POLYNOMIAL VALUES AND ALMOST POWERS

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1. Introduction. For $n \in \mathbf{Z}$ and $m \in \mathbf{N}$ with $m \geqslant 2$ the m-free part of n is the smallest positive integer a with the property that $\pm n = ay^m$ for some $y \in \mathbf{Z}$. Let $F \in \mathbf{Z}[X]$ have at least two distinct zeros. We prove that if $a > 1$ is the m-free part of $F(x)$ for some $x \in \mathbf{Z}$ and some $m \in \mathbf{N}$ with $m \geqslant 2$ then a has a certain multiplicative structure (see Corollary 1), e.g. the greatest prime divisor $P(a)$ of a exceeds $c_1 \log\log a$, where c_1 is a positive number depending only on F. This includes the well known fact that $P(F(x)) > c_1 \log\log|F(x)|$ for $|F(x)| > 1$. Let $F \in \mathbf{Z}[X]$ have at least three simple zeros and suppose that $F(x) = \pm ab$, where $x \in \mathbf{Z}$, $a \in \mathbf{N}$ and b is some power, i.e. $b \in \{y^m \mid y \in \mathbf{Z}, m \in \mathbf{N}, m \geqslant 2\}$. We prove that a cannot be small in comparison to $|F(x)| : a > \frac{1}{2}\exp(c_1(\log\log(|F(x)|+3))^{1/5-\epsilon})$, where c_1 denotes a positive constant depending only on F and $\epsilon > 0$. This includes the well-known fact that there exist only finitely many $x \in \mathbf{Z}$ such that $F(x)$ is a power. We also show that these numbers a are not a product of primes which are very small with respect to $|F(x)| : P(a) > c_3 \log\log\log(|F(x)|+3)$, where $c_3 > 0$ depends only on F.

2. Let $F \in \mathbf{Z}[X]$ have at least two distinct zeros and let $a \in \mathbf{Z}$, $a \neq 0$.

It is well known (see, e.g., [1]) that there exist positive numbers ϵ_F and $c_F(a)$ such that $P(F(x)) > \epsilon_F \log\log|F(x)|$ for $|F(x)| > 1$ and such that if $F(x) = ay^m$, for certain $x, y \in \mathbf{Z}$ with $|y| > 1$ and some $m \in \mathbf{N}$, then $m \leqslant c_F(a)$. In the existing proofs for the second result, the first result is used. This is unnecessary, in fact the first result can be proved in the same manner as the second one, as can be seen in the proof of Theorem 1 in this section. We also give an upper bound $c_F(a)$ for m which is explicit in a. For completeness we state the results that we use in the proof of Theorem 1. These results can be found in [1] as Theorem A (= Proposition), Lemma C (= Lemma 1), while Lemma 2 is implicit in the proof of Theorem 1 in [1] (see also [2], Lemma 4.3).

PROPOSITION. *Let $\alpha_1, \ldots, \alpha_N$, where $N \geqslant 2$, be nonzero algebraic numbers. Let K be the smallest normal field containing $\alpha_1, \ldots, \alpha_N$ and put $d = [K:\mathbf{Q}]$. Let $A_1, \ldots, A_N$ ($\geqslant 3$) be upper bounds for the heights of $\alpha_1, \ldots, \alpha_N$, respectively. Put $\Omega' = \prod_{j=1}^{N-1} \log A_j$, $\Omega = \Omega' \log A_N$. There exist positive numbers C_1 and C_2 such that for every $B \geqslant 2$ the inequalities*

$$0 < |\alpha_1^{b_1} \ldots \alpha_N^{b_N} - 1| < \exp(-(C_1 N d)^{C_2 N} \Omega \log \Omega' \log B)$$

have no solution in rational integers $b_1, \ldots, b_N$ with absolute values at most B.

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LEMMA 1. *Let γ_1 and γ_2 be algebraic integers in a field K of degree d. Then*

$$H\left(\frac{\gamma_1}{\gamma_2}\right) \leqslant 2d\cdot 2^d \prod_\sigma \max\{|\sigma\gamma_1|, |\sigma\gamma_2|\}$$

where σ runs through all isomorphic injections of K into $\mathbf{C}$ and $H(\alpha)$ denotes the height of α.

