DO ALMOST FLAT MANIFOLDS BOUND?

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0. Introduction. A closed connected Riemannian manifold M^n is said to be flat if its sectional curvatures are all zero. Since the (real) Pontrjagin classes are computable in terms of the sectional curvatures, a flat Riemannian manifold M has zero (real) Pontrjagin classes and hence zero Pontrjagin numbers when M is orientable. This observation was part of the reason for the conjecture that such a manifold is a boundary, i.e., that there exists a compact manifold W^{n+1} such that $\partial W = M$. (Recall that Thom had shown that a manifold bounds if and only if all its Stiefel–Whitney numbers vanish and Wall showed an orientable manifold is an oriented boundary if in addition all its Pontrjagin numbers vanish.) Recently, Hamrick and Royster, after partial results by Marc Gordon (cf. [6], [3]), have verified this conjecture.

In this paper, we conjecture that any almost flat manifold (whose definition is recalled below) bounds and give some partial results on this extended conjecture. We will rely heavily on the earlier methods of Gordon, Hamrick and Royster. Let $b(\ ,\ )$ be a Riemannian metric on a compact manifold M^n, let $d(M^n, b)$ denote the diameter of M^n with respect to $b(\ ,\ )$ and let $c(M^n, b)$ denote the maximum of the absolute values of the sectional curvatures of M^n relative to $b(\ ,\ )$. Following the terminology introduced by Gromov [5], an almost flat structure on a closed connected smooth manifold M^n is a sequence of Riemannian metrics $b_i(\ ,\ )$, where $i = 1, 2, \ldots$, such that

(0.1) (a) $\lim_{i \to \infty} c(M^n, b_i) = 0$ and

(b) $\{d(M^n, b_i) : i = 1, 2, \ldots\}$ has a finite upper bound.

CONJECTURE 1. *If M^n supports an almost flat structure, then there exists a compact smooth manifold W^{n+1} such that $\partial W^{n+1} = M^n$.*

This conjecture is geometrically motivated by work of Gromov [5]. He showed that if M^n supports an almost flat structure then M^n has a finite sheeted cover that is a nilmanifold and consequently the Pontrjagin classes of M^n vanish since nilmanifolds are parallelizable. Recall a nilmanifold is the quotient of a (connected) simply connected nilpotent Lie group by a discrete cocompact subgroup. In fact, a second result of Gromov [4] (which uses Margulis' lemma) suggests the above conjecture could possibly be strengthened as follows.

CONJECTURE 2. (a) *If M^n is a flat Riemannian manifold, then $M^n = \partial W^{n+1}$, where $W - \partial W$ supports a complete hyperbolic structure (constant negative sectional curvatures) with finite volume.*

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(b) *If M^n supports an almost flat structure, then $M^n = \partial W^{n+1}$, where $W - \partial W$ supports a complete Riemannian metric with finite volume all of whose sectional curvatures are negative.*

We do not know how to approach Conjecture 2; hence the rest of this paper is devoted to partial results about Conjecture 1.

We recall that a closed connected manifold M^n is aspherical if $\pi_i(M) = 0$ for all $i \geqslant 2$. Also if Γ is a finitely generated torsion-free virtually nilpotent group, then it is a consequence of Mal'cev's work (cf. [2: Lemma 1.2]) that there exists a smooth aspherical manifold M^n with fundamental group Γ. In fact, M^n can be chosen to be an infranilmanifold, i.e., M^n is a double coset space $\Gamma \backslash L \rtimes G/G$, where L is a (connected) simply connected nilpotent Lie group, $L \rtimes G$ is the semidirect product with respect to a faithful representation of a finite group G into $\mathrm{Aut}(L)$ and Γ is a discrete cocompact subgroup of $L \rtimes G$. Since any infranilmanifold supports an almost flat structure, the homotopy type of an aspherical manifold is determined by its fundamental group, and Stiefel–Whitney classes are homotopy type invariants, we have that Conjecture 1 is equivalent to the following.

CONJECTURE 1′. *Let M^n be a closed smooth aspherical manifold such that $\pi_1(M)$ contains a nilpotent subgroup with finite index; then there exists a compact smooth manifold W^{n+1} such that $\partial W = M$.*

We will prove the following partial result concerning Conjecture 1′.

