ON CLOSE-TO-CONVEX UNIVALENT FUNCTIONS

Maxwell O. Reade

1. INTRODUCTION

In a recent paper, W. Kaplan [3] introduced a class of univalent functions which he called close-to-convex. This class, which we shall call K, includes as proper subclasses the well-known convex functions, star maps, and Robertson's functions which are convex in a given direction [6]. In this note, we shall show that the normalized functions of class K have coefficients that verify the so-called Bieberbach conjecture, and we shall obtain a theorem of Study-type which is analogous to a theorem due to Carathéodory [2]. In addition, we introduce a class of functions we shall call close-to-star functions; these bear the same relation to Kaplan's close-to-convex functions that Robertson's analytic functions star-like in one direction bear to his analytic functions convex in some direction.

2. THE BIEBERBACH CONJECTURE FOR CLOSE-TO-CONVEX FUNCTIONS

If $f(z)$ is analytic for $|z| < 1$, and if $f'(z) \neq 0$ for $|z| < 1$, then $f(z)$ is said to be close-to-convex if and only if there exists a univalent convex function $\phi(z)$ such that

$$\Re \frac{f'(z)}{\phi'(z)} \geq 0 \tag{1}$$

holds in $|z| < 1$. The class of close-to-convex functions we shall denote by K. It was shown by Kaplan that (1) may be replaced by

$$\int_{\theta_1}^{\theta_2} \Re \left(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta > -\pi, \tag{2}$$

which must hold for all $\theta_1 < \theta_2$ and for all $0 \leq r < 1$. For these functions we have the following result.

THEOREM 1. *Let* $f(z) \equiv \Sigma_0^\infty a_n z^n$ *be close-to-convex for* $|z| < 1$. *Then the coefficients satisfy the inequality*

$$|a_n| \leq n|a_1| \quad (n = 1, 2, 3, \cdots). \tag{3}$$

Proof. Since $f(z)$ is close-to-convex, there exists a univalent convex function $\phi(z)$ satisfying (1). Without loss of generality, we may assume that $f(z)$ and $\phi(z)$ are normalized, that is

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$$f(z) \equiv z + \sum_2^\infty \alpha_n z^n \quad \text{and} \quad \phi(z) \equiv z + \sum_2^\infty c_n z^n.$$

If we write

$$\frac{f'(z)}{\phi'(z)} \equiv g(z) \equiv 1 + \sum_1^\infty b_n z^n, \tag{4}$$

we obtain the following relations among the coefficients:

$$(n+1)\alpha_{n+1} = b_n + \sum_{k=1}^{n-1} (k+1) c_{k+1} b_{n-k} + (n+1) c_{n+1}. \tag{5}$$

For the function (4), we have the well-known bounds [1]

$$|b_n| \leq 2 \quad (n = 1, 2, 3, \cdots), \tag{6}$$

and for the convex function $\phi(z)$ we have the equally well-known inequalitie

$$|c_n| \leq 1 \quad (n = 1, 2, 3, \cdots). \tag{7}$$

From (5), (6) and (7) we obtain $|\alpha_n| \leq n$ $(n \geq 1)$. This can immediately be lated into (3) for the non-normalized close-to-convex functions. For in the case we need only consider $\left|\frac{c_1}{a_1}\right| \left[\frac{f'(z)}{\phi'(z)} + \left|\frac{a_1}{c_1}\right| - \frac{a_1}{c_1}\right]$ instead of (4).

We point out that the preceding result, obtained by the author in 1952, n obtained by using the very general set of inequalities obtained by Tammi [7 too used (5) after (6) and (7) had been established. We also remark that the ceding result was known to M. S. Robertson (oral communication to this au from Kaplan) as well as to A. W. Goodman (written communication to this a of recent date).

Although Theorem 1 contains the known results for star mappings and f pings which are convex in one direction, it does not seem to contain the ve of the Bieberbach conjecture for another set of mappings introduced by Ta

3. A STUDY-TYPE RESULT

From the definition of close-to-convex mappings and from the classic S theorem, it follows that if $f(z)$ is close-to-convex in $|z| < 1$, then $f(z)$ is close-to-convex for $|z| < r_0$, for each $r_0 < 1$. We shall generalize this re

THEOREM 2. *Let* C *denote the perimeter of a closed disc* D *lying in If* $f(z)$ *is close-to-convex in* $|z| < 1$, *then* $f(z)$ *is close-to-convex in* D.

