $\zeta\in T$, the unit circle,

$$|\varphi'(\zeta)|=\lim_{r\to 1}\frac{1-|\varphi(r\zeta)|}{1-r}.$$

We rely on this in the proof of Lemma 1 in Section 2.

As is usual, we use the notation $A\doteq B$ to denote that $c_1A\leqslant B\leqslant c_2A$ for absolute constants c_1 and c_2. A constant c which appears in one inequality may change its value in the next inequality. Finally, l^p denotes the set of sequences $\{\beta_k\}$ for which $\sum|\beta_k|^p<\infty$.

1. Interpolating Blaschke products. A Blaschke product

$$B(z)=\prod_{k=1}^{\infty}\frac{\bar a_k}{a_k}\,\frac{a_k-z}{1-\bar a_k z}$$

is called an *interpolating Blaschke product* (i.B.p.) if the zero sequence $\{a_k\}$ is an interpolation sequence. We will need the following results about interpolation sequences. We let $d_k=1-|a_k|$.

THEOREM A. *If* $\{a_k\}$ *is an interpolation sequence and* $f\in H^p$, *then*

$$\sum_{k=1}^{\infty}|f(a_k)|^p d_k\leqslant\gamma\|f\|_p^p$$

where γ *is an absolute constant.*

THEOREM B. *If* $\{a_k\}$ *is an interpolation sequence then the functions*

$$f_k(z)=\frac{d_k^{1/2}}{1-\bar a_k z}$$

form an unconditional basis for their closed linear span in H^2. *Thus the series* $\sum\beta_k f_k$ *converges to an* H^2 *function if and only if* $\sum|\beta_k|^2<\infty$.

For Theorem A see [6: Chapter 9]. For Theorem B see [8]. We prove our first result.

THEOREM 1. *Let* $1/2<p<1$. *Suppose* B *is an* i.B.p. *Then* $B'\in H^p$ *if and only if* $\sum d_k^{1-p}<\infty$.

Proof. We need only show the necessity. If $B'\in H^p$, Theorem A implies

$$\sum_{k=1}^{\infty}|B'(a_k)|^p d_k<\infty.$$

However a familiar calculation establishes that $|B'(a_k)|\geqslant c/(1-|a_k|)$, where c is independent of k. This proves the theorem. □

We now turn our attention to the subspace of H^2 spanned by the functions f_k defined above. It is well known that the closed linear span of the f_k is simply $(BH^2)^\perp = H^2 \ominus BH^2$. See [1] for exact details. We will need the following results about $(BH^2)^\perp$.

THEOREM C. *If* $\{a_k\}$ *is an interpolation sequence then the operator* $T\colon (BH^2)^\perp \to l^2$ *defined by* $Tf = \{f(a_k)d_k^{1/2}\}$ *is* 1:1 *and onto.*

THEOREM D. *Let* φ *be any inner function and let* $f \in (\varphi H^2)^\perp$. *Then there is an* $h \in (\varphi H^2)^\perp$ *such that*

$$f(e^{i\theta}) = \varphi(e^{i\theta})\overline{e^{i\theta}h(e^{i\theta})} \text{ a.e. } [d\theta].$$

In the special case that $\varphi = B$, *a Blaschke product, both* f *and* h *are analytic on* $\mathbf{C}\setminus\overline{\{1/\bar{a}_k\}}$, *where* $\{a_k\}$ *is the zero set of* B *and for* $z \in \mathbf{C}\setminus\overline{\{1/\bar{a}_k\}\cup\{a_k\}}$

$$f(z) = B(z)\frac{1}{z}\overline{h(1/\bar{z})}.$$

For Theorem C see [6: Chapter 9]. For Theorem D see [5].

We now prove a theorem about the derivative of a $(BH^2)^\perp$ function.

THEOREM 2. *Let* B *be an* i.B.p. *Suppose* $\frac{2}{3} < p < 1$. *Then the following conditions are equivalent:*

(i) $B' \in H^p$.