THEOREM 1. *Conjecture* 1′ *is true when $\pi_1(M^n)$ contains a nilpotent subgroup of index* 2.

Actually we prove a somewhat stronger result (cf. Proposition 1.3).

Recall that if N^n is an m-sheeted cover of a closed smooth manifold M^n, where m is an odd integer, then M^n has the same Stiefel-Whitney numbers as N^n; consequently, M^n bounds if and only if N^n bounds.

COROLLARY 2. *Conjecture* 1′ *is true when $\pi_1(M)$ contains a nilpotent subgroup of index either m or $2m$, where m is an odd integer.*

This follows from the above recollection, Theorem 1, the fact that nilmanifolds are parallelizable and Mal'cev's result that any torsion-free finitely generated nilpotent group is the fundamental group of a closed nilmanifold.

1. Proof of Theorem 1. First case. Assume Γ is a finitely generated torsion-free group containing a nilpotent subgroup N of finite index a power of 2 so that

$$1 \to N \to \Gamma \to G \to 1 \tag{1.1}$$

is exact, and the action of G on N is effective. In this section we show, following closely the method of Hamrick and Royster, that if the action in (1.1) of G on the center $Z(N)$ of N is effective then a closed $K(\Gamma, 1)$ manifold M bounds. Thus we will have reduced the proof of Theorem 1 to the case when $G = Z_2$ acts trivially on $Z(N)$, which we tackle in Section 2.

The method for showing that M bounds consists in making an elementary abelian 2-group act on it and analyzing the action's fixed.point set together with its normal bundle. Once we have the set-up to apply Hamrick and Royster's techniques, we freely refer to their paper [6] for the proofs.

As noted in the introduction we may assume that

$$M=\Gamma\backslash L\rtimes G/G \tag{1.2}$$

for some (connected) simply connected nilpotent Lie group L with discrete cocompact subgroup N. To see this, consider a (connected) simply connected nilpotent Lie group L containing N as a discrete cocompact subgroup. Such a group exists by the work of Mal'cev [8]. Form the pushout diagram

$$\begin{array}{ccc} N & \to & \Gamma \\ \downarrow & & \downarrow \\ L & \to & L', \end{array} \tag{1.3}$$

and write L' as the semidirect product $L'=L\rtimes G$, where $G=N\backslash\Gamma\simeq L\backslash L'$ is a maximal compact subgroup of L. Since $\Gamma\backslash L'/G$ is a $K(\Gamma,1)$ it is homotopy equivalent to M; therefore it has the same Stiefel-Whitney numbers as M and it bounds if and only if M does.

There is a regular covering $p\colon\tilde{M}\to M$, where $\tilde{M}$ is the nilmanifold $N\backslash L$, defined by $p(Nx)=\Gamma xG$ for $x\in L$. The group G acts freely on $\tilde{M}$ as the group of covering transformations by

$$g(Nx)=N\gamma xg^{-1}, \tag{1.4}$$

where γ is any element of Γ such that $\gamma xg^{-1}\in L$.

Denote by $Z(L)$ the center of L. It is known that $Z(L)$ is nontrivial and that $Z(N)$ is a cocompact subgroup of $Z(L)$. The latter acts by multiplication on L, inducing an action of $Z(N)\backslash Z(L)$ on $\tilde{M}$. Let Σ denote the subgroup of $Z(N)\backslash Z(L)$ consisting of elements of order $\leqslant 2$, and Σ_G the subgroup of elements in Σ fixed under the conjugation action of G on Σ. Since G is a 2-group by assumption, Σ_G is nontrivial: the orbits of G in Σ consist of either one or an even number of elements, $1\in\Sigma$ is the single element in its orbit, $|\Sigma|$ is even, hence there is at least one other element of Σ fixed under G.

The action of Σ_G on $\tilde{M}$ passes to an action of Σ_G on M:

$$Z(N)s(\Gamma xG)=\Gamma sxG \tag{1.5}$$

for $x\in L$ and $s\in Z(L)$ such that $2s\in Z(N)$ (we use additive notation within $Z(L)$) and $Z(N)gsg^{-1}=Z(N)s$ for all $g\in G$.

