A DUALITY THEOREM FOR HARMONIC FUNCTIONS

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Let D be a bounded open subset of $\mathbf{R}^n$ with C^∞ boundary, and let $h^\infty(\bar{D})$ denote the space of complex valued harmonic functions on D which are in $C^\infty(\bar{D})$. In this paper, we prove that the dual of the Frechet space $h^\infty(\bar{D})$ is the space $h^{-\infty}(D)$ of harmonic functions on D which satisfy finite growth conditions at the boundary. More precisely, a harmonic function g is in $h^{-\infty}(D)$ if and only if there are positive constants m and C such that $\mathrm{Sup}\{|g(z)|d(z)^m : z\in D\}<C$ where $d(z)$ is the distance of z to bD, the boundary of D. In fact, we prove that $h^\infty(\bar{D})$ and $h^{-\infty}(D)$ are mutually dual via an extension of the usual $L^2(D)$ pairing.

This duality in conjunction with some classical results from potential theory allows us to prove an interesting theorem about the Poisson kernel $P(x,\theta)$ of the domain D. It is a classical fact that the operator $\phi\mapsto\int_{bD}P(x,\theta)\phi(\theta)\,d\sigma_\theta$ maps $C^\infty(bD)$ isomorphically onto $h^\infty(\bar{D})$. In this paper, we prove that the operator

$$h\mapsto\int_D h(x)P(x,\theta)\,dV_x,$$

when defined correctly, is an isomorphism between $h^{-\infty}(D)$ and $\mathfrak{D}'(bD)$.

A key step toward proving these results is the establishment of

LEMMA 1. *Suppose D is a smooth bounded domain in $\mathbf{R}^n$ and s is a positive integer. There is a positive integer $m=m(s)$ and a constant $C=C(s)$ such that if f and g are harmonic functions in $L^2(D)$, then*

$$\left|\int_D fg\right|\le C\Big(\sup_{z\in D,\,|\alpha|\le m}|\partial^\alpha f(z)|\Big)\Big(\sup_{z\in D}|g(z)|d(z)^s\Big).$$

Here, the symbol ∂^α is defined when $\alpha=(a_1,a_2,\dots,a_n)$ is a multi-index as the differential operator

$$\partial^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{a_1}\partial x_2^{a_2}\cdots\partial x_n^{a_n}}\,.$$

The constants m and C do not depend on f or g.

This lemma leads to the remarkable conclusion that if $f\in h^\infty(\bar{D})$ and $g\in h^{-\infty}(D)$, then $\int_D fg$ is a well defined quantity, even though $|fg|$ may be far from integrable.

Before we can state and prove our main theorem, we must establish some definitions and recall some facts from potential theory.

Throughout this paper, D will be a smooth bounded domain contained in $\mathbf{R}^n$. If s is a positive integer, we let $W^s(D)$ denote the usual Sobolev space of complex valued functions on D with norm $\|\ \|_s$ induced by the inner product

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$$\langle u, v\rangle_s = \sum_{|\alpha|\le s} \int_D \partial^\alpha u \overline{\partial^\alpha v}\ .$$

We let $W_0^s(D)$ denote the closure of $C_0^\infty(D)$ in $W^s(D)$. A function u will be said to vanish to order t on bD if $\partial^\alpha u(z)=0$ for all multi-indices α with $|\alpha|\le t$ and all $z\in bD$. It is easily proved that if v is a function in $C^s(\bar D)$ which vanishes to order $s-1$ on bD, then $v\in W_0^s(D)$.

The dual space of $W_0^s(D)$ will be written $W^{-s}(D)$ and can be identified as a Banach space with the space of distributions λ supported on D such that

$$\|\lambda\|_{-s} = \mathrm{Sup}\{|\lambda(\phi)| : \phi\in C_0^\infty(D);\ \ \|\phi\|_s = 1\}$$

is finite.

We define $h^s(D)$ and $h^{-s}(D)$ to be the corresponding subspaces of $W^s(D)$ and $W^{-s}(D)$ consisting of harmonic functions.

An alternate and, for our purposes, more useful description of $h^{-\infty}(D)$ is $h^{-\infty}(D)=\bigcup_{s=1}^\infty h^{-s}(D)$. This equality is valid because of

LEMMA 2. *Let s be a positive integer with $s>n$. There are positive constants c_1 and c_2 such that if g is harmonic on D, then*

$$c_1\|g\|_{-s-n} \le \mathrm{Sup}_{z\in D}|g(z)|d(z)^s \le c_2\|g\|_{-s+n}.$$

Proof of Lemma 2. Sobolev's lemma and Taylor's formula imply that $|\phi(x)|\le C\|\phi\|_{s+n}d(z)^s$ for ϕ in $W_0^{s+n}(D)$. Hence, the left-hand side of the inequality is true.