(ii) $f' \in H^{2p/(2+p)}$ *for all* $f \in (BH^2)^\perp$.

Proof. Suppose $B' \in H^p$. We will show that for $f = \sum_{k=1}^n \beta_k f_k$,

$$\|f'\|_\alpha \leq \gamma\left(\sum_{k=1}^n |\beta_k|^2\right)^{1/2}$$

where $\alpha = 2p/(p+2)$ and γ is independent of n. Theorem B and the completeness of H^p will then imply $\|f'\|_\alpha \leq c\|f\|_2$ for all $f \in (BH^2)^\perp$, for some constant c.

If $f = \sum_{k=1}^n \beta_k f_k$ then

$$f'(z) = \sum \beta_k \bar{a}_k d_k^{1/2} \frac{1}{(1-\bar{a}_k z)^2}$$

and

$$|f'(z)| \leq \sum |\beta_k| d_k^{1/2} \frac{1}{|1-\bar{a}_k z|^2}.$$

Since the sum on the right is finite we may estimate $\|f'\|_\alpha$ by setting $z = e^{i\theta}$ and integrating. Since $\frac{1}{2} < \alpha < 1$,

$$\begin{aligned}\|f'\|_\alpha^\alpha &\leq \sum |\beta_k d_k^{1/2}|^\alpha \int_{-\pi}^{\pi} \frac{d\theta}{|1-\bar{a}_k e^{i\theta}|^{2\alpha}} \\ &\leq c\sum |\beta_k d_k^{1/2}|^\alpha (1-|a_k|)^{1-2\alpha} \\ &= c\sum |\beta_k|^{2p/(p+2)} d_k^{1-3p/(p+2)},\end{aligned}$$

where c is independent of n.

Since $(p+2)/p$ and $(p+2)/2$ are conjugate exponents, Hölder's inequality shows that

$$\|f'\|_\alpha^\alpha \leqslant c\cdot\left(\sum |\beta_k|^2\right)^{p/(p+2)}\left(\sum d_k^{1-p}\right)^{2/(2+p)}.$$

Since $B'\in H^p$, Theorem 1 shows that

$$\|f'\|_{2p/(p+2)} \leqslant \gamma\cdot\left(\sum |\beta_k|^2\right)^{1/2}$$

as desired.

Now suppose $f'\in H^\alpha$ for every $f\in(BH^2)^\perp$ where $\alpha=2p/(p+2)$. The closed graph theorem implies then that $\|f'\|_\alpha \leqslant \gamma\|f\|_2$ for an absolute constant γ. Using the relation $f(z)=B(z)(1/z)\overline{h(1/\bar{z})}$ of Theorem D yields

$$f'(z)=B'(z)\frac{1}{z}\overline{h(1/\bar{z})}-B(z)\frac{1}{z^2}\overline{h(1/\bar{z})}-B(z)\frac{1}{z^3}\overline{h'(1/\bar{z})}.$$

If $f(z)=\sum_{k=1}^n \beta_k f_k$ we may set $z=e^{i\theta}$ in the above equality. It is then an easy observation that $\|B'h\|_\alpha \leqslant 3\gamma\|f\|_2$. Since the functions h which arise from the functions f form a dense subset of $(BH^2)^\perp$, we see that $B'h\in H^\alpha$ for all $h\in(BH^2)^\perp$. We now apply Theorem A and get

$$\sum_{k=1}^{\infty} |h(a_k)B'(a_k)|^{2p/(p+2)}d_k<\infty.$$

Thus, for all $h\in(BH^2)^\perp$,

$$\sum |h(a_k)|^{2p/(p+2)}d_k^{1-2p/(p+2)}<\infty.$$

Next observe that by Theorem C, as h ranges over all of $(BH^2)^\perp$, the sequences

$$\{\beta_k\}=\{h(a_k)^{2p/(2+p)}d_k^{p/(2+p)}\}$$

range over all of $l^{(2+p)/p}$. Thus

$$\sum |\beta_k|\, d_k^{1-3p/(p+2)}<\infty$$

for all $\{\beta_k\}\in l^{(2+p)/p}$. A simple argument shows that $\sum d_k^{1-p}<\infty$, and thus $B'\in H^p$, as desired. □

2. General Blaschke products. In this section we extend the results of Section 1 to arbitrary Blaschke products. Our main tool is the following lemma.

