ON A PROBLEM OF LUSIN

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1. Let the function f(z) be regular for $|z| < 1$. Let us denote by $\mathfrak{L}(f)$ the set of those points ζ on $|z| = 1$ with the property that the interior of any circle in $|z| \leq 1$ and tangent to $|z| = 1$ at ζ is mapped by f onto a Riemann domain of infinite area. Lusin [5] conjectured the existence of a bounded function for which every point of $|z| = 1$ belongs to $\mathfrak{L}(f)$. Lohwater and Piranian [4] have given an example of such a function, their method being based on the use of lacunary Taylor series. In this paper we construct by geometric methods a function with the properties stated in the following

THEOREM. *There exists a function* f(z), *regular and bounded in* $|z| < 1$, *such that*

(i) *the set* $\mathfrak{L}(f)$ *constitutes the whole circumference* $|z| = 1$, *apart possibly from a set of measure zero and of the first category,*

(ii) *wherever the radial limit* $\lim_{r \to 1} f(re^{i\theta})$ *exists, it has modulus* 1.

2. Let D denote the domain obtained from the unit circle $|w| < 1$ by inserting at each point $\exp(\pi i r/s)$ a radial slit of logarithmic length $s^{-1/2}$, where r, s are integers, $s > 0$, $|r| \leq s$, and the fraction r/s is in its lowest terms. We see at once that no point of such a slit in $|w| < 1$ is a limit point of points on other slits. Further, no point on $|w| = 1$ is a vertex of a triangle in D. Let D be mapped conformally onto $|z| < 1$ so that the point $w = 0$ goes into the point $z = 0$. Then the set of boundary points of D on $|w| = 1$ goes into a set E on $|z| = 1$ which by a result of Lavrentiev [3, p. 822] has angular Lebesgue measure zero. The use of the result of Lavrentiev in the construction of D was suggested to me by G. Piranian. I had previously used a potential theoretic method. Domains of this general type were earlier considered in other connections by Green [1], Schaeffer [7] and af Hällström [2].

3. The boundary of the domain D consists of the circle $|w| = 1$, together with a countable set of radial slits s_i $(i = 1, 2, \cdots)$. We obtain from it a Riemann surface $\mathfrak{R}$ by the following construction. For each slit s_i $(i = 1, 2, \cdots)$ we take a domain D_i obtained from D by reflection in the diameter bearing s_i. Then we cross-join D_i to D along s_i and the corresponding slit for D_i. Let the remaining boundary slits of D_i be denoted by s_{ij} $(j = 1, 2, \cdots; j \neq i)$ corresponding to s_j. Let D_{ij} be obtained from D by rotation about the origin in such a way that the slit s_j is carried into the slit s_{ij}. We cross-join D_{ij} to D_i along this slit for each admissible value of j. We continue in this way, the images of D being obtained alternately by sense-reversing and sense-preserving operations. By this process we obtain a Riemann surface $\mathfrak{R}$ which has no relative boundary over $|w| < 1$. It is evidently simply-connected and of hyperbolic type. Thus there exists a function f(Z) regular for $|Z| < 1$ and mapping $|Z| < 1$ in a (1, 1) manner on the surface $\mathfrak{R}$. We may assume that this function carries the origin $Z = 0$ into the point covering the origin $w = 0$ in the initial sheet D of $\mathfrak{R}$.

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The surface $\mathfrak{R}$ is invariant under the following transformations: consider any radial segment s on $\mathfrak{R}$ along which two replicas D' and D" of D have been cross-joined; with any point P in D' we associate the point in D" which covers the point in the w-plane obtained from the point in the w-plane covered by P by reflection in the line above which the radial segment s lies, and conversely. The points of the replicas of D cross-joined to D' and D" along corresponding slits are associated in the same manner. Continuing in this way and extending the mapping to the points on the cross-joins by continuity we obtain the transformations in question. In particular, the points of the cross-join corresponding to the initial segment s are each invariant.

Let $\mathfrak{T}$ denote the set of corresponding transformations in $|Z| < 1$. Each of these is an anticonformal transformation of $|Z| < 1$ onto itself. It is thus the conjugate of a linear transformation. Also, it leaves pointwise invariant a cross-cut running from $|Z| = 1$ back to $|Z| = 1$ inside $|Z| < 1$ corresponding to the appropriate cross-join on $\mathfrak{R}$. The conjugate of a linear transformation carrying $|Z| < 1$ onto itself can leave such an arc pointwise invariant only if the arc lies on a circle orthogonal to $|Z| = 1$ and the mapping in question is inversion in this circle.

