PROJECTIVE MODULES OVER GROUP-ALGEBRAS OF TORSION-FREE GROUPS

Jacques Lewin

To the memory of David L. Williams.

1. Introduction. Let Q be the field of rationals and let G be a torsion-free group. If every finitely generated projective QG-module is free we say that G is Q-projective-free.

Little seems to be known about the class of Q-projective-free groups. The known examples fall into three classes: $G=F$, a free group, $G=F\times C$, the direct product of a free group and an infinite cycle, and $G=A$, a free abelian group. The first two results are due to H. Bass [2] and P. M. Cohn [6]. The abelian case to Swan [11, p. 144]. This set of examples can be enlarged slightly since the class of Q-projective-free groups is closed under direct limits and, from G. Bergman's coproduct theorems [3], free products.

There are two examples of groups which are not Q-projective-free in the literature: $G=\langle x, y \mid x^2=y^3\rangle$, the group of the trefoil knot, and G/G'', its metabelian version. These examples are due to M. J. Dunwoody [8] and P. Berridge and M. J. Dunwoody [4].

We exhibit here a class of groups which are not projective-free.

THEOREM. *Let H be a group with a subgroup G such that*

a) KH is a domain when $K=Q$ and $K=Z/pZ$.

b) G has two generators and is not free.

c) G/G' is not free abelian of rank 2.

Then QH contains a two-generator nonfree projective left ideal P. However, $P\oplus QH=QH^2$.

COROLLARY 1. *If G is a one-relator, two-generator group whose relation is neither a power nor a commutator relation, then G is not projective-free.*

COROLLARY 2. *If G is a torsion-free polycyclic-by-finite group which is projective-free, then G is nilpotent.*

As another special case, we have

COROLLARY 3. *If G is a torsion-free abelian-by-finite group which is projective free, then G is abelian.*

This generalizes a result of D. Farkas and the author and answers a question of Farkas [9, question #21].

Note in particular that the group $G=\langle x, y \mid x^2=y^2\rangle$ is not projective-free. But G has a free abelian subgroup of index two, and QG is a skew Laurent polynomial

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extension of a principal ideal domain. This is easily seen since G also has the presentation $G=\langle a, b \mid aba^{-1}=b^{-1}\rangle$.

Corollary 1 follows immediately from the theorem since KG is a domain for any field K [12].

If G is polycyclic-by-finite and torsion free then G satisfies a) by theorems of D. Farkas and R. Snider [10] and G. Cliff [5]. Corollary 2 thus follows from the following proposition.

PROPOSITION. *Let G be a polycyclic-by-finite group such that every noncyclic, two-generator subgroup H of G has H/H' free abelian of rank 2. Then G is nilpotent.*

Proof. It is enough, by a theorem of R. Baer [1] to show that the two-generator subgroups of G are nilpotent. By induction on the Hirsch number n of H (the number of infinite factors in a polycyclic series for H), we may assume that for x in H then $gp(H',x)$, which has Hirsch number at most $n-1$, is nilpotent. Thus every x in H belongs to a normal nilpotent subgroup of H and hence H is nilpotent. □

I thank Warren Dicks for a substantial simplification of my original proof and I thank Ralph Strebel for showing me how to remove from the statement of the theorem the unnecessary assumption that G have finite cohomological dimension.

2. To prove the theorem, notice first that the conditions c) and d) on G amount to saying that we can find generators x_1 and x_2 for G that satisfy a relation $r(x_1,x_2)$ that has positive exponent sum on x_1.

We begin more generally with a group G, generated by elements x_i, $i=1,2,\dots$, presented via a free group F freely generated by a set $X=\{X_i,\ i=1,2,\dots\}$: $1 \longrightarrow N \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1$. We assume that the kernel N contains an element $r(X_1,X_2,\dots)$ with $r(X_1,1,1,\dots)=X_1^n$, $n>0$.

Let u be any positive integer different from 1 and 2 and let p be a prime divisor of $u-1$. Let R be the local ring $Z_{(p)}$ of integers localized at the prime ideal pZ. Then $Q=R[p^{-1}]$. u is invertible in R and $s=1-u^n$ is in $pZ_{(p)}$. If $R \longrightarrow \bar{R}=Z_{(p)}/pZ_{(p)}=Z/pZ$ is the natural map, then $\bar{u}=1$ and $\bar{s}=0$.