Choose a representative $s\in Z(L)$ for each element of Σ_G and to save notation consider s also as an element of Σ_G. If there is no point of M fixed under Σ_G, then by a theorem of Conner and Floyd (Theorem 30.1 in [1]) M bounds. So assume there is $x\in L$ such that $\Gamma sxG=\Gamma xG$ for each $s\in\Sigma_G$. Then there is an injective homomorphism $\phi\colon\Sigma_G\to G$ defined by $\phi(s)=g$ if and only if $sx=\gamma xg$ for some $\gamma\in\Gamma$. The last equation determines g uniquely for if $\gamma xg=\gamma_1 xg_1$ then $xg_1g^{-1}\in\Gamma$ has finite order and since Γ is torsion-free $g_1=g$.

For each $s \in \Sigma_G$, the action (1.4) on M of the subgroup $\langle \phi(s) \rangle$ of G generated by $\phi(s)$ gives rise to an action on $\langle N, s \rangle \backslash L$, where $\langle N, s \rangle$ is the subgroup of L generated by N and s. It has a fixed point, namely Nx.

We will need the following fact: If $\langle t \rangle = Z_2$ acts on the nilmanifold $N \backslash L$ and $U = \{f: L \to L: qf = q \text{ or } qf = tq$, where $q: L \to N \backslash L$ is the projection$\}$ is the group of liftings to L, then

$$(1.6) \qquad 1 \to N \to U \to Z_2 = \langle t \rangle \to 1$$

splits if and only if t has a fixed point in $N \backslash L$.

In our situation, restriction of (1.1) to $\langle \phi(s) \rangle$ and projection give rise to the following commutative diagram

$$\begin{array}{ccccccccc} 1 \to & \pi_1(N \backslash L) & \to & p^{-1}(\langle \phi(s) \rangle) & \to & \langle \phi(s) \rangle & \to 1 \\ & \downarrow & & \downarrow & & \downarrow \; = & \\ 1 \to & \pi_1(\langle N, s \rangle \backslash L) & \to & U & \to & \langle \phi(s) \rangle & \to 1. \end{array}$$

By the fact above, the bottom sequence splits. The top sequence corresponds to the restriction of the cohomology class of (1.1) in $H^2(G; Z(N))$ to a class $[\alpha_{\phi(s)}] \in H^2(\langle \phi(s) \rangle; Z(N))$ such that $\alpha_{\phi(s)}$ becomes a boundary when mapped to $H^2(\langle \phi(s) \rangle; Z(\langle N, s \rangle))$. Thus $\alpha_{\phi(s)} = (1 + \phi(s))a$ for some $a \in Z(\langle N, s \rangle)$. Let $\beta(s) = 2s \in N = \pi_1(N \backslash L)$. The above discussion implies that the cocycle $\alpha_{\phi(s)}$ can be chosen to be $(1 + \phi(s))\beta(s)/2$.

Let $W = \{w \in Z(N): w$ conjugated by $\phi(s)$ is $\pm w$ for all $s \in \Sigma_G$, $w \notin 2Z(N)$ and $[w/2] \in Z(N) \backslash Z(L)$ is fixed by $G\}$. Now Lemmas 6 and 7 from [6] read as follows.

LEMMA 1.1. *Suppose Σ_G has a fixed point on M determining an injective homomorphism $\phi: \Sigma_G \to G$.*

(a) *Let $\{w_i: i = 1, 2, \ldots, k\} \subset W$ be such that $\{[w_i/2]: i = 1, 2, \ldots, k\}$ form a linearly independent subset in $\Sigma \subset Z(N) \backslash Z(L)$. Then some w_i is fixed by every element of $\phi(\langle [w_i]: i = 1, 2, \ldots, k \rangle)$.*

(b) *For any subgroup $S \neq \{e\}$ of Σ_G there is a $g \in \phi(S \backslash \{e\})$ such that $gw = w$ for all $w \in W$ with $[w/2] \in S$.*

The proofs from [6] can be copied verbatim, so we omit them.

