ON MEROMORPHIC FUNCTIONS WHOSE IMAGES CONTAIN A GIVEN DISC

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1. Introduction. Let S denote the well-known family of functions, f, that are analytic and univalent in $D = \{z : |z| < 1\}$ and have the normalization $f(0) = 0$, $f'(0) = 1$. Netanyahu [6], [7] introduced the classes $S(d) = \{f \in S : d = d_f\}$ where d_f is defined by

$$d_f = \inf\{|\alpha| : f(z) \neq \alpha\}. \tag{1.1}$$

He studied the relation between the second coefficient and d. This concept has been extended in several directions in [3], [8].

We define M to be the set of all normalized univalent meromorphic functions $f(z) = z + a_2 z^2 + \cdots$ in D. For each function f in M, we define d_f by (1.1), and let $M(d) = \{f \in M : d \leqslant d_f\}$.

For each $p \in (0,1)$, Goodman [2] introduced the classes

$$U(p) = \{f \in M : f(p) = \infty\}.$$

He conjectured that $k_p(z) = pz/[(p-z)(1-pz)]$ is the extremal function for the coefficient problem in $U(p)$. Ever since, several papers have been written on this conjecture, each obtaining partial results. Among these, we mention the ones by Jenkins [4], and Kirwan and Schober [5].

It follows from the minimum modulus problem of Fenchel [1], that $U(p)$ is a subset of $M(d)$ when $d = p/(1+p)^2$, and for all d we have $S(d) \subseteq M(d)$. Therefore, it seems natural to study the classes $M(d)$.

In this paper, we solve the coefficient problem for $M(d)$, and moreover we are able to find the extreme points of the closed convex hull of $M(d)$. The coefficient bounds we obtain for $M(d)$ are valid but, of course, are not sharp for the subclasses $U(p)$ and $S(d)$.

2. Extreme points of the closed convex hull of $M(d)$. We begin with some preliminary results.

LEMMA 1. *Suppose $f \in M(d)$, then $d|z| \leqslant |f(z)|$ for all $|z| < 1$.*

Proof. Let us define $\phi(\eta) = f^{-1}(d\eta)$. Then $\phi(0) = 0$ and $|\phi(\eta)| \leqslant 1$ for $|\eta| < 1$. So, by the Schwarz Lemma, we obtain $|\phi(\eta)| \leqslant |\eta|$ for all $|\eta| < 1$. However, this is equivalent to the conclusion of the lemma when $|f(z)| < d$. To see this, let $d\eta = f(z)$. Note that when $|f(z)| \geqslant d$, the conclusion holds trivially. □

LEMMA 2. *If $f \in M(d)$ has a pole at the point a, then $|a| \geqslant d$.*

Received February 16, 1981.
Michigan Math. J. 29 (1982).

Proof. Consider the function $\phi(z)$ defined by

$$\phi(z) = \frac{1-\bar{a}z}{z-a}\frac{dz}{f(z)}.$$

Note that $\phi(z)$ is a regular analytic function in D and on ∂D we have $|\phi(z)| \leqslant 1$. Therefore, by the Maximum Principle, $|\phi(z)| \leqslant 1$ holds for all $|z| < 1$. In particular, $|\phi(0)| \leqslant 1$ and this is the conclusion of the lemma. □

COROLLARY. *If $f \in M(d)$ has a pole at the point a, and $\alpha(f)$ is its residue at a, then $d|a|(1-|a|^2) \leqslant |\alpha(f)|$.*

Proof. This result follows from the fact that $|\phi(a)| \leqslant 1$, where ϕ is the function defined in the proof of Lemma 2. □

THEOREM 1. *Suppose $f \in M(d)$, then the function $h(z)$ defined by*

$$h(z) = \frac{2f(dz)}{(1-d^2)dz} - \frac{1+d^2}{1-d^2} \tag{2.1}$$

belongs to the class P of analytic functions which are subordinate to $(1+z)/(1-z)$.

Proof. It follows from Lemma 1 that

$$\rho(z) = \frac{d-(dz/f(z))}{1-(d^2z/f(z))}$$

is a regular analytic function bounded by 1, and takes zero to zero. Now, let $\psi(z) = \rho(dz)/d$. Then $|\psi(z)| \leqslant |z|$ for all $z \in D$, and $h(z) = (1+\psi(z))/(1-\psi(z))$. Hence, the theorem follows. □

COROLLARY. *Suppose $f \in M(d)$. Then $|a_{n+1}| \leqslant (1-d^2)/d^n$ for all n, and the bound is sharp.*

Proof. It is well-known that the coefficients of the functions which belong to P are bounded by 2. Note that the nth coefficient of the function $h(z)$ defined by (2.1) is $2a_{n+1}d^n/(1-d^2)$. Hence, $|a_{n+1}| \leqslant (1-d^2)/d^n$. Also, note that these bounds are assumed by the coefficients of the function $f(z) = dz(1-dz)/(d-z)$ which is in $M(d)$. □