Let A be a commutative ring and let Ω_A be the direct sum $\Omega_A=\bigoplus_{X_i\in X} AG\delta X_i$ of copies of AG. Then Ω_A is also a right A- module. Let α be a homomorphism of F into the group of units of A. The map $X_i \longrightarrow \delta X_i$ extends to a $\pi-\alpha$ derivation $\delta: AF \longrightarrow \Omega_A$. i.e. is A-linear, vanishes on A, and, for $f_1, f_2 \in F$,

$$\delta(f_1 f_2) = \delta(f_1)\alpha(f_2) + \pi(f_1)\delta(f_2).$$

Since Ω_A is free on $\{\delta X_i,\ X_i \in X\}$, the map $\delta X_i \longrightarrow x_i - \alpha(x_i)$ extends to a homomorphism $\beta_A : \Omega_A \longrightarrow AG$. Since $f \longrightarrow \pi(f)-\alpha(f)$ is also a $\pi-\alpha$ derivation $AF \longrightarrow AG$, it follows that $\beta\delta f=\pi(f)-\alpha(f)$. If $A=Z$ and α is the usual augmentation which maps the free generators of F to 1, then we have the exact sequence of ZG-modules (Lyndon [13]),

$$(1) \qquad 0 \longrightarrow N/N' \longrightarrow \Omega_Z \xrightarrow{\beta_Z} ZG \xrightarrow{\epsilon} Z \longrightarrow 0.$$

This sequence splits as Z modules so that applying $\bar{R} \otimes_Z -$ gives the exact sequence:

(2) $$0 \longrightarrow \bar{R} \otimes_Z N/N' \longrightarrow \Omega_{\bar{R}} \xrightarrow{\beta_{\bar{R}}} \bar{R}G \xrightarrow{\bar{\epsilon}} \bar{R} \longrightarrow 0.$$

We now consider the exact sequence of RG-modules which arises from the map $\alpha: F \longrightarrow A$ given by $X_1 \longrightarrow u$, $X_i \longrightarrow 1$ for $i>1$:

(3) $$0 \longrightarrow S \longrightarrow \Omega_R \xrightarrow{\beta_R} RG \longrightarrow \operatorname{Coker}\beta_R \longrightarrow 0.$$

The image of β_R, the two-sided ideal $RG(x_1-u)+\Sigma RG(x_i-1)$ generated as R-module by the elements $\pi(f)-\alpha(f)$ with f in F is contained, modulo p, in the augmentation ideal of RG. There is then a surjection $\operatorname{Coker}\beta_R \longrightarrow \bar{R}$, and thus $\operatorname{Coker}\beta_R \neq 0$. Further, $\beta_R \delta r(X_1, X_2, \ldots) = \pi(r) - u^n = 1 - u^n = s$ and so there is a surjection $R/sR \longrightarrow \operatorname{Coker}\beta_R$. Thus $\operatorname{Coker}\beta_R = R/tR$ with $p \mid t$ and $t \mid s$. Now apply $\bar{R} \otimes_R -$ to the sequence (3). The resulting sequence

(4) $$0 \longrightarrow \bar{R} \otimes_R S \longrightarrow \bar{R} \otimes_R \Omega_R \longrightarrow \bar{R} \otimes_R RG \longrightarrow \bar{R} \otimes_R \operatorname{Coker}\beta_R \longrightarrow 0$$

is

(5) $$0 \longrightarrow \bar{R} \otimes_R S \longrightarrow \Omega_{\bar{R}} \xrightarrow{\beta_{\bar{R}}} \bar{R}G \xrightarrow{\bar{\epsilon}} \bar{R} \longrightarrow 0,$$

since $\bar{u}=1$. Since R, a *PID*, has global dimension 1, (4) is exact except at $\Omega_{\bar{R}}$, where its homology is $\operatorname{Tor}_R(R/tR, \bar{R})$. From the sequence of R-modules

$$0 \longrightarrow R \xrightarrow{t} R \longrightarrow R/tR \longrightarrow 0,$$

we get

$$0 \longrightarrow \operatorname{Tor}_R(R/tR, \bar{R}) \longrightarrow \bar{R} \xrightarrow{t} \bar{R} \longrightarrow \bar{R} \longrightarrow 0$$

and hence $\operatorname{Tor}(R/tR, \bar{R}) = \bar{R}$. By (2), $\operatorname{Ker}\beta_{\bar{R}} = \bar{R} \otimes_Z N/N'$ so that we have an exact sequence

(6) $$0 \longrightarrow \bar{R} \otimes_R S \longrightarrow \bar{R} \otimes_Z N/N' \xrightarrow{\tau} \bar{R} \longrightarrow 0$$