Proof. To prove the theorem, we must exhibit a convex mapping $\phi_0(z)$ such that

$$\Re \frac{f'(z)}{\phi_0'(z)} \geq 0 \tag{8}$$

holds in D. To this end, we use the close-to-convex transformation $f(z)$ a sociated convex map $\phi(z)$. Carathéodory [2] has shown that *the image of* C *a convex curve*, so that by Study's classic result [1] $\phi(z)$ is convex in D. Hence the function $\phi_0(z)$ in (8) may be chosen to be the same $\phi(z)$ as for (1).

We wish to thank Professor J. L. Walsh for calling our attention to the preceding result due to Carathéodory.

4. CLOSE-TO-STAR FUNCTIONS

Let $f(z) \equiv \Sigma_1^\infty a_n z^n$ $(a_1 \neq 0)$ be analytic for $|z| < 1$, and let $f(z) \neq 0$ for $z \neq 0$. Then we say that $f(z)$ is a close-to-star function if and only if there exists a univalent star map $\psi(z) \equiv \Sigma_1^\infty \sigma_n z^n$, star with respect to $w = 0$, such that

$$\Re \frac{f(z)}{\psi(z)} \geq 0 \tag{9}$$

holds for $|z| < 1$. The close-to-star maps, constituting what we shall call class K*, evidently bear the same relation to the class K of close-to-convex maps that the ordinary star maps bear to the usual convex maps. Unlike the usual star maps, but like Robertson's functions star-like in one direction, the functions of class K* are not necessarily univalent. But as for the other maps, we have the following results.

(A). If $f(z) \equiv \Sigma_1^\infty a_n z^n$ $(a_1 \neq 0)$ is analytic for $|z| < 1$, then $f(z)$ is in class K* if and only if $F(z) \equiv \int_0^z \frac{f(z)}{z} dz$ is in K.

(B). If $F(z)$ is analytic for $|z| < 1$, with $F'(z) \neq 0$ for $|z| < 1$, then $F(z)$ is in K if and only if $f(z) \equiv z F'(z)$ is in K*.

The preceding two results can be proved in the usual way [1, 6]. Indeed, we can also obtain the following results, which have their analogues in the other "star" classes to which we have referred.

THEOREM 3. *If* $f(z) \equiv \Sigma_1^\infty a_n z^n$ $(a_1 \neq 0)$ *is analytic for* $|z| < 1$, *with* $f(z) \neq 0$ *for* $z \neq 0$, *then* $f(z)$ *is close-to-star if and only if the inequality*

$$\int_{\theta_1}^{\theta_2} \Re \left[re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right] d\theta > -\pi, \tag{10}$$

holds for all $\theta_1 < \theta_2$ *and for all* $0 \leq r < 1$.

THEOREM 4. *If* $f(z) \equiv z + \Sigma_2^\infty a_n z^n$ *is close-to-star, then the coefficients satisfy the inequality*

$$|a_n| \leq n^2 \quad (n = 2, 3, \cdots), \tag{11}$$

with equality for the Robertson functions star-like in one direction [6].

Proofs of Theorems 3 *and* 4. If we use the remark (A) above, then (10) follows from (2) applied to $F(z)$; similarly, (11) follows from the inequalities (3) applied to $F(z)$. As for equality, we merely know that the functions mentioned in Theorem 4 actually satisfy the stated equalities [6].

We include the geometric interpretation of (10) to show that our class is a little more inclusive than Robertson's "star" class. The inequality (10) states that the radius vector to the image of $|z| = r < 1$, under close-to-star maps $f(z)$, never turns back by an amount as much as π radians *on any arc*.

We should like to point out that Tammi [7, 8] has also thoroughly exploi device of this paragraph to obtain a larger class of analytic functions relat certain univalent functions.

5. CONCLUSION

In a recent note, Renyi [5] showed that if f(z) is analytic in $|z| < 1$, if in $|z| < 1$, and if the inequality

$$\text{(12)} \qquad \underset{0 \leq r < 1}{\text{l. u. b.}} \int_0^{2\pi} \left| \Re\left(1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) \right| d\theta \leq 4\pi$$

holds, then f(z) is univalent and the coefficients satisfy the inequality (3). first part of the result was established earlier by Paatero [4]; but Renyi ba proof upon a demonstration that the functions satisfying (12) are indeed cor some direction, and hence he can use Robertson's results [6]. We point ou condition (12) is stronger by far than Kaplan's condition (2), so that univale lows from Kaplan's work. In addition, our Theorem 1 is thus seen to conta Renyi's result.

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Paris, France
and
The University of Michigan