LEMMA 2. *Let K be a field of degree $[K:\mathbf{Q}]=d$. By definition, the units of K are the algebraic integers ϵ in K with $|N\epsilon|=1$, where $N=N_{K/\mathbf{Q}}$ is the norm map from K to $\mathbf{Q}$. There exist an integer $r=r(K)\in\{0,1,\dots,d-1\}$ and units $\eta_0,\dots,\eta_r$ of K, with η_0 a root of unity, such that every unit ϵ of K is of the form $\epsilon=\prod_{j=0}^r \eta_j^{b_j}$ for certain $b_j\in\mathbf{Z}$ $(0\leqslant j\leqslant r)$, while $|b_0|\leqslant c_0(K)$, some suitable number depending only on K. Moreover, there exists a number $c=c(K)$ such that for every $\alpha\in K$ there exists a unit ϵ of K such that $\beta=\epsilon\alpha$ satisfies $c^{-1}|N\beta|^{1/d}\leqslant|\sigma\beta|\leqslant c|N\beta|^{1/d}$ for every isomorphic injection σ of K into C.*

THEOREM 1. *Let $F\in\mathbf{Z}[X]$ have at least two distinct zeros and let $a\in\mathbf{Z}$, $a\neq 0$. There exist positive numbers $x_F, c_1=c_1(F)$ and $c_2=c_2(F)$, depending only on F, such that if*

$$F(x) = ay^m \tag{1}$$

with $x,y,m\in\mathbf{N}$ and $x\geqslant x_F$, with the proviso that $m\leqslant \log|F(x)|$ if $y=1$, then

$$m \leqslant (2(\omega(a)+1))^{c_1\cdot(\omega(a)+1)}\left(\prod_{p|a}\log p\right)^{c_2} =: c_F(a), \tag{2}$$

where $\omega(a)$ denotes the number of distinct primes dividing a.

Proof. Write $F(x)=a_n\prod_{i=1}^\nu (X-\alpha_i)^{k_i}$ with $\alpha_1,\dots,\alpha_\nu$ distinct and $k_1,\dots,k_\nu\in\mathbf{N}$. We may assume that F is monic $(a_n=1)$ in view of the following argument. Firstly, we may assume without loss of generality that a_n is positive. It follows from (1) that $G(a_nx):=a_n^{n-1}F(x)=(a_n^{n-1}a)y^m$, where n is the degree of F. If the theorem has been proved for monic polynomials then, provided $x\geqslant x_F:=a_n^{-1}x_G$, since G is monic, $m\leqslant c_G(a_n^{n-1}a)$ and the contribution of the primes dividing a_n can be incorporated in $c_1=c_1(F)$. Since $a_n=1$, the $\alpha_1,\dots,\alpha_\nu$ are algebraic integers. Let K be the (normal) field generated by $\alpha_1,\dots,\alpha_\nu$, put $d=[K:\mathbf{Q}]$ and let $\mathcal{P}_1,\dots,\mathcal{P}_s$ be the distinct prime ideals of K which divide the ideal generated by $a\cdot\prod_{i<j}(\alpha_i-\alpha_j)$. Assume that (1) holds for some $x\geqslant x_F$, where $x_F\geqslant 2$ is sufficiently large, depending only on F (how large will be apparent from the sequel). Then the prime ideal decomposition of the integral ideal $[x-\alpha_i]$ generated by $x-\alpha_i$ has the form

$$[x-\alpha_i] = \prod_{k=1}^s \mathcal{P}_k^{w_k(i)}\Gamma_i^{m_i} \quad (i=1,\dots,\nu)$$

for certain $w_k(i)\geqslant 0$, $w_k(i)\in\mathbf{Z}$, where $m_i=m/(m,k_i)$, with $\Gamma_i=[1]$ if $y=1$. Choose α_i and α_j with $\alpha_i\neq\alpha_j$, say α_1 and α_2. Note that m_1 and m_2 are divisible by $m^*=m/(m,[k_1,k_2])$. Hence there exist integral ideals Γ_1^*, Γ_2^* $(=[1]$ if $y=1)$ in K with

(3) $$[x-\alpha_i] = \prod_{k=1}^{s} \mathcal{P}_k^{w_k(i)}(\Gamma_i^*)^{m^*} \quad (i=1,2).$$