4. In the mapping f of $|Z| < 1$ onto $\mathfrak{R}$, the set going into the initial sheet D of $\mathfrak{R}$ is a subdomain Δ of $|Z| < 1$ bounded by a countable set of open arcs c_i $(i = 1, 2, \cdots)$ on circles orthogonal to $|Z| = 1$ (one for each slit s_i, $i = 1, 2, \cdots$) together with a set H on $|Z| = 1$. If we combine the inverse of this mapping with the mapping of $|z| < 1$ onto D, we obtain a mapping ϕ of $|z| < 1$ onto Δ. The homeomorphic extension of the mapping ϕ to $|z| = 1$ parametrizes the boundary C of Δ as a Jordan curve. Since the length of the arc of a circle orthogonal to $|Z| = 1$ and contained in $|Z| \leq 1$ is for a suitable constant q less than q times the length of the arc which the circle intercepts on $|Z| = 1$, the curve C is rectifiable. By a theorem of F. and M. Riesz [6] the set H, as the image under ϕ of the set E of angular measure zero on $|z| = 1$, has linear measure zero on C and thus angular measure zero on $|Z| = 1$.

We consider now the transformations in the family $\mathfrak{T}$ which carry Δ into domains adjacent to it across the arcs c_i $(i = 1, 2, \cdots)$. Next we take the transformations in $\mathfrak{T}$ carrying these domains into further adjacent domains. Continuing indefinitely in this way we sweep out the domain $|Z| < 1$. The images of H under the compositions of these transformations form a countable set of sets, each of angular measure zero. Thus their union K has angular measure zero. Further, H and each of its images is nowhere dense on $|Z| = 1$; thus K is of the first category.

5. Let Q be any point in the complement L of the set K on $|Z| = 1$. We will now show that the interior $\mathfrak{J}$ of any circle lying in $|Z| \leq 1$ and tangent to $|Z| = 1$ at Q is mapped by the function f(z) onto a Riemann domain of infinite area. There exists a sequence Δ_n $(n = 1, 2, \cdots)$ of distinct domains such that $\Delta_1 = \Delta$, Δ_{n+1} is the image of Δ_n under a transformation in $\mathfrak{T}$, and such that all domains of sufficiently large index lie in a given neighborhood of Q relative to $|Z| < 1$.

Let $|Z| < 1$ be mapped conformally onto the upper half-plane $\Im W > 0$ in such a way that Q goes into the point at infinity. The domains Δ_n are carried into domains Λ_n $(n = 1, 2, \cdots)$, where Λ_n and Λ_{n+1} have as common boundary a semi-circular arc Γ_n centered on the real W-axis. The domains Λ_n and Λ_{n+1} are interchanged by inversion in Γ_n, and the distance of Γ_n from $W = 0$ tends to infinity with n. The interior $\mathfrak{J}$ of a circle lying in $|Z| \leq 1$ and tangent to $|Z| = 1$ at Q goes into a half-plane $\Im W > t$, $t > 0$.

For n large enough, Λ_n will meet this half-plane. In the intersection we take some set M_n of positive measure, say a closed circular disc. Let the successive images of M_n under inversion in the circles Γ_m $(m \geq n)$ be denoted by M_m $(m > n)$. These sets are disjoint and all lie in the half-plane $\Im W > t$. Let the corresponding sets in $|Z| < 1$ be denoted by N_m $(m \geq n)$. These sets are disjoint and lie in $\Im$. The images $f(N_m)$ of the sets N_m $(m \geq n)$ are disjoint on $\mathfrak{R}$ and congruent under Euclidean transformations. Thus they all have a fixed positive area. This proves that the image of $\Im$ has infinite area. Hence the set L is contained in $\mathfrak{L}(f)$. Since L has angular Lebesgue measure 2π, so does $\mathfrak{L}(f)$.

Finally, suppose that $\lim_{r \to 1} f(re^{i\theta})$ exists for some real θ and has the value α. Then from the form of the surface $\mathfrak{R}$ it is clear that $|\alpha| < 1$ is impossible. Thus we must have $|\alpha| = 1$. Hence the function f provides the desired example.

It should be remarked that, if we use in addition the symmetry of the domain D constructed in §2 under reflection in the real axis, we can arrange that the function $f(Z)$ should have real coefficients in its Taylor expansion about $Z = 0$.

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