LEMMA 1. *Let* $0<\epsilon<1$. *Suppose* B *is an* i.B.p. *with zeros* $\{a_k\}$, *and* φ *is an arbitrary inner function. If* $\sup_k|\varphi(a_k)|<\epsilon$, *then there is a constant* γ *such that*

$$|B'(e^{i\theta})|\leqslant\gamma|\varphi'(e^{i\theta})| \text{ a.e. } [d\theta].$$

Proof. We may choose r so small that the set $\{z\colon |B(z)|<r\}$ is a union of disjoint components R_k where R_k contains the single zero a_k. It can be shown that there is an s, going to zero as r goes to zero, such that

$$R_k\subseteq\left\{z\colon \left|\frac{z-a_k}{1-\bar{a}_k z}\right|<s\right\},\quad \text{for all } k.$$

Since $|\varphi(a_k)|<\epsilon$, we may choose r and s so small that there is a $\delta<1$ such that $|\varphi(z)|<\delta$ for $z\in\bigcup R_k$. Thus we have the containment $\bigcup R_k\subseteq\{z: |\varphi(z)|<\delta\}$.

Let $G=\Delta\cap\{z: |\varphi(z)|>\delta\}$. Since $\{z: |B(z)|<r\}\subseteq\Delta\backslash G$, $-\log|B(z)|\leqslant-\log r$, for $z\in G$. On $\partial G\cap\Delta$, $|\varphi|=\delta$. Thus

$$(*)\qquad -\log|B(z)|\leqslant-\gamma\log|\varphi(z)|$$

for $z\in\partial G\cap\Delta$, where $\gamma=\log r/\log\delta$. If $z\in\partial G\cap T$ then

$$(**)\qquad -\log|B(z)|=-\log|\varphi(z)|=0 \text{ a.e. } [d\theta].$$

Since $-\log|B|$ and $-\log|\varphi|$ are bounded harmonic functions on G, (*) and (**) imply that $-\log|B(z)|\leqslant\gamma\log|\varphi(z)|$ for all $z\in G$.

Now, for almost all θ, $re^{i\theta}\in G$ if r is close enough to 1. For such a θ and r we have

$$\frac{-\log|B(re^{i\theta})|}{1-r}\leqslant-\gamma\frac{\log|\varphi(re^{i\theta})|}{1-r}$$

and letting $r\to1$ we get

$$|B'(e^{i\theta})|\leqslant\gamma|\varphi'(e^{i\theta})| \text{ a.e. } [d\theta].$$

This completes the proof. □

In order to generalize the results of Section 1, we let φ be an arbitrary inner function. We plan to find an i.B.p., B, such that $\varphi'\in H^p$ if and only if $B'\in H^p$.

To this end, let Γ_ϵ be the "Carleson curve" surrounding the set $\{z: |\varphi(z)|<\epsilon\}$. Recall that Γ_ϵ consists of certain curvilinear polygonal segments and that:

(i) arc length on Γ_ϵ is a Carleson measure;

(ii) $\epsilon<|\varphi(z)|<\epsilon'<1$, for $z\in\Gamma_\epsilon$; and

(iii) $\{z: |\varphi(z)|\subseteq\operatorname{Int}\Gamma_\epsilon\}$.

We need the following theorem.