Let $\phi: \Sigma_G \to G$ be an arbitrary injective homomorphism, and

$$(1.7) \qquad E_\phi = \{x \in \tilde{M}: xs = \phi(s)x \quad \text{for all } s \in \Sigma_G\}.$$

LEMMA 1.2. *If $\phi(\Sigma_G) \subset Z(G)$ and the action of G on $Z(N)$ in* (1.1) *is effective, then E_ϕ is empty.*

To show that, note that by Lemma 1.1(b) there is an element $\phi(s_1) = g \in \phi(\Sigma_G \backslash \{e\})$ such that $gw = w$ for all $w \in W$. But since g acts nontrivially on $Z(N)$, the eigenspace A_1 corresponding to the value -1 is nontrivial. Let $\Sigma_G = \{e, s_1, \ldots, s_k\}$. Now $\phi(s_2)$ acts on A_1. If it leaves it fixed, take $A_2 = A_1$; if not, let A_2 be the subspace of A_1 corresponding to the eigenvalue -1 of $\phi(s_2)$, etc. The group G acts on each A_i since all $\phi(s_i)$ are central in G. Then A_k determines a nontrivial subgroup S of Σ on which G acts. A counting argument again shows that at least one nontrivial element of S is fixed by G, i.e., belongs to Σ_G. This element defines a $w \in W$ such that $gw = -w$, which contradicts the choice of g.

Finally, the passage from Lemma 1.2 to the proof of Proposition 1.3 below is the same as the one used in [6].

PROPOSITION 1.3. *If M is a closed $K(\Gamma, 1)$ manifold, where Γ is as in* (1.1) *and the 2-group G acts effectively on $Z(N)$, then M bounds.*

Hence, to prove Theorem 1 we may assume that the action of G on $Z(N)$ is not effective. In case $G = Z_2$, this means that the action is trivial. Note that then $\Sigma_G = \Sigma$. Should there be a point $x \in M$ fixed under Σ, the injective homomorphism ϕ determined by x implies that $\Sigma_G = Z_2$, so the center of L is one-dimensional. We turn to that case in the following section.

2. Proof of Theorem 1. Final case. Our proof will be based on the following result.

PROPOSITION 2.1. *Let Γ be a finitely generated torsion-free group and N be a nilpotent subgroup of index 2. Assume that the center of $N, Z(N)$ is ∞-cyclic; furthermore, assume that $Z(N)$ is the center of Γ. Then there exists a closed smooth aspherical manifold M^n and an involution $T: M \to M$ such that*

(a) $\pi_1(M) \simeq \Gamma$;

(b) *the fixed point set F of T is the disjoint union of a finite number of closed manifolds F_i, all having the same dimension $n-k$;*

(c) *the normal bundle ν of F in M is isomorphic to the Whitney sum of the same line bundle η with itself k times, i.e., $\nu = \eta \oplus \cdots \oplus \eta$ (k summands);*

(d) *the Stiefel-Whitney classes $\omega_i(F) = 0$ for all $i > 0$.*

We defer the proof of this proposition to the end of this section after we have used it to complete the proof of Theorem 1. Note we may assume that the manifold M^n referred to in Theorem 1 is the one given by Proposition 2.1.

Let $\psi: F \to \mathrm{BO}(k)$ be the classifying map for the normal bundle ν of F in M. By a theorem of Conner and Floyd [1], it suffices to show that $[\psi]$(= the cobordism class of ψ) is the zero element in $\mathfrak{N}_{n-k}(\mathrm{BO}(k))$. By a result of Conner and Floyd [1], $[\psi] = 0$ if and only if all its Stiefel-Whitney numbers vanish, i.e., for each cohomology class $x \in H^i(\mathrm{BO}(k); Z_2)$ and each monomial v in the Stiefel-Whitney classes $1, \omega_1(F), \ldots, \omega_{n-k}(F)$ of degree $(n-k)-i$, we must show that the Kronecker product $\langle \psi^*(x) \cup v, [F] \rangle = 0$. But by Proposition 2.1(d), $\omega_j(F) = 0$ if $j > 0$; hence, we may assume that $\omega = 1$. Also note that $\psi^*(x)$ is a monomial in the Stiefel-Whitney classes of ν. Let ω denote the total Stiefel-Whitney class. Since

$$(2.1) \qquad \omega(\nu) = \omega(\eta \oplus \cdots \oplus \eta) = (\omega(\eta))^k = (1 + \omega_1(\eta))^k = \sum_{i=1}^{k} \binom{k}{i} (\omega_1(\eta))^i,$$

we may assume that $\psi^*(x) = (\omega_1(\eta))^{n-k}$. Therefore, it suffices to show that

$$(2.2) \qquad \langle (\omega_1(\eta))^{n-k}, [F] \rangle = 0.$$

First, we will assume that $n-k$ is even; i.e., $n-k=2j$ for some integer j. Since $\omega_i(F)=0$ for $i>0$, it is a consequence of Wu's formula that

$$\langle \mathrm{Sq}^i(y), [F]\rangle = 0, \quad \text{for all } i>0,\ y\in H^*(F;Z_2). \tag{2.3}$$