To prove the right-hand side of the inequality, let θ be a C^∞ radially symmetric function supported in the unit ball with $\int\theta=1$. Define $\theta_z(x)=\epsilon^{-n}\theta((x-z)/\epsilon)$ where $\epsilon=d(z)$. Then

$$|g(z)| = \left|\int_D g\theta_z\right| \le \|g\|_{-s+n}\|\theta_z\|_{s-n} \le C\|g\|_{-s+n}d(z)^{-s+(n/2)}.$$

This completes the proof of lemma 2. □

$h^{-\infty}(D)$ now obtains its topology as the inductive limit of the Banach spaces $h^{-s}(D)$. With this topology, a linear functional T on $h^{-\infty}(D)$ is continuous if and only if T restricts to be a continuous linear functional on $h^{-s}(D)$ for each positive integer s.

Before we can state our main theorem, we must describe a special differential operator. Let P denote the orthogonal projection of $L^2(D)$ onto its subspace $h^0(D)$ consisting of harmonic functions.

LEMMA 3. *For each positive integer s, there is a linear differential operator L^s of order $N=(s/2)(s+3)$ with $C^\infty(\bar D)$ coefficients which maps $W^{s+N}(D)$ to $W_0^s(D)$ such that $PL^s=P$.*

This lemma equips us to define the extension of the $L^2(D)$ pairing $\langle f,g\rangle_0=\int_D f\bar g$ for functions $f\in h^\infty(\bar D)$ and $g\in h^{-\infty}(D)$. We simply define $\langle f,g\rangle_0=\int_D(L^sf)\bar g$

whenever $g \in h^{-s}(D)$. We will prove that this pairing is well defined and nondegenerate in the course of the proof of

THEOREM 1. *If D is a smooth bounded domain in $\mathbf{R}^n$, then the spaces $h^{\infty}(\bar{D})$ and $h^{-\infty}(D)$ are mutually dual via the nondegenerate sesquilinear pairing $\langle\ ,\ \rangle_0$.*

Proof of Lemma 3. Let r be a smooth defining function for D, i.e., let r be a C^∞ function such that $D=\{r<0\}$, $bD=\{r=0\}$, and $\nabla r \neq 0$ on bD. The operator L^s will have the form

$$L^s u = u - \Delta\left(\sum_{k=0}^{s-1} \theta_k r^{k+2}\right)$$

where the functions θ_k depend on u and will be determined inductively. Note that $PL^s = P$ because the subtracted term is the Laplacian of a function which vanishes to second order on bD and is therefore orthogonal to $h^0(D)$. Let X be a C^∞ function which is equal to 1 in a neighborhood of bD and which is zero in a neighborhood of $\{\nabla r=0\}$.

For $u \in C^\infty(\bar{D})$, set $L^1 u = u - \Delta(\theta_0 r^2)$ where $\theta_0 = \frac{1}{2}|\nabla r|^{-2} X u$. We have chosen θ_0 in this manner so that $L^1 u$ vanishes on bD. Hence $L^1 u \in W_0^1(D)$. Note that L^1 is a linear differential operator of order 2 with coefficients in $C^\infty(\bar{D})$.

Suppose that L^t has been constructed so that $L^t u$ vanishes to order $t-1$ on bD and in such a way that L^t is a linear differential operator of order $(t/2)(t+3)$ with $C^\infty(\bar{D})$ coefficients. Set $L^{t+1}u = L^t u - \Delta(\theta_t r^{t+2})$ where θ_t is to be determined. Note that $L^{t+1}u$ vanishes to order $t-1$ on bD no matter what we choose θ_t to be. Set $(\partial/\partial\eta) = \dfrac{\nabla r \cdot \nabla}{|\nabla r|^2}$. $(\partial/\partial\eta)$ is a vector field which points in the normal direction on bD. To make $L^{t+1}u$ vanish to order t on bD, it suffices to choose θ_t so that $(\partial/\partial\eta)^t L^{t+1} u = 0$ on bD. To this end, we set

$$\theta_t = \frac{X}{(t+2)!}\,|\nabla r|^{-2}\left(\frac{\partial}{\partial\eta}\right)^t L^t u.$$

Now L^{t+1} is a linear differential operator of order $(t/2)(t+3)+t+2=(t+1)(t+4)/2$ with coefficients in $C^\infty(\bar{D})$. Furthermore, $L^{t+1}u$ vanishes to order t on bD when $u \in C^\infty(\bar{D})$. Therefore, L^{t+1} extends to be a bounded operator from $W^{t+1+N}(D)$ to $W_0^{t+1}(D)$ where $N=(t+1)(t+4)/2$. This completes the induction.

Proof of Lemma 1. If $f \in W^{s+N}(D)$ where $N=(s/2)(s+3)$ and $g \in h^0(D)$, then

$$\left|\int_D fg\right| = \left|\int_D (L^s f) g\right| \le \|L^s f\|_s \|g\|_{-s} \le C\|f\|_{s+N}\|g\|_{-s}.$$

Lemma 1 now follows from lemma 2. □

Proof of Theorem 1. Let G be the solution operator to the Dirichlet problem $\Delta^2 \phi = v$ with $\phi = \dfrac{\partial\phi}{\partial\eta} = 0$ on bD. Here $\partial\phi/\partial\eta$ denotes the normal derivative of ϕ and $\phi = Gv$. It is easy to verify that $P = I - \Delta G \Delta$ where I is the identity operator. Since G

maps $W^s(D)$ to $W^{s+4}(D)$ in a bounded way, it follows that P is bounded from $W^s(D)$ to $W^s(D)$ for each positive integer s and that P maps $C^\infty(\bar{D})$ to $C^\infty(\bar{D})$. This fact will be very important in what follows.