Note that it follows from (1) and $y>1$ that $m^* \leqslant m \leqslant (\log|F(x)|)(\log 2)^{-1} \leqslant c_3 \log x$ for some $c_3 = c_3(F)$. If $y=1$ in (1) then

(4) $$m^* \leqslant m \leqslant c_3 \log x$$

also holds, by assumption. Taking $x_F \geqslant 2\max_{1\leqslant i\leqslant \nu}|\alpha_i|$, we have by (3) and $x \geqslant x_F$,

(5) $$\sum_{k=1}^{s} w_k(i)\log N\mathcal{P}_k + m^* \log N\Gamma_i^* = \log N[x-\alpha_i] \leqslant \log(2x)^d \ll_F \log x \quad (i=1,2).$$

Let h be the class number of K. By Lemma 2 there exist algebraic integers $\pi_1,\ldots,\pi_s,\gamma_1,\gamma_2$ in K (with $\gamma_i=1$ if $\Gamma_i^*=[1]$) and a number $c=c(K)$ such that

$$[\pi_k] = \mathcal{P}_k^h, \quad [\gamma_i] = (\Gamma_i^*)^h, \quad \text{for} \quad k=1,\ldots,s \quad \text{and} \quad i=1,2 \quad \text{and}$$

(6) $$c^{-1} \leqslant |\sigma\alpha||N\alpha|^{-1/d} \leqslant c \quad \text{for every automorphism } \sigma \text{ of } K \text{ and } \alpha \in \{\pi_1,\ldots,\pi_s,\gamma_1,\gamma_2\}.$$

It follows from (3) and (6) that $(x-\alpha_i)^h = \epsilon_i \prod_{k=1}^{s} \pi_k^{w_k(i)}\gamma_i^{m^*}$ for some unit ϵ_i of K $(i=1,2)$. Hence, by Lemma 2,

(7) $$(x-\alpha_i)^h = \prod_{j=0}^{r} \eta_j^{b_j(i)} \prod_{k=1}^{s} \pi_k^{w_k(i)}\gamma_i^{m^*} \quad (i=1,2),$$

for certain $b_j(i) \in \mathbf{Z}$, with $|b_0(i)| \leqslant c_0(K)$. We now show that

(8) $$|b_j(i)| \ll_F \log x \quad \text{for} \quad j=0,1,\ldots,r \quad \text{and} \quad i=1,2.$$

Since $\sigma x = x$ and $|\sigma\eta_0|=1$ for $\sigma \in \mathrm{Aut}(K)$ we infer from (7) that

(9) $$\sum_{j=1}^{r} b_j(i)\log|\sigma\eta_j| = h\log|x-\sigma\alpha_i| - \sum_{k=1}^{s} w_k(i)\log|\sigma\pi_k| - m^*\log|\sigma\gamma_i| =: \lambda_\sigma(i) \quad \text{for} \quad \sigma \in \mathrm{Aut}(K) \quad \text{and} \quad i=1,2.$$

It follows from (6), $N\mathcal{P}_k \geqslant 2$, $N\Gamma_i^* \geqslant 1$ that $\log|\sigma\pi_k| \geqslant -\log c$ and $\log|\sigma\gamma_i| \geqslant -\log c$. With the use of (5) we obtain that $|\lambda_\sigma(i)| \ll_F \log x$ for every σ and $i=1,2$. From the equations (9) with r distinct σ's and Cramer's rule it follows that (8) holds for $j=1,\ldots,r$ and $i=1,2$. As observed already, (8) also holds for $j=0$. From (7) we conclude that

(10) $$(x-\alpha_i)^h = \prod_{j=0}^{r} \eta_j^{\beta_j(i)} \prod_{k=1}^{s} \pi_k^{\omega_k(i)}\delta_i^{m^*} \quad (i=1,2),$$

where $\omega_k(i) \in \{0,1,\ldots,m^*-1\}$ with $\omega_k(i) \equiv w_k(i) \bmod m^*$ and $|\beta_j(i)| < m^*$ with $\beta_j(i) \equiv b_j(i) \bmod m^*$ and $\mathrm{sgn}(\beta_j(i)) = \mathrm{sgn}(b_j(i))$, with

$$\delta_i = \gamma_i \left\{ \prod_{j=0}^{r} \eta_j^{b_j(i)-\beta_j(i)} \prod_{k=1}^{s} \pi_k^{w_k(i)-\omega_k(i)} \right\}^{1/m^*} \quad (i=1,2).$$