Let Sq denote $\mathrm{Sq}^0+\mathrm{Sq}^1+\mathrm{Sq}^2+\cdots$; we have

$$\begin{aligned}\mathrm{Sq}((\omega_1(\eta))^j) &= \mathrm{Sq}(\omega_1(\eta))^j \\ &= [\omega_1(\eta)+(\omega_1(\eta))^2]^j \\ &= (\omega_1(\eta))^j(1+\omega_1(\eta))^j \\ &= (\omega_1(\eta))^j\left[\sum_{i=1}^{j}\binom{j}{i}(\omega_1(\eta))^i\right] \\ &= \sum_{i=1}^{j}\binom{j}{i}(\omega_1(\eta))^{i+j}.\end{aligned} \tag{2.4}$$

Consequently,

$$\mathrm{Sq}^j((\omega_1(\eta))^j) = (\omega_1(\eta))^{2j} = (\omega_1(\eta))^{n-k}. \tag{2.5}$$

Formulas (2.3) (with $y=(\omega_1(\eta))^j$ and $i=j$) and (2.5) yield that assertion (2.2) is true when $n-k$ is even.

Hence for the rest of the argument we will assume that $n-k$ is odd. Next assume that k is even; i.e., $k=2j$ for some integer j. Then

$$\omega(\nu) = [\omega(\eta\oplus\eta)]^j = [1+2\omega_1(\eta)+(\omega_1(\eta))^2]^j = [1+(\omega_1(\eta)^2)]^j; \tag{2.6}$$

hence $\omega_s(\nu)=0$ if s is odd. Consequently, $\psi^*(x)$ has even degree but M has odd dimension. Therefore, $\langle\psi^*(x), [F]\rangle=0$ and M bounds if either k is even or $n-k$ is even.

Thus we will assume that both k and $n-k$ are odd. To finish the proof of Theorem 1, we make use of the main result of the Kosniowski-Stong paper [7]. Referring to this result, let $f(x_1,\ldots,x_n)$ be the Newton polynomial $x_1^{k-1}+\cdots+x_n^{k-1}$. Since $k-1<n$, the Stiefel-Whitney number of M associated to f is 0. Hence, their formula becomes

$$0=\left\langle \frac{f(1+\omega_1(\eta),\ldots,1+\omega_1(\eta),0,0,\ldots,0)}{(1+\omega_1(\eta))^k}, [F]\right\rangle. \tag{2.7}$$

In (2.7), the first k variables $x_1,\ldots,x_k$ have been replaced by $1+\omega_1(\eta)$ and $x_{k+1},\ldots,x_n$ by 0. Consequently, (2.7) becomes

$$\left\langle \frac{k(1+\omega_1(\eta))^{k-1}}{(1+\omega_1(\eta))^k}, [F]\right\rangle = 0. \tag{2.8}$$

Since k is odd, we obtain

$$\left\langle \frac{1}{(1+\omega_1(\eta))}, [F]\right\rangle = 0. \tag{2.9}$$

But

$$\frac{1}{1+\omega_1(\eta)}=1+\omega_1(\eta)+\cdots+(\omega_1(\eta))^{n-k}; \tag{2.10}$$

hence, (2.9) and (2.10) imply (2.2), showing that M bounds. This completes the proof of Theorem 1.