THEOREM 2. *Extreme points of the closed convex hull, $\overline{\text{co}}\, M(d)$, of $M(d)$ consist of functions*

$$f_\eta(z) = dz\frac{1-dz\eta}{d-z\eta} \qquad (|\eta| = 1).$$

Proof. The mapping Λ defined by

$$\Lambda(f)(z) = 2f(dz)/(1-d^2)dz$$

is a linear homeomorphism of the space $X = \{f : f(0) = 0,\ f \text{ is analytic in } |z| < d\}$ into the space $Y = \{f : f \text{ is analytic in } |z| < 1\}$. We define $L: X \to Y$ by $L(f)(z) = \Lambda(f)(z) - (1+d^2)/(1-d^2)$, so that L is a linear homeomorphism followed by a translation. By Lemma 2, we know that $M(d) \subseteq X$, therefore,

$$\overline{\text{co}}\, L(M(d)) = L(\overline{\text{co}}\, M(d)) \quad \text{and} \quad E_{\overline{\text{co}}\, L(M(d))} = L(E_{\overline{\text{co}}\, M(d)}), \tag{2.2}$$

where E_A denotes the extreme points of the set A (see, for example, Schober [9, p. 172]). By Theorem 1, $L(M(d)) \subseteq P$. Therefore it follows that

$$\overline{\text{co}}\, L(M(d)) \subseteq \overline{\text{co}}\, P = P, \tag{2.3}$$

since P is a compact convex subset of Y. It is known that

$$E_P = \{(1+\eta z)/(1-\eta z) : |\eta| = 1\}$$

and it can easily be verified that $E_P \subseteq L(M(d))$. From this we obtain

$$\overline{\text{co}}\, E_P \subseteq \overline{\text{co}}\, L(M(d)). \tag{2.4}$$

As noted above, P is a compact convex subset of the locally convex space Y, so by the Krein-Milman Theorem $\overline{\text{co}}\, E_P = \overline{\text{co}}\, P = P$. Comparing this with the relations (2.3) and (2.4), we arrive at $P = \overline{\text{co}}\, L(M(d))$. This last equality and the equalities (2.2) permit us to conclude that $P = L(\overline{\text{co}}\, M(d))$ and $E_P = L(E_{\overline{\text{co}}\, M(d)})$. Finally, from the knowledge of E_P, we obtain $E_{\overline{\text{co}}\, M(d)} = L^{-1}\{E_P\} = \{f_\eta(z) : |\eta| = 1\}$, and this is the conclusion of the theorem. □

COROLLARY. *Given $f \in M(d)$, there exists a probability measure μ on $|z| = 1$ such that*

$$f(z) = \int_{|\eta|=1} dz \frac{1 - d\eta z}{d - \eta z}\, d\mu_\eta.$$

3. Additional coefficient estimates. In this section we obtain bounds on the coefficients c_n defined by

$$\log \frac{f(z)}{z} = c_1 z + \cdots + c_n z^n + \cdots \tag{3.1}$$

where z is small.

THEOREM 3. *Suppose $f \in M(d)$ and define c_n by* (3.1). *Then if f has a pole at the point a, we have*

$$|c_n| \leqslant 2 \log \frac{|a|}{d} + \frac{1}{n}\left(\frac{1}{|a|^n} - |a|^n\right) \leqslant \frac{1}{n}\left(\frac{1}{d^n} - d^n\right) \tag{3.2}$$

for all n. In particular, if $f \in S(d)$, we have $|c_n| \leqslant 2 \log \frac{1}{d}$ for all n. The inequalities (3.2) *are sharp.*

Proof. Consider the function $g(z)$ defined by

$$g(z) = \log\left(\frac{f(z)}{dz} \cdot \frac{z-a}{1-\bar{a}z}\right),$$

where the branch of $\log w$ is chosen so that $\log 1 = 0$. It is a consequence of Lemma 1 that $g(z)$ is a regular analytic function in D with positive real part. Also note that

$\operatorname{Re} g(0) = \log(|a|/d)$. It follows from the theory of functions with positive real part that the nth coefficient of $g(z)$ is bounded by $2\log(|a|/d)$, i.e.,

$$\left| c_n - \frac{1}{n}\left\{\frac{1}{a^n} - \bar{a}^n\right\}\right| \leqslant 2\log\frac{|a|}{d}.$$

Hence (3.2) follows. Note that the extreme points of $\overline{\text{co}}\, M(d)$ show these inequalities are sharp. If $f \in S(d)$, a similar argument establishes the inequality $|c_n| \leqslant 2\log(1/d)$.

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