We now show that the algebraic integers $\delta_i \in K$ satisfy, for some $c_4 = c_4(F)$,

(11) $$|\sigma\delta_i| \leqslant \exp((c_4 \log x)/m^*) \quad \text{for} \quad \sigma \in \operatorname{Aut}(K) \quad \text{and} \quad i = 1,2.$$

By (6) and (5) we have $\log|\sigma\gamma_i| \ll_F \log N\Gamma_i^* \ll_F (\log x)/m^*$. By (8) we have

$$\sum_{j=0}^{r} (b_j(i) - \beta_j(i)) \log|\sigma\eta_j| \ll_F \log x.$$

By (6) and (5) we have

$$\begin{aligned}\sum_{k=1}^{s} (w_k(i) - \omega_k(i)) \log|\sigma\pi_k| &\ll_F \sum_{k=1}^{s} (w_k(i) - \omega_k(i)) \log N\mathcal{P}_k \\ &\leqslant \sum_{k=1}^{s} w_k(i) \log N\mathcal{P}_k \ll_F \log x.\end{aligned}$$

This proves (11). From (10) we obtain

(12) $$\left(\frac{x-\alpha_2}{x-\alpha_1}\right)^h - 1 = \prod_{j=0}^{r} \eta_j^{\beta_j(2)-\beta_j(1)} \prod_{k=1}^{s} \pi_k^{\omega_k(2)-\omega_k(1)} (\delta_2/\delta_1)^{m^*} - 1.$$

Taking $x_F > |(\alpha_2 - \alpha_1\zeta)(1-\zeta)^{-1}|$ for every $\zeta \neq 1$ with $\zeta^h = 1$, we have, for $x \geqslant x_F$, that the expression in (12) is nonzero. Assuming that $m^* > 1$ we apply the proposition to the right-hand side of (12), with $N = r+s+2$, $\alpha_i = \eta_{i-1}$ for $1 \leqslant i \leqslant r+1$, $\alpha_{r+2}, \ldots, \alpha_{N-1} = \pi_1, \ldots, \pi_s$, $\alpha_N = \delta_2/\delta_1$ and $B = m^*$. Put $\omega = \omega(aD_F)$, where $D_F = N\left(\prod_{i<j}(\alpha_i - \alpha_j)\right)$. Then $N \leqslant d - 1 + d\omega + 2 \leqslant (d+1)(\omega+1)$. Also $H(\alpha_i) \leqslant c_5 = c_5(K) =: A_i$ for $1 \leqslant i \leqslant r+1$. By Lemma 1 and (6) we have

$$\begin{aligned}H(\pi_k) &\leqslant 2d \cdot 2^d \prod_{\sigma} \max\{|\sigma\pi_k|, 1\} \leqslant 2d \cdot 2^d \cdot c^d (N\mathcal{P}_k)^h \\ &\leqslant p(k)^{c_6} =: A_{r+k+1} \quad (1 \leqslant k \leqslant s)\end{aligned}$$

for some $c_6 = c_6(K)$, where $p(k)$ is the rational prime in $\mathcal{P}_k$. Note that the number of distinct k with $p(k) = p$ is at most d for every prime p. By Lemma 1 and (11) we have

$$\begin{aligned}H(\delta_2/\delta_1) &\leqslant 2d \cdot 2^d \prod_{\sigma} \max\{|\sigma\delta_2|, |\sigma\delta_1|\} \leqslant 2d \cdot 2^d \exp((dc_4 \log x)/m^*) \\ &\leqslant \exp((c_7 \log x)/m^*) =: A_N,\end{aligned}$$

where we used (4) in the last inequality. It follows from the proposition that

$$\begin{aligned}\left|\left(\frac{x-\alpha_2}{x-\alpha_1}\right)^h - 1\right| &> \exp(-(C_1 d(d+1)(\omega+1))^{C_2(d+1)(\omega+1)} c_8 \cdot \Big(\prod_{p \mid aD_F} \log p\Big)^d \\ &\cdot \Big(\log c_8 + d \sum_{p \mid aD_F} \log\log p\Big) \cdot (\log x)(m^*)^{-1} \log m^*)\end{aligned}$$

for some $c_8 = c_8(K)$. On the other hand, for $x \geqslant x_F$, with x_F sufficiently large,

$$\left|\left(\frac{x-\alpha_2}{x-\alpha_1}\right)^h - 1\right| = \left|\left(1 + \frac{\alpha_1 - \alpha_2}{x-\alpha_1}\right)^h - 1\right| < c_9 x^{-1} < \exp(-\tfrac{1}{2}\log x).$$