LEMMA. *If $\bar{R} \otimes_R S$ is projective, then G is free.*

Proof. Note first that the Tor term $\bar{R}$ in the sequence (6) is a G-module with trivial action. For tensoring the sequence

$$0 \longrightarrow \operatorname{Im}\beta_R \longrightarrow RG \longrightarrow \operatorname{Coker}\beta_R \longrightarrow 0$$

with $\bar{R}$ gives the sequence

$$0 \longrightarrow \bar{R} = \operatorname{Tor}_R(R/tR, \bar{R}) \longrightarrow \operatorname{Im}\beta_R \otimes \bar{R} \xrightarrow{j} \bar{R}G \longrightarrow \bar{R} \longrightarrow 0.$$

If $a \in \operatorname{Im}\beta_R$ and $j(a \otimes 1) = 0$, then $a = pa'$ for some a' in RG. If $g = \pi(f)$, for $f \in F$, then, since $u = 1 \bmod p$, $\alpha(f) = 1 \bmod p$. Thus

$$(g-1)(a \otimes 1) = (g-\alpha(f))(a \otimes 1) \in \operatorname{Im}\beta_R \cdot a \otimes 1 = \operatorname{Im}\beta_R a' p \otimes 1 = \operatorname{Im}\beta_R a' \otimes p = 0.$$

Let now $n \in \bar{R} \otimes_Z N/N'$ with $\tau(n) = 1$, and consider the (inner) derivation $D': G \longrightarrow \bar{R} \otimes N/N'$ given by $g \longrightarrow (g-1)n$. Then the image of D' is in $\ker \tau$, hence in $\bar{R} \otimes S$. Thus D' defines a derivation $D: G \longrightarrow \bar{R} \otimes_R S$ which is no longer inner. For if $Dg=(g-1)m$ for all $g\in G$, then $(g-1)(n-m)=0$ for all g, which forces $n=m$ since $\bar{R}\otimes N/N'$ is a submodule of a free $\bar{R}G$-module and G is infinite. Thus if $\bar{R}\otimes S$ is projective we can construct, by composing D with one of the projections $\bar{R}\otimes S\longrightarrow \bar{R}G$, a non-inner derivation $G\longrightarrow \bar{R}G$ from which it follows immediately that $H^1(G, \bar{R}G)\neq 0$. It then follows from the Stallings-Swan theorems [7, III.4] that G is free. □

Now, apply $QH\otimes_{RG}—=Q\otimes_R RH\otimes_{RG}—$ which is exact (since $Q\otimes_R—$ is exact) to (3) to obtain

$$(7)\qquad 0\longrightarrow QH\otimes_{RG}S\longrightarrow QH\otimes_{RG}\Omega_R\longrightarrow QH\longrightarrow 0,$$

since t is invertible in Q.

Thus $P=QH\otimes_{RG}S$ is QH-projective. This is the module which, in the case that G has two generators, we show is not free.

Suppose then that F has two generators X_1, X_2 and that P is free. Then, since QH has the invariant basis property, we find that, from (7), P is cyclic, say with generator a. Since $Q=R[p^{-1}]$, we may choose $a\in RH\otimes_{RG}S$, $a\notin p(RH\otimes_{RG}S)$. Thus the image $\bar{a}$ of a in $\bar{R}H\otimes_{RG}S$ is nonzero. Let v also be in $RH\otimes_{RG}S$ with $\bar{v}\neq 0$. Then there is $u\in QH$ with $ua=v$. Write $u=p^k u'$ with $u\in RH$, $h'\notin pRH$. Then $k\leq 0$ since $\bar{v}\neq 0$. If $k<0$ then $\overline{u'a}=p^{-k}v=0$. However, from (5) we see that $\bar{R}H\otimes_{RG}S=\bar{R}H\otimes_{\bar{R}G}\bar{R}G\otimes_{RG}S=\bar{R}H\otimes_{\bar{R}G}(\bar{R}\otimes_R S)$ is a submodule of the free $\bar{R}H$-module $\bar{R}H\otimes_{\bar{R}G}\Omega_R$. Since by assumption a) $\bar{R}H$ is a domain, then $\bar{a}\neq 0$ implies $\bar{u}'=0$, a contradiction to the choice of u'. Thus $k=0$. Hence v is in the RH-submodule of P generated by a and it follows that $\bar{R}H\otimes_{RG}S$ is the cyclic $\bar{R}H$-module generated by $\bar{a}$. Since $\bar{R}H$ is a domain, $\bar{a}$ generates a free submodule, so that $\bar{R}H\otimes_{RG}S$ is free on the basis $\{\bar{a}\}$. If $\{1\}\cup\{t_j\}$ is a left transversal for G in H, then $\bar{R}H\otimes_{RG}S$ is a free RG-module with basis $\{\bar{a}\}\cup\{t_j\bar{a}\}$. Also, $\bar{R}H=\bar{R}G\oplus\sum t_j\bar{R}G$ as RG-bimodules and thus $\bar{R}G\otimes_{RG}S=\bar{R}\otimes_{\bar{R}}S$ is a summand of $\bar{R}H\otimes_{RG}S$ and hence is a projective $\bar{R}G$ module.

