NOTE ON CYLINDRIC ALGEBRAS AND POLYADIC ALGEBRAS

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Both the cylindric algebras of Tarski and Thompson [2] and the polyadic algebras of Halmos [1] are algebraic counterparts of the first-order functional calculi. Polyadic algebras are defined in terms of an infinite set of operators. In this note, we define a class of algebras closely related to polyadic algebras but somewhat simpler in that the operators are produced by six generators and the algebras are defined in terms of these generators. The introduction of these generators requires us to specialize our system so that it is based on a denumerable rather than an arbitrary index set; but the specialization does not restrict the functional calculi which are represented by the algebras of our system. We axiomatize the substitution and quantification operators of Halmos in terms of the six generators and by means of postulates expressed in equational form. Our algebras are also related to cylindric algebras in that they contain diagonal elements. These elements are generated by the operators.

We are concerned with an abstract boolean algebra A with operators and their defining relations. We denote *and, or, not,* respectively, by $\wedge$, $\vee$, $\sim$, and the zero and unit elements by 0, 1. We interpret appropriate combinations of the generators as producing the substitutions and quantifications of the functional calculi. In order to create an intuitive picture of the manner in which the substitutions and quantifications are produced, we imagine that the elements of A are functions of variables $\cdots, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \cdots$, and that the values of the functions are propositions. We introduce the operator $\exists$ such that $\exists x$ is interpreted as the existential quantification of the function x with respect to the variable ξ_0. The symbol Q represents an operator such that Qx is interpreted as $x \wedge (\xi_0 = \xi_1)$. In particular Q1, interpreted as $\xi_0 = \xi_1$, is a diagonal element. One can now see that $\exists Qx$ is interpreted as the function which results when ξ_1 is substituted in place of ξ_0 in the function x.

The remaining substitutions and quantifications and the remaining diagonal elements are produced by combining the operations above with permutations of the variables. Thus P is such that Px is interpreted as the function resulting from the interchange of ξ_0 with ξ_1 in the function x, and T is such that Tx is interpreted as the function resulting when each ξ_k in x is replaced by ξ_{k+1}. The inverse of T and the identity operator are denoted respectively by T^{-1}, I. Then P, T, T^{-1}, I generate a group which contains a subgroup isomorphic to the group of finite permutations of the integers. The semigroup of operators of A is generated by $\exists$, Q, P, T, T^{-1}, I.

The operators which produce the substitutions and quantifications are combinations of the generators. We shall introduce symbols for these combinations and symbols for the diagonal elements. Interpretations follow the definitions of the symbols.

D1. ${}_oP_n = (PT)^{n-1} P\, T^{-n+1} = {}_nP_o$ if $n \neq 0$; ${}_nP_n = I$;

${}_mP_n = {}_oP_m\, {}_oP_n\, {}_oP_m$ if $m \neq n$ and $m, n \neq 0$.

D2. $\exists_n = {}_oP_n\, \exists\, {}_oP_n$.

D3. ${}_nQ_n = I$; ${}_mQ_n = {}_oP_m\, P\, {}_oP_n\, Q\, {}_oP_n\, P\, {}_oP_m$ if $m \neq n$.

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D4. ${}_m e_n = {}_m Q_n 1$.

D5. ${}_m S_m = I$; ${}_m S_n = \exists_m {}_m Q_n$ if $m \neq n$.

We interpret ${}_m P_n x$ as the function resulting from the interchange of ξ_m with ξ_n in x, and $\exists_n x$ as the existential quantification of x with respect to ξ_n. The diagonal element ${}_m e_n$ is interpreted as $\xi_m = \xi_n$. It follows readily from the postulates that ${}_m Q_n x = x \wedge {}_m e_n$, and this equation furnishes an interpretation of ${}_m Q_n x$. We interpret ${}_m S_n x$ as the function which results when ξ_m is substituted in place of ξ_n in x.

The algebras A are defined by means of the following twelve postulates. Interpretations are given below.

P1. A is a boolean algebra.

P2. I, T, T^{-1}, P are endomorphisms of A, and I is the identity.

P3. $T T^{-1} = P^2 = I$.

P4. ${}_o P_n^2 = ({}_o P_n \, {}_o P_m)^3 = ({}_o P_n \, {}_o P_m \, {}_o P_n \, {}_o P_k)^2$ if $m, n \neq 0$ and $m \neq n$ and $k \neq m, n$.