Combining these estimates for $|((x-\alpha_2)/(x-\alpha_1))^h-1|$ we obtain, for some large $c_{10}=c_{10}(F)$,

$$m^*=1 \quad \text{or} \quad m^*/\log m^* < (2(\omega+1))^{c_{10}(\omega+1)}\Big(\prod_{p\mid aD_F}\log p\Big)^d\Big(1+\sum_{p\mid aD_F}\log\log p\Big).$$

Finally observe that $m\leqslant [k_1,k_2]m^*=c_{11}m^*$. This implies that (2) holds for every $c_2>d$, provided that $c_1=c_1(F)$ is sufficiently large (we also incorporate the primes dividing D_F in $c_1(F)$). □

REMARK. If $y=1$ in (1) then one can, more naturally, apply the proposition to the right hand side of (7) with $\gamma_i=1$, with $N=r+s+1$, $B=c_F\log x$ (cf. (8) and (5)).

The condition $x\geqslant x_F$ in Theorem 1 can actually be omitted if one takes $c_1(F)$ sufficiently large: if (1) holds with $x<x_F$ then

$$m\leqslant c_{12}(F) := \max_{1\leqslant x\leqslant x_F}\log|F(x)|/\log 2.$$

3. Definition. Let $m\in\mathbf{N}$ with $m\geqslant 2$. The m-free part of an integer n is the smallest positive integer a with the property that $\pm n=ay^m$ for some $y\in\mathbf{Z}$.

We recall that the number of distinct prime divisors of a positive integer a is denoted by $\omega(a)$ and the greatest prime divisor of $a>1$ by $P(a)$, while $P(1)=1$.

COROLLARY 1. *Let $F\in\mathbf{Z}[X]$ have at least two distinct zeros. There exist positive numbers ϵ_1 and δ_1, depending only on F, with the following property. Let $a\in\mathbf{N}$ with $a\geqslant 3$ be the m-free part of $F(x)$ for some $x\in\mathbf{Z}$ and some $m\in\mathbf{N}$ with $m\geqslant 2$. Then*

$$\text{(i)}\quad \omega(a) > \delta_1(\log\log a)(\log\log\log a)^{-1}$$

(13) $$\text{or}$$

$$\text{(ii)}\quad P(a) > \exp(\delta_1(\log\log a)(\log\log\log a)^{-1}).$$

In particular

(14) $$P(a) > \epsilon_1\log\log a.$$

Proof. Since $\omega(a)\geqslant 1$ and $P(a)\geqslant 2$ we may assume that $a\geqslant a_0$, where a_0 is some large number depending only on F, since for the remaining values of a the inequalities (13) and (14) are valid if we take $\delta_1>0$ and $\epsilon_1>0$ sufficiently small. Observe that $c_F(a)$ in (2) satisfies

(15) $$c_F(a) \leqslant ((\omega(a)+1)\log 3P(a))^{c_0\cdot(\omega(a)+1)} \quad (\text{for every } a\in\mathbf{Z},\ a\neq 0)$$

for some $c_0=c_0(F)$. Since a is m-free we have

(16) $$a\leqslant \prod_{p\mid a}p^{m-1}\leqslant P(a)^{(m-1)\omega(a)}.$$

We have $F(x)=\pm ay^m$ for some $x,y\in\mathbf{Z}$. We may assume that $F(x)=ay^m$ with $x\geqslant 0$ and $y\in\mathbf{N}$ (by considering also $\pm F(-x)$). Since $a\geqslant a_0$, we have $x\geqslant x_F$. If $y>1$ then, by Theorem 1, $m\leqslant c_F(a)$ and it follows from (15) and (16) that