THEOREM E. *Let $0<\sigma<1$. Then if σ is small enough, there is a uniformly separated sequence $\{\omega_k\}$ on Γ_ϵ with the property that if Γ_j is any component of Γ*

$$\Gamma_j=\bigcup_{k=n_j}^{N_j}[\omega_k,\omega_{k+1}]\quad\text{where}\quad\frac{\omega_k-\omega_{k+1}}{1-\bar\omega_k\omega_{k+1}}<\sigma,$$

and $[\omega_k,\omega_{k+1}]$ is the segment on Γ_j connecting ω_k and ω_{k+1}. For the proof see [7: p. 341]. Essentially, one just distributes the ω_k evenly (with respect to the non-Euclidean distance) over Γ_ϵ. Thus $\{\omega_k\}$ is a separated sequence. Since arc length on Γ_ϵ is a Carleson measure, $\{\omega_k\}$ is easily shown to be uniformly separated.

The i.B.p. B we choose will be the one with zeros $\{\omega_k\}$, where ϵ and σ are chosen as above.

We may now generalize Theorem 1.

THEOREM 3. *Let $\frac{1}{2}<p<1$. Suppose φ is inner. Let B be the* i.B.p. *related to φ as above. Then the following conditions are equivalent:*

(i) $\varphi'\in H^p$.

(ii) $B'\in H^p$.

(iii) $\int_{\Gamma_\epsilon}|dz|/(1-|z|)^p<\infty$, *where Γ_ϵ is the Carleson curve of Theorem* E.

Proof. That (i) implies (ii) follows from Lemma 1. To see that (ii) is equivalent to (iii) we write

$$\int_{\Gamma_\epsilon} \frac{|dz|}{(1-|z|)^p} = \sum_j \int_{\Gamma_j} \frac{|dz|}{(1-|z|)^p}$$

where the Γ_j are the components of Γ_ϵ. By Theorem E

$$\int_{\Gamma_j} \frac{|dz|}{(1-|z|)^p} = \sum_{k=n_j}^{N_j} \int_{[\omega_k, \omega_{k+1}]} \frac{|dz|}{(1-|z|)^p}$$
$$\doteq \sum_{k=n_j}^{N_j} (1-|\omega_k|)^{1-p}.$$

Since $B' \in H^p$ and B is an i.B.p., we see that (ii) and (iii) are equivalent.

We finally show that (ii) implies (i). It is not hard to see that for some $\delta < 1$, $|B| < \delta$ on the curve Γ_ϵ. Thus

$$\{z\colon |\varphi(z)| < \epsilon\} \subseteq \operatorname{Int} \Gamma_\epsilon \subseteq \{z\colon |B(z)| < \delta\}.$$

We may now argue as in Lemma 1 and find a constant γ such that

$$|\varphi'(e^{i\theta})| \leqslant \gamma |B'(e^{i\theta})| \text{ a.e. } [d\theta].$$

This completes the proof. □

We turn now to functions in $(\varphi H^2)^\perp$.

THEOREM 4. *Let $\frac{2}{3} < p < 1$. Let φ be an inner function. The following conditions are equivalent:*

(i) $\varphi' \in H^p$.

(ii) $f' \in H^{2p/(p+2)}$, *for all* $f \in (\varphi H^2)^\perp$.

Proof. Assume $\varphi' \in H^p$. By Theorem 3 in [3], φ is a Blaschke product. Let $\{a_k\}$ be the zero set of φ. Then it will be enough to show that $\|f'\|_\alpha \leqslant \gamma \cdot \|f\|_2$ for all f in $(B_k H^2)^\perp$, where B_k is a finite subproduct of B and γ is an absolute constant.