P5. $\exists(x \wedge \exists y) = (\exists x) \wedge (\exists y)$.

P6. $x \wedge \exists x = x$.

P7. $\exists 0 = 0$.

P8. $(\exists P)^2 = (P \exists)^2 = \exists P \exists$.

P9. ${}_m P_n \exists = \exists \, {}_m P_n$ if $m, n \neq 0$.

P10. $Q(x \wedge y) = x \wedge Qy = x \wedge QPy$.

P11. $\exists Q \sim x = \sim \exists Q x$.

P12. ${}_o P_n Q \exists = \exists Q \, {}_o P_n P Q \exists$.

Postulates 1 to 7 and Postulate 9 are self-explanatory. By D2, Postulate 8 can be written in the form

$$\exists_1 \exists_0 = \exists_0 \exists_1 = P \exists_1 \exists_0 = \exists_1 \exists_0 P .$$

If we set $y = 1$ in Postulate 10, we get $Qx = x \wedge {}_0 e_1$; and if we set $x = 1$, we get ${}_0 e_1 \wedge y = {}_0 e_1 \wedge y = {}_0 e_1 \wedge Py$ (see D4). From this the general case follows. Postulate 11 can be written in the form ${}_0 S_1 \sim x = \sim {}_0 S_1 x$ (see D5). Postulate 12, together with Postulate 10 and Definition 4, implies that ${}_1 e_n = \exists_0 e_1 \wedge {}_0 e_n$.

The following theorems display the important properties of the algebras we have defined. We omit the proofs, since they require patience rather than ingenuity.

T1. The operators ${}_0 P_n$ generate a group which is isomorphic to the group of finite permutations of the integers.

T2. Postulates 5, 6, 7 hold if $\exists$ is replaced by $\exists_n$.

T3. $\exists_n^2 = \exists_n$, and $\exists_n 1 = 1$.

T4. $x \wedge \exists_n y = x$ implies $(\exists_n x) \wedge (\exists_n y) = \exists_n x$.

T5. $\exists_n (x \vee y) = (\exists_n x) \vee (\exists_n y)$.

T6. ${}_m P_n \exists_k = \exists_k \, {}_m P_n$ if $k \neq m, n$; ${}_n P_m \exists_m = \exists_n \, {}_n P_m$.

T7. ${}_m P_n \exists_m \exists_n = \exists_m \exists_n = \exists_n \exists_m = \exists_n \exists_m \, {}_n P_m$.

T8. The operators ${}_mS_n$ are endomorphisms of the boolean algebra.

T9. ${}_mS_n^2 = {}_mS_n$.

T10. ${}_{m'}P_{n'}\,{}_mS_n = {}_mS_n\,{}_{m'}P_{n'}$ if m, n, m', n' are distinct.

T11. ${}_mP_n\,{}_mS_n = {}_nS_m = {}_nS_m\,{}_mP_n$.

T12. ${}_mP_n\,\exists_n\,{}_mS_n = \exists_n\,{}_mS_n$.

T13. $\exists_m\,{}_me_n = 1$; $\exists_k\,{}_me_n = {}_me_n$ if $k \neq m, n$.

T14. ${}_kP_m\,{}_me_n = {}_ke_n$.

T15. ${}_mS_n\,{}_ne_m = {}_ne_n = 1$.

T16. ${}_me_n = {}_ne_m$.

T17. ${}_me_n = \exists_k\,{}_me_k \wedge {}_ke_n$.

T18. $({}_me_k \wedge {}_ke_n) \wedge {}_me_n = {}_me_k \wedge {}_ke_n$.

We can interpret T 15, 16 and 18 as stating the reflexivity, symmetry and transitivity of the equality. The condition $\exists_n x = x$ can be interpreted as stating that x is independent of ξ_n. Thus we interpret the first half of T3 as implying that $\exists_n x$ is independent of ξ_n, and the second half of T13 as implying that ${}_me_n$ depends only on ξ_m and ξ_n. A locally finite algebra A (see [1]) can be interpreted as one in which each function depends only on a finite number of variables. The formal definition of local finiteness is the following.

D6. An algebra A is said to be locally finite if for each x in A there is an integer N such that $\exists_n x = x$ whenever $|n| > N$.

REFERENCES

1. P. Halmos, *Polyadic Boolean algebras*, Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 296-301.

2. A. Tarski and F. B. Thompson, *Some general properties of cylindric algebras* (abstract), Bull. Amer. Math. Soc. 58 (1952), 65.

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