PROOF OF PROPOSITION 2.1. As in Section 1, take $L'=L\rtimes G$ and $M=\Gamma\backslash L'/G$. The center $Z(L)$ intersects N in $Z(N)$, and $Z(L)$ is isomorphic to the additive group of real numbers. Note that G is a cyclic group of order 2; let $a\in G$ be its generator. Conjugation by a acts trivially on $Z(L)$, hence $Z(L)$ is contained in the center of L'. Let $b\in Z(L)$ be an element such that b^2 generates $Z(N)$. We define $T: M\to M$ by $T(\Gamma xG)=\Gamma bxG$. Since L is diffeomorphic to $\mathbf{R}^n$ (for some n), M is a closed aspherical manifold and $\pi_1(M)\simeq\Gamma$. Since b is in the center of L', T is well defined and is clearly a smooth involution on M. Note that F, which is the fixed point set of T, is a closed smooth manifold. Suppose $T(\Gamma xG)=\Gamma xG$; this happens if and only if $bx=\gamma xa$ for some $\gamma\in\Gamma$; i.e.,

$$F=\{\Gamma xG\colon b(xax^{-1})\in\Gamma\}. \tag{2.11}$$

If F is empty, then Proposition 2.1 is verified. Hence we may assume that $F\neq\varnothing$. Then there is an $x\in L'$ such that $b(xax^{-1})\in\Gamma$. But $\{e, xax^{-1}\}$ is another maximal compact subgroup of L'. Hence we may assume that G was chosen so that $ba\in\Gamma$.

Consider the closed nilmanifold $\tilde{M}=N\backslash L$; the map $p\colon\tilde{M}\to M$ given by $p(Nx)=\Gamma xG$ is a 2-sheeted covering. There are involutions $f, g\colon\tilde{M}\to\tilde{M}$ described by

$$f(Nx)=Nxb \quad\text{and}\quad g(Nx)=Naxa, \tag{2.12}$$

and the group of four motions $\{\text{id}, f, g, fg\}$ is isomorphic to the Klein 4-group. Also, $\{\text{id}, fg\}$ is the group of covering transformations for $p\colon\tilde{M}\to\tilde{M}$. Let $\tilde{F}=$ the fixed point set of g; then $p\colon\tilde{F}\to F$ is a 2-sheeted covering. The line bundle η posited in Proposition 2.1(d) will be the associated line bundle to the principal O(1)-bundle $p\colon\tilde{F}\to F$. The group $H=\{\text{id}, f\}$ acts freely on $\tilde{M}$ with orbit space $M'=\tilde{M}/H$. Note that M' can be viewed in another way. Namely, let $N'=N\cup Nb$; then N' is a discrete torsion-free cocompact subgroup of L which contains N as a subgroup of index 2. Then $M'=N'\backslash L$ and hence is a closed nilmanifold. Note that $\tilde{F}$ is left invariant by f since $fg=gf$. Hence $F'=\tilde{F}/H$ is a submanifold of M'. In fact, F' is diffeomorphic to F since f and fg agree on $\tilde{F}$. Also, $F'=$ fixed point set of $h\colon N'\backslash L\to N'\backslash L$, where h is described by

$$h(N'x)=N'axa. \tag{2.13}$$

To complete the proof of Proposition 2.1, we need the following result.

LEMMA 2.2. *Let K be a (connected) simply connected nilpotent Lie group and S be a discrete cocompact torsion-free subgroup of K. Also, let $\alpha\colon K\to K$ be an automorphism such that $\alpha^2=\text{id}$ and $\alpha(S)=S$. Consider the involution $\hat{\alpha}\colon S\backslash K\to S\backslash K$ given by $\hat{\alpha}(Sx)=S\alpha(x)$. Then*

(i) *all components of the fixed point set of* $\hat{\alpha}$ *have the same dimension, and*
(ii) *the normal bundle of the inclusion of the fixed point set of* $\hat{\alpha}$ *into* $S\backslash K$ *is trivial.*

Let us complete the proof of Proposition 2.1 before proving Lemma 2.2. Apply Lemma 2.2 in the case where $K=L$, $S=N'$ and $\hat{\alpha}=h$. This immediately yields part (b) of Proposition 2.1. To obtain part (d), recall that $N'\backslash L$ is parallelizable. But $\tau(F')\oplus\nu'$ is the pullback of $\tau(N'\backslash L)$ under the inclusion map, where ν' denotes the normal bundle and $\tau(\ )$ the tangent bundles. Since both ν' and $\tau(N'\backslash L)$ are trivial bundles and the Stiefel-Whitney classes are stable characteristic classes, (d) follows.