(17) $$((\omega(a)+1)\log 3P(a))^{c\cdot(\omega(a)+1)}\geqslant \log a,$$

where $c = c_0 + 1$. If $y=1$ then it follows from (2) with $m = [\log F(x)] = [\log a]$ and (15), that (17) also holds in this case. If $\omega(a) \leqslant \log P(a)$ then it follows from (17) that (13)(ii) holds and if $\omega(a) > \log P(a)$ then (13)(i) follows from (17). The inequality (14) is a direct consequence of (13) and $\omega(a) \leqslant \pi(P(a)) < 2P(a)(\log P(a))^{-1}$ for $a > 1$, where $\pi(x)$ denotes the number of primes not exceeding x. □

REMARKS. If $a \in \mathbf{N}$ is the m-free part of some integer for some $m \geqslant 2$ then $P(a) > \frac{1}{2}(m-1)^{-1}\log a$, since $a \leqslant \prod_{p \mid a} p^{m-1} \leqslant \prod_{p \leqslant P(a)} p^{m-1} < e^{2(m-1)P(a)}$ (the constant $\frac{1}{2}$ cannot be replaced by a constant larger than 1 in view of the m-free numbers of the form $a = \prod_{p \leqslant P} p^{m-1}$). So if a is the m-free part of some integer for some m ($\geqslant 2$) which is small with respect to a then we have a nontrivial lower bound for $P(a)$, e.g., if $m < (2\epsilon)^{-1}(\log a)(\log\log a)^{-1}$ then $P(a) > \epsilon \log\log a$. The inequality (14) of Corollary 1 states that if $a > 1$ is the m-free part of an integer of the form $F(x)$, where $F \in \mathbf{Z}[X]$ has at least two distinct zeros, then $P(a) > \epsilon \log\log a$ holds regardless of the value of m ($\geqslant 2$), with $\epsilon = \epsilon_1(F)$. Note that this includes that $P(F(x)) > \epsilon_1(F)\log\log|F(x)|$ for $|F(x)| > 1$ (take $a = |F(x)|$ and $m = [(\log|F(x)|)(\log 2)^{-1}] + 1$). As observed already, the inequalities for $\omega(a)$ and $P(a)$ in (13) and (14) are trivial for small values of a, i.e. $a \leqslant a_0(F)$. By the well-known theorem of Schinzel and Tijdeman [1] these values for a can only occur for $|x| \leqslant x_F$, provided that F has at least three simple zeros. See also, Theorem 2, (22).

4. Let $F \in \mathbf{Z}[X]$ have at least three simple zeros.
Suppose that

$$F(x) = \pm a y^m \tag{18}$$

where $x, y, a, m \in \mathbf{N}$ with $m \geqslant 2$. Let a_m denote the m-free part of a. Sprindžuk (see [1]) obtained the following upper bound for x when $m=2$ in (18):

$$x < \exp((2a_2)^{C(F)}) \quad \text{when} \quad m = 2 \quad \text{in (18)}. \tag{19}$$

In [2] Sprindžuk investigated the equations (18) with $m \geqslant 3$. It follows from his results that when $m \geqslant 3$ in (18):

$$x < \exp(c(2a_m)^{4m^3 n}) \tag{20}$$

where n is the degree of F and c is an effectively computable number depending only on F and m. We need to know how c depends on m. Estimating the relevant constants occurring in his proof we find that (20) holds with $c = c(F)^{m^5}$, e.g., with $c(F) = (2H)^{C_0 n^4}$, where H is the height of F and C_0 is an absolute constant. The dependency of the bound in (20) on m can be removed with the use of Theorem 1.

THEOREM 2. *Let $F \in \mathbf{Z}[X]$ have at least three simple zeros. There exist positive numbers C, c_1, c_2, c_3, depending only on F, with the following properties. Let $x \in \mathbf{Z}$ and suppose that $F(x) = \pm ab$ where $a \in \mathbf{N}$ and b is some power. Then*

$$((\omega(a)+1)\log 3P(a))^{C\cdot(\omega(a)+1)} > \log\log(|F(x)|+3). \tag{21}$$

In particular

$$a > \tfrac{1}{2}(\log\log(|F(x)|+3))^{c_1} \tag{22}$$

(23) $$P(a) > c_3 \log\log\log(|F(x)|+3).$$

Proof. To prove (21) we may assume that $x \geqslant x_F$ (by taking $c=c(F)$ sufficiently large) and that $F(x)=\pm ay^m$ for some $m, y \in \mathbf{N}$ with $y>1$ and $m>1$ (if $y=1$ then (21) follows from Theorem 1 and (15)). The inequality (21) follows now from (20) and $a_m \leqslant P(a)^{m\omega(a)}$ (see (16)) and (2) and (15). To prove (22) from (21) it is sufficient to observe that $P(a) \leqslant a$ and $\omega(a) \ll (\log a)(\log\log a)^{-1}$ for $a>2$. Finally, (23) follows from (21) noting that $\omega(a) \leqslant \pi(P(a)) \ll P(a)(\log P(a))^{-1}$ for $a>1$. □

Finally we formulate some of our results somewhat differently.

