SOME CONSTANTS ASSOCIATED WITH THE RIEMANN ZETA-FUNCTION

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The following proposition was stated without proof by Ramanujan [3, p. 134] and Hardy [2]; two proofs have recently been given by Chowla and the author [1].

THEOREM: *The Riemann zeta-function has the representation*

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n,$$

where

$$\gamma_n = \lim_{N\to\infty}\left[\sum_{t=1}^{N} \frac{\log^n t}{t} - \int_1^N \frac{\log^n x}{x}dx\right].$$

The purpose of this paper is to investigate the magnitudes and signs of the constants γ_n.

1. THE SIGNS OF THE CONSTANTS

THEOREM 1. *Infinitely many* γ_n *are positive, and infinitely many are negative.*

From the identity

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{s\pi}{2}\,\Gamma(1-s)\zeta(1-s)$$

it follows that

(1) $$\zeta(1-2m) = (-1)^m \frac{2(2m-1)!}{(2\pi)^m}\zeta(2m) \quad (m = 1, 2, 3, \cdots).$$

Comparing this with the power series representation of $\zeta(s)$, one obtains the relation

(2) $$\zeta(1-2m) = -\frac{1}{2m} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(-2m)^n = -\frac{1}{2m} + \sum_{n=0}^{\infty} \frac{\gamma_n}{n!}(2m)^n.$$

From (1) it follows that the sign of $\zeta(1-2m)$ is $(-1)^m$, since all other factors are positive. Hence, if m is positive and even, (2) shows that the γ_n can not all be non-positive. Assume that there exist only a finite number of positive γ_n, and that N is

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the largest integer for which $\gamma_n > 0$. Let M be the smallest integer larger than N for which $\gamma_M < 0$. Then each nonzero term in the series

$$\sum_{n=M}^{\infty} \frac{\gamma_n}{n!}(2m)^n$$

is negative, and in particular

$$\frac{\gamma_M (2m)^M}{M!} < 0. \tag{3}$$

On the other hand

$$-\frac{1}{2m} + \sum_{n=0}^{N} \frac{\gamma_n}{n!}(2m)^n < k(N+1)(2m)^N, \quad \text{where } k = \max_{0<n<N} \gamma_n;$$

here k is positive, since $\gamma_N > 0$. Now let $k' = -\gamma_M > 0$, and choose m large enough so that

$$2m > \frac{k}{k'}(N+1)M!.$$

Then

$$(2m)^{M-N} > \frac{k}{k'}(N+1)M!,$$

and

$$\frac{k'(2m)^M}{M!} > k(N+1)(2m)^N,$$

which shows that $\zeta(1-2m) < 0$, which in turn contradicts (1) for even m. Hence there exists an infinite number of positive γ_n. A similar argument holds under the assumption that there exists only a finite number of negative γ_n, and a contradiction is reached by taking m odd. This proves the theorem.

Approximate values of γ_n can be computed from the expression

$$\sum_{t=1}^{r} \frac{\log^n t}{t} - \frac{\log^{n+1} r}{n+1} - \frac{\log^n r}{2r},$$

where r is chosen large enough so that the second derivative of $(\log^n x)/x$ is positive for $x > r$. This gives the following values:

$$\gamma_1 = -0.073, \quad \gamma_2 = -0.516, \quad \gamma_3 = -0.147, \quad \gamma_4 = 0.002.$$

2. AN UPPER BOUND FOR γ_n

Various representations of γ_n can be obtained by applying the following easily derived sum formula [5, p. 31]:

Let f(x) have a continuous second derivative in $Q \leq x \leq R$. Set $\rho(x) = 1/2 - \{x\}$, where $\{x\}$ denotes the fractional part of x, and let $\sigma(x) = \int_0^x \rho(t)\,dt$. Then

$$\sum_{Q<x\leq R} f(x) = \int_Q^R f(x)\,dx + \rho(R)f(R) - \rho(Q)f(Q) - \sigma(R)f'(R) + \sigma(Q)f'(Q) + \int_Q^R \sigma(x)f''(x)\,dx.$$

In the present case, let $Q = 1$ and $f_n(x) = (\log^n x)/x$, and take the limit as $R \to \infty$ to obtain the formula

$$\gamma_n = \int_1^\infty \sigma(x) f_n''(x)\,dx. \tag{4}$$

Since

$$\rho(t) = \sum_{k=1}^\infty \frac{\sin 2k\pi t}{k\pi}$$

when t is not an integer,

$$\sigma(x) = -\frac{1}{2\pi^2}\sum_{k=1}^\infty \frac{\cos 2k\pi x}{k^2} + \frac{1}{12}.$$

Substituting this in (4) and integrating by parts twice, one obtains the representation

$$\gamma_n = 2\sum_{k=1}^\infty \int_1^\infty \cos(2k\pi x)\, f_n(x)\,dx. \tag{5}$$

Another representation can be obtained by considering the function $(\log^{n+1} x)/(n+1)$. By a mean-value theorem,

$$\frac{1}{n+1}\left[\log^{n+1}(k+1) - \log^{n+1} k\right] = \frac{\log^n k}{k} + \frac{1}{2}\left[\frac{n\log^{n-1} x_k - \log^n x_k}{x_k^2}\right],$$

where $x_k = k + \theta_k$ $(0 < \theta_k < 1)$. On summing both sides from $k = 1$ to $k = x - 1$ and taking the limit as $x \to \infty$, one gets the representation

$$\gamma_n = \frac{1}{2}\sum_{k=1}^\infty \frac{\log^n x_k - n\log^{n-1} x_k}{x_k^2}. \tag{6}$$

LEMMA.

$$\sum_{k=1}^{\infty} \frac{\log^n x_k}{x_k^2} = n! + O\left\{\left(\frac{n}{2e}\right)^n\right\} \quad (x_k = k + \theta_k,\ 0 \leq \theta_k \leq 1).$$

This lemma follows from the relation

$$\sum_{k=1}^{\infty} \frac{\log^n x_k}{x_k^2} = \int_1^{\infty} \frac{\log^n x}{x^2} dx + O\left\{\max_{1<x<\infty} \left(\frac{\log^n x}{x^2}\right)\right\},$$

since the integral equals $n!$ and the maximum involved occurs at $\exp n/2$.

THEOREM 2.

$$\gamma_n = \left(\frac{n}{2e}\right)^n \epsilon(n) \quad (|\epsilon(n)| < 1).$$

The theorem follows from (6) and the lemma. It can also be obtained by considering the absolute value of the integral in (4).

Infinitely many of the constants γ_n are much smaller than Theorem 2 indicates. Using the fact that $\zeta(s) - 1/(s - 1)$ is an entire function of order one, and applying the theorem [4, p. 253] which states that the necessary and sufficient condition that $\sum_0^{\infty} a_n z^n$ be an entire function of order r is that

$$\liminf_{n \to \infty} \frac{\log 1/|a_n|}{n \log n} = \frac{1}{r},$$

one obtains the result

$$\liminf_{n \to \infty} \frac{\log n! - \log |\gamma_n|}{n \log n} = 1.$$

This implies that

$$\liminf_{n \to \infty} \frac{\log |\gamma_n|}{n \log n} = 0,$$

which in turn implies

THEOREM 3. *If* $\epsilon > 0$, *then the inequalities*

$$n^{-\epsilon n} < |\gamma_n| < n^{\epsilon n}$$

hold for infinitely many n.

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