To see (c), use the fact that ν' is trivial to construct everywhere nonzero cross sections $X_1,\ldots,X_k$ to $\tilde{\nu}=$ the normal bundle of $\tilde{F}$ in $\tilde{M}$ such that

$$X_i(f(Nx))=df(X_i(Nx)) \tag{2.14}$$

for each $Nx\in\tilde{F}$ (and $i=1,2,\ldots,k$), where df denotes the derivative of f. Then (2.14) implies

$$X_i(gf(Nx))=-d(gf)(X_i(Nx)) \tag{2.15}$$

for each $Nx\in\tilde{F}$ and $i=1,2,\ldots,k$. But (2.15) immediately implies (c). This completes the proof of Proposition 2.1 once we have proved Lemma 2.2 which we now proceed to do.

Let F denote the fixed point set of $\hat{\alpha}$. We proceed by induction on $n=\dim K$, i.e., suppose Lemma 2.2 is true if $\dim K<n$. Let D denote the center of K and C denote the center of S; then $C=D\cap S$ and D is isomorphic to the additive group of the real vector space $\mathbf{R}^m$ for some $m>0$. Consequently, $C\backslash D$ is an m-dimensional torus denoted by T^m. Consider the principal T^m-bundle

$$T^m\to S\backslash K\xrightarrow{p}\bar{S}\backslash\bar{K}, \tag{2.16}$$

where $\bar{K}$ is the (connected) simply connected nilpotent Lie group $D\backslash K$ and $\bar{S}$ is the cocompact discrete torsion-free subgroup $C\backslash S$. Consider the automorphism $\beta\colon D\backslash K\to D\backslash K$ defined by $\beta(Dx)=D\alpha(x)$. Clearly, $\beta^2=\mathrm{id}$ and $\beta(C\backslash S)=C\backslash S$. Hence, by our inductive assumption, the fixed point set $\bar{F}$ of $\hat{\beta}\colon\bar{S}\backslash\bar{K}\to\bar{S}\backslash\bar{K}$ satisfies the conclusion of Lemma 2.2.

Note that $p(F)\subset\bar{F}$; we proceed to show that $p(F)$ is the union of connected components of $\bar{F}$. Since $p(F)$ is compact, it suffices to show that $p(F)$ is open in $\bar{F}$.

Let $x_0\in F$ and U be an open neighborhood of $p(x_0)$ in $\bar{F}$ such that the bundle $p\colon S\backslash K\to\bar{S}\backslash\bar{K}$ is trivial when restricted to U. In particular, let $s\colon U\to S\backslash K$ be a cross section to p such that $s(p(x_0))=x_0$; then the composite $\hat{\alpha}s\colon U\to S\backslash K$ is a second cross section such that $\hat{\alpha}(s(p(x_0)))=x_0$. Hence, there exists a continuous function $\phi\colon U\to T^m$ such that

$$\hat{\alpha}(s(y))=s(y)\phi(y) \tag{2.17}$$

for all $y\in U$ and $\phi(p(x_0))=e=$ the identity element of the group T^m. Applying $\hat{\alpha}$ to (2.17) yields

(2.18) $$s(y)=\hat{\alpha}(s(y))\,\hat{\alpha}(\phi(y))$$

for all $y\in U$. Combining (2.17) and (2.18) yields

(2.19) $$\hat{\alpha}(\phi(y))=\phi(y)^{-1}$$

for all $y\in U$. In a sufficiently small neighborhood of $p(x_0)$, $s(y)$ has a unique small square root; this fact and (2.19) together imply there is an open neighborhood V of $p(x)$ in U and a continuous function $\psi\colon V\to T^m$ such that

(2.20)
$$\text{(i)}\quad \psi(p(x_0))=e,$$
$$\text{(ii)}\quad (\psi(y))^2=\phi(y)\quad\text{and}$$
$$\text{(iii)}\quad \hat{\alpha}(\psi(y))=\psi(y)^{-1}$$

for all $y\in V$. Consider the point $s(y)\psi(y)\in S\backslash K$ for each $y\in V$ and note that

(2.21)
$$(i)\quad p(s(y)\psi(y))=p(s(y))=y\quad\text{and}$$
$$\begin{aligned}(ii)\quad \hat{\alpha}(s(y)\psi(y))&=\hat{\alpha}(s(y))\hat{\alpha}(\psi(y))\\ &=s(y)\phi(y)\hat{\alpha}(\psi(y))\quad\text{(by (2.17))}\\ &=s(y)\phi(y)\psi(y)^{-1}\quad\text{(by (2.20.iii))}\\ &=s(y)(\psi(y))^2\psi(y)^{-1}\quad\text{(by (2.20.ii))}\\ &=s(y)\psi(y).\end{aligned}$$

Consequently, $V\subset p(F)$; hence, we have shown that $p(F)$ is a union of components of $\bar{F}$. In particular, $p(F)$ is a closed manifold and each component of $p(F)$ has the same dimension.