Without loss of generality, we may assume that $\varphi(0) = 0$. This in turn implies that the curve Γ_ϵ will not contain 0. Now let Γ_ϵ^* be the reflection of Γ_ϵ across the unit circle. Since the only singularities of f occur at $\{1/\bar{a}_k\}$, and since 0 is in the interior of Γ_ϵ, we have the integral formula

$$f'(z) = \frac{1}{2\pi i} \int_{\Gamma_\epsilon^*} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Let $z = e^{i\theta}$. Since Γ_ϵ is a union of "intervals" of the form $[\omega_k, \omega_{k+1}]$ where the ω_k are as in Theorem E, an elementary estimate gives, for a c independent of n,

$$|f'(e^{i\theta})| \leqslant c \sum |f(t_k^*)| \frac{1-|\omega_k|}{|\omega_k - e^{i\theta}|^2}$$

where t_k^* is some point on the reflection of the interval $[\omega_k, \omega_{k+1}]$. The relation $f(z) = B(z)\overline{h(1/\bar{z})}/z$ of Theorem D shows that

$$|f'(e^{i\theta})| \leqslant c \cdot \sum_{k=1}^{n} |h(t_k)| \frac{1-|\omega_k|^2}{|\omega_k - e^{i\theta}|^2}.$$

It is quite easy to see that $\sum |h(t_k)|^2(1-|\omega_k|) \leqslant \gamma \cdot \|h\|^2$ and that $\|h\|_2 = \|f\|_2$. Thus if $\beta_k = h(t_k)(1-|\omega_k|)^{1/2}$, then $\{\beta_k\} \in l^2$ and we have

$$|f'(e^{i\theta})| \leqslant c \cdot \sum \frac{\beta_k(1-|\omega_k|)^{1/2}}{|1-e^{-i\theta}\omega_k|^2}.$$

It follows now that $\|f'\|_\alpha \leqslant \gamma \cdot \|f\|_2$ exactly as in the proof of Theorem 2, since $\sum (1-|\omega_k|)^{1-p} < \infty$. □

To prove the converse, we will need the following result, and its corollary.

THEOREM F. *Let $\{\omega_k\}$ be an interpolation sequence and suppose* $\sup_k |\varphi(\omega_k)| < 1$. *Then for some n, the functions*

$$\left\{ \frac{1-\overline{\varphi^n(\omega_k)}\varphi^n(z)}{1-\bar{\omega}_k z} \right\}$$

form an unconditional basis for their closed linear span in H^2.

COROLLARY. *If $\{\omega_k\}$, φ and n are as above, then the map T: $(\varphi^n H^2)^\perp \to l^2$ defined by $Tf = \{f(\omega_k)(1-|\omega_k|)^{1/2}\}$ is onto.*

See [8: p. 265, 275] for the discussions.

Now assume $f' \in H^\alpha$, for all $f \in (\varphi H^2)^\perp$, where $\alpha = 2p/(p+2)$. As in the proof of Theorem 2 it follows immediately that $\varphi' f \in H^\alpha$ for any f in $(\varphi H^2)^\perp$. In [1] it is shown that

$$(\varphi^n H^2)^\perp = (\varphi H^2)^\perp \oplus \varphi(\varphi H^2)^\perp \oplus \cdots \oplus \varphi^{n-1}(\varphi H^2)^\perp.$$

An easy calculation now shows that if $g \in (\varphi H^2)^\perp$, then $g' \in H^\alpha$ and $\varphi' g \in H^\alpha$.

Now let B be the i.B.p. of Theorem E. Clearly, $B'g \in H^\alpha$ for any $g \in (\varphi^n H^2)^\perp$. Thus

$$\sum_{k=1}^{\infty} |B'(\omega_k) g(\omega_k)|^\alpha (1-|\omega_k|) < \infty.$$

We may now use Theorem F and its corollary and the argument of Theorem 2 to conclude that $\sum (1-|\omega_k|)^{1-p} < \infty$ and that $B' \in H^p$. An application of Theorem 3 completes the proof. □

3. Other results. If B is a Blaschke product with zero set $\{a_k\}$, define $(BH^n)^\perp$ to be the closure in H^n of $H^\infty \cap (BH^2)^\perp$. It is shown in [8] that for $1 < n < \infty$, $\{1/(1-\bar{a}_k z)\}$ is an unconditional basis for its span in H^n if and only if $\{a_k\}$ is an interpolation sequence. The methods of Section 1 and Section 2 easily yield the following theorem.