DEFINITION. For nonnegative integers n let $a(n)$ be the smallest positive integer a with the property that $n=ab$ for some power b, i.e. for some $b=y^m$ with $y \in \mathbf{Z}$ and $m \in \mathbf{N}$ with $m \geqslant 2$.

Clearly, for $n \in \mathbf{N}$ we have $1 \leqslant a(n) \leqslant n$ and $a(n)=1$ if and only if n is a power, while $a(n)=n$ if and only if n is not divisible by a power exceeding 1 (i.e. n is square free).

DEFINITION. Let ϕ be a nondecreasing function defined on $\mathbf{Z}$. An integer $n \geqslant 0$ is called a ϕ-almost power if $1<a(n) \leqslant \phi(n)$, i.e. if there exist a power $b=y^m$ ($m \geqslant 2$) and an (m-free) integer a with $1<a \leqslant \phi(n)$ such that $n=ab$.

A different, perhaps more natural, definition of a ϕ-almost power results if we replace $1<a \leqslant \phi(n)$ by $1<P(a) \leqslant \phi(n)$. (Equivalently, n is a ϕ-almost power if there is a $m \in \mathbf{N}$ with $m \geqslant 2$ such that $v_p(n) \in m\mathbf{Z}$ for $p>\phi(n)$ but not for all primes p).

We shall refer to these notions as ϕ-almost power in the first sense and ϕ-almost power in the second sense, respectively.

The following corollary shows that the function $n \to a(n)$ restricted to the integers $n=|F(x)|$, $x \in \mathbf{Z}$, where $F \in \mathbf{Z}[X]$ has at least two distinct zeros, has in its range no integers a which are a product of primes that are small in comparison to a. Also, if F has at least three simple zeros, $|F(x)|$, $x \in \mathbf{Z}$, is never a ϕ-almost power, in both senses, for appropriate ϕ.

COROLLARY 2. *Let $F \in \mathbf{Z}[X]$ have at least two distinct zeros. Then*

(24) $$P(a(n)) > c_0 \log\log(a(n)+1) \quad \text{for} \quad n \in \{|F(x)| : x \in \mathbf{Z}\},$$

where c_0 is a positive number depending only on F.

Let $F \in \mathbf{Z}[X]$ have at least three simple zeros. For $n \in \mathbf{Z}$, $n \geqslant 0$ we write

$$\phi_1(n) = \tfrac{1}{2}(\log\log(n+3))^{c_1}, \quad \phi_2(n) = c_3 \log\log\log(n+3).$$

Let $x \in \mathbf{Z}$. Then

$|F(x)|$ is not an ϕ_1-almost power in the first sense.

$|F(x)|$ is not an ϕ_2-almost power in the second sense.

Proof. If $a(n) \geqslant 4$ then (24), with $c_0 \leqslant \frac{1}{2}\epsilon_1$, follows from Corollary 1, (14), since $a(n)$ is the m-free part of n for some $m \geqslant 2$. For $1 \leqslant a(n) \leqslant 3$ the inequality (24) trivially holds if $c_0 < 1$, say. The other assertions are merely restatements of (22) and (23). □

Added in proof. The bound $c_F(a)$ for m given in Theorem 1 depends only on the prime divisors of a. One also has the bound $m \leqslant c_F \log 3|a| \log\log 3|a|$, which can be obtained likewise by not splitting a in its prime factors. Consequently the bound in (22) can be improved to $a > \frac{1}{2}\exp(c_1(\log\log(|F(x)|+3))^{1/5-\epsilon})$, and one can take ϕ_1 in Corollary 2 accordingly. I am grateful to Prof. V. G. Sprindžuk for pointing this out to me.

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