Note that $\hat{\alpha}\colon T^m\to T^m$ is a group homomorphism; let A be the fixed point set of $\hat{\alpha}\,|\,T^m$. Then A is a compact abelian Lie subgroup of T^m; consequently, all components of A have the same dimension. But

(2.22) $$A\to F\xrightarrow{p}p(F)$$

is a principal A-bundle; hence all components of F have the same dimension which verifies part (i) of Lemma 2.2.

Consider the diagram

(2.23)
$$\begin{array}{ccc} A & \subset & T\\ \downarrow & & \downarrow\\ F & \subset & S\backslash K\\ \downarrow & & \downarrow\\ \bar{F} & \subset & \bar{S}\backslash\bar{K}.\end{array}$$

Let $\bar{\nu}$ denote the normal bundle of $\bar{F}$ in $\bar{S}\backslash\bar{K}$. By our inductive assumption $\bar{\nu}$ is a trivial bundle. But $\nu=p^*(\bar{\nu})\oplus\bar{\nu}_f$, where $\bar{\nu}_f$ denotes the normal bundle along the fibers, i.e., for each point $Sx\in F$, the fiber $\bar{\nu}_f(Sx)$ of $\bar{\nu}_f$ over Sx is defined as the following quotient vector space. Let k_{Sx} and k'_{Sx} denote the kernels of

(2.24) $$dp\colon \tau_{Sx}(S\backslash K) \to \tau_{p(Sx)}(\bar{S}\backslash\bar{K}) \quad \text{and}$$
$$dp\colon \tau_{Sx}(F) \to \tau_{p(Sx)}(\bar{F}), \quad \text{respectively.}$$

Then $\bar{\nu}_f(Sx)$ is the quotient vector space defined by

(2.25) $$\bar{\nu}_f(Sx) = k_{Sx}/k'_{Sx}.$$

It remains to show that $\bar{\nu}_f$ is the trivial bundle. Let $v_1, \dots, v_k, v_{k+1}, \dots, v_m$ be a basis for $\tau_e(T^m)$ such that $v_{k+1}, \dots, v_m$ is a basis for $\tau_e(A)$. Define k linearly independent cross sections $X_1, \dots, X_k$ to $\bar{\nu}_f$ by

(2.26) $$X_i(Sx) = dh_{Sx}(v_i) + k'_{Sx}$$

for each $Sx \in F$, where $h_{Sx}\colon T^m \to S\backslash K$ is defined by

(2.27) $$h_{Sx}(t) = Sxt, \text{ for each } t \in T^m$$

and $dh_{Sx}\colon \tau_e(T^m) \to \tau_{Sx}(S\backslash K)$ is the derivative of h_{Sx} at $e \in T^m$. This completes the proof of Lemma 2.2. □

REFERENCES

1. P. E. Conner and E. E. Floyd, *Differentiable periodic maps,* Springer, Berlin, 1964.
2. F. T. Farrell and W. C. Hsiang, *The Whitehead group of poly-(finite or cyclic) groups,* J. London Math. Soc. (2) 24 (1982), 308–324.
3. M. W. Gordon, *The unoriented cobordism classes of compact flat Riemannian manifolds,* J. Differential Geom. 15 (1980), 81–90.
4. M. L. Gromov, *Manifolds of negative curvature,* J. Differential Geom. 13 (1978), 223–230.
5. ———, *Almost flat manifolds,* J. Differential Geom. 13 (1978), 231–241.
6. G. C. Hamrick and D. C. Royster, *Flat Riemannian manifolds are boundaries,* Invent. Math. 66 (1982), 405–413.
7. C. Kosniowski and R. E. Stong, *Involutions and characteristic numbers,* Topology 17 (1978), 309–330.
8. A. I. Mal'cev, *On a class of homogeneous spaces,* Izv. Akad. Nauk SSSR Ser. Mat. 13 (1949), 9–32.

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