We conclude from the lemma that G is free, a contradiction to our hypothesis. Thus P is not a free module.

It remains to show that P is a left ideal of QH. This is almost immediate for, by (7), $QH(x_1-u)+(QH)(x_2-1)=QH$. If $L_Q=QH(x_1-u)\cap QH(x_2-1)$, then the exact sequence

$$(8)\qquad 0\longrightarrow L_Q\longrightarrow QH(x_1-u)\oplus QH(x_2-1)\longrightarrow QH\longrightarrow 0$$

gives an isomorphism $P\cong L_Q$.

If $A\leq Q$, it is clear that $AH(x_1-u)+AH(x_2-1)=AH$, as long as both u and s are invertible in A, and thus that the corresponding intersection L_A is projective. Since $L_Q=L_A\otimes Q$ is not free, neither is L_A.

It may not even be necessary for u to be invertible on A. Suppose that $\delta r(X_1,X_2)=r_{x_1}\delta X_1+r_{x_2}\delta X_2$ lies in $ZG\delta X_1+ZG\delta X_2$. Then

$$r_{x_1}(x_1-u)+r_{x_2}(x_2-1)=s$$

in ZG so that $L_{Z[s^{-1}]}$ is a projective $Z[s^{-1}]$ G-module. This module is not free since we have again that $L_{Z[s^{-1}]}\otimes Q=L_Q\simeq P$. This occurs if for example the relation

$r(X_1, X_2)$ is of the form $p_1(X_1, X_2) = p_2(X_1, X_2)$ where $p_1(X_1, X_2)$ and $p_2(X_1, X_2)$ are words in which X_1 and X_2 occur only with positive exponents. The simplest such relations are $X_1^n = X_2^m$.

Specific relations sometimes lend themselves to specific computations.

Example. Let G be a group generated by two elements x and y which satisfy a relation $y^{-1}x^n y = x^m$ with $m > n$ and $m > 0$. Suppose also that QG is a domain and that there is a nontrivial homomorphism $G \longrightarrow Z$. Then QG has a nonfree projective left ideal. If further $m - n = 1$, then ZG has a nonfree projective left ideal which induces a nonfree QG-projective module.

Note that we need not assume that G is a one-relator group.

Apply the sequence (8) with $x_1 = x, x_2 = y$ and $u = 2$. Let $r(X, Y) = YX^{-n}Y^{-1}X^m$. Then $r_y = 2^{m-n} - x^{-m}2^m$. From the splitting of (8) it is easily verified that L_Q is generated by the two elements $(x-2)r_y\ (y-1)$ and $[s - (y-1)r_y](y-1)$

Thus, since $y-1$ is not a zero divisor, L_Q is isomorphic to the left ideal generated by $(x-2)r_y$ and $s - (y-1)r_y$.

If N is the normal closure of x in G, N is in the kernel of any homomorphism $G \longrightarrow Z$. Our assumptions then show that y is of infinite order modulo N. QG is then a skew Laurent polynomial extension of QN via the automorphism induced on N by conjugation by y. If $p(y) = p_n y^n + \cdots + p_k y^k$ with p_i in QN, let $d(p(y)) = n - k$. Since QN is a domain, $d(p(y)q(y)) = d(p(y)) + d(q(y))$. Suppose that P' is principal, say $P' = QGa$. Since $d((x-2)r_y) = 0$, then $d(a) = 0$ as well. But then a divides both $s + r_y$ and r_y and it follows that $P' = QGs = QG$. Thus $L_Q = QG(y-1)$ and since L_Q is also contained in $QG(x-2)$ we find that $QG(y-1) \leq QG(x-2)$. Since $QG = QG(y-1) + QG(x-2)$ it follows that $QG = QG(x-2)$ so that $x-2$ is a unit. This cannot happen since x has infinite order. Thus P' is not principal and so neither is L_Q.

If $m - n = 1$, then $s = -1$ is a unit in Z so that L_Z is indeed projective.

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Department of Mathematics
Syracuse University
Syracuse, N.Y. 13210