THEOREM 5. *Let $\frac{1}{2} < np/(p+n) < p < 1$. If φ is a Blaschke product the following conditions are equivalent:*

(i) $\varphi' \in H^p$.

(ii) $f' \in H^{np/(p+n)}$, *for all $f \in (\varphi H^n)^\perp$.*

Note that if $\frac{1}{2}<p<1$, then for large enough n, $\frac{1}{2}<np/(p+n)$, and so the theorem applies.

The limiting case, when $n=\infty$, is also of interest. We may define $(BH^{\infty})^{\perp}$ to be equal to $(BH^2)^{\perp}\cap H^{\infty}$. We have the following theorem.

THEOREM 6. *Let $\frac{1}{2}<p<1$. Then if φ is a Blaschke product, the following conditions are equivalent:*

(i) $\varphi'\in H^p$.

(ii) $f'\in H^p$, *for all* $f\in(\varphi H^{\infty})^{\perp}$.

Proof. We let B and $\{\omega_k\}$ be as in the first part of proof of Theorem 4. Repeating the argument, if $\varphi'\in H^p$ and f is in $(\varphi H^{\infty})^{\perp}$,

$$|f'(e^{i\theta})|\leqslant c\cdot\sum|h(t_k)|\,\frac{1-|\omega_k|^2}{|\omega_k-e^{i\theta}|^2},$$

where $t_k\in[\omega_k,\omega_{k+1}]$ and $f=\varphi\overline{e^{i\theta}h}$ a.e. $[d\theta]$. Since $f\in H^{\infty}$, so does h. Thus

$$\begin{aligned}|f'(e^{i\theta})|&\leqslant c\cdot\sum\frac{1-|\omega_k|^2}{|\omega-e^{i\theta}|^2}\\&\leqslant c|B'(e^{i\theta})|.\end{aligned}$$

Thus Theorem 3 implies $f'\in H^p$.

In the other direction, recall that for any $\zeta\in\Delta$, the function

$$k_{\zeta}(z)=\frac{1-\overline{\varphi(\zeta)}\varphi(z)}{1-\bar{\zeta}z}$$

is in $(\varphi H^2)^{\perp}\cap H^{\infty}$. If $k_{\zeta}'\in H^p$, it follows easily that $\varphi'\in H^p$, provided ζ is not a zero of φ. This completes the proof. □

REFERENCES

1. P. R. Ahern, *The mean modulus and the derivative of an inner function,* Indiana Univ. Math. J. 28 (1979), 311-347.
2. P. R. Ahern and D. N. Clark, *On functions orthogonal to invariant subspaces,* Acta Math. 124 (1970), 191-204.
3. ———, *On inner functions with H^p derivative,* Michigan Math. J. 21 (1974), 115-127.
4. ———, *On inner functions with B^p derivative,* Michigan Math. J. 23 (1976), 107-118.
5. W. Cohn, *Carleson measures for functions orthogonal to invariant subspaces,* Pacific J. Math., 103 (1982), 347-364.
6. P. L. Duren, *Theory of H^p spaces,* Academic Press, New York, 1970.
7. J. B. Garnett, *Bounded analytic functions,* Academic Press, New York, 1981.
8. S. V. Hruscev, N. K. Nikol'skii and B. S. Pavlov, *Unconditional bases of exponentials and of reproducing kernels,* Complex Analysis and Spectral Theory (Leningrad, 1979/1980), 214-335, Lecture Notes in Math., 864, Springer, Berlin, 1981.

9. D. Protas, *Blaschke products with derivative in H^p and B^p,* Michigan Math. J. 20 (1973), 393-396.
10. D. Sarason, *Function theory on the unit circle,* Virginia Poly. Inst. and State Univ., Blacksburg, Virginia, 1978.

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