Let r be the defining function for D used in the proof of lemma 3. We let $r_\epsilon = r+\epsilon$ and $D_\epsilon = \{r_\epsilon < 0\}$ for small $\epsilon > 0$. It is a standard result from the theory of partial differential equations that the operator G_ϵ which is the solution operator to the Dirichlet problem $\Delta^2\phi = v$ on D_ϵ with $\phi = \dfrac{\partial\phi}{\partial\eta_\epsilon} = 0$ on bD_ϵ is such that for any given $v \in W^s(D)$,

$$\|Gv - G_\epsilon v\|_{W^{s+4}(D_\epsilon)} \longrightarrow 0$$

as $\epsilon \longrightarrow 0$. From this, it follows that for any given $v \in W^s(D)$, the operator $P_\epsilon = I - \Delta G_\epsilon \Delta$ is such that

$$\|Pv - P_\epsilon v\|_{W^s(D_\epsilon)} \longrightarrow 0$$

as $\epsilon \longrightarrow 0$.

Using the defining function r_ϵ for D_ϵ and the procedure outlined in the proof of lemma 3, we can construct operators

$$L_\epsilon^s \colon W^{s+N}(D_\epsilon) \longrightarrow W_0^s(D_\epsilon) \subseteq W_0^s(D)$$

such that $P_\epsilon L_\epsilon^s = P_\epsilon$. It is not hard to check that

$$\|L_\epsilon^s v - L^s v\|_{W^s(D)} \leq C_\epsilon \|v\|_{W^{s+N}(D)}$$

where $C_\epsilon \longrightarrow 0$ as $\epsilon \longrightarrow 0$.

We are now in a position to prove that the pairing $\langle\ ,\ \rangle_0$ is well defined and non-degenerate. Suppose that $u \in h^\infty(\bar{D})$ and $v \in h^{-s}(D) \cap h^{-t}(D)$. Then

$$\langle L^s u, v\rangle_0 - \langle L^t u, v\rangle_0 = \lim_{\epsilon\to 0}\langle L_\epsilon^s u - L_\epsilon^t u, v\rangle_0 = 0$$

because $P_\epsilon L_\epsilon^s = P_\epsilon L_\epsilon^t = P_\epsilon$. Hence the pairing is well defined. It is clear that there is no nonzero function $u \in h^\infty(\bar{D})$ such that $\langle u, v\rangle_0 = 0$ for all $v \in h^{-\infty}(D)$. Suppose that $v \in h^{-s}(D)$ and that $\langle u, v\rangle_0 = 0$ for all $u \in h^\infty(\bar{D})$. We will prove that v must be the zero function. For $z \in D$, let θ_z be a $C_0^\infty(D)$ function which is radially symmetric about z with $\int \theta_z = 1$. Then $P\theta_z \in h^\infty(\bar{D})$ and $0 = \langle P\theta_z, v\rangle_0 = \langle L^s P\theta_z, v\rangle_0 = \lim_{\epsilon\to 0}\langle L_\epsilon^s P_\epsilon \theta_z, v\rangle_0 = \langle \theta_z, v\rangle_0 = v(z)$. Hence $v \equiv 0$ and the pairing is nondegenerate.

The inequality

$$\left|\int_D u\bar{v}\right| \leq C\|u\|_{s+N}\|v\|_{-s}$$

where $N = (s/2)(s+3)$ which appears in the proof of lemma 1 reveals that each function $v \in h^{-\infty}(D)$ defines a continuous linear functional on $h^\infty(\bar{D})$ via $u \mapsto \langle u, v\rangle_0$ and similarly, each $u \in h^\infty(\bar{D})$ defines a continuous linear functional on $h^{-\infty}(D)$. We must prove that all continuous linear functionals on $h^\infty(\bar{D})$ and $h^{-\infty}(D)$ are among these. In order to accomplish this, we now define an operator E^s which maps $h^s(D)$ to $h^{-s}(D)$ boundedly such that $\langle u, v\rangle_s = \langle u, E^s v\rangle_0$ for all $u \in h^\infty(\bar{D})$ and $v \in h^s(D)$, and such that E^s maps $h^\infty(\bar{D})$ to $h^\infty(\bar{D})$. Note that if u and v are in $h^\infty(\bar{D})$, then

$$\langle u, v\rangle_s = \sum_{|\alpha|\le s} \langle \partial^\alpha u, \partial^\alpha v\rangle_0 = \sum \langle \partial^\alpha u, L^s \partial^\alpha v\rangle_0$$
$$= \sum (-1)^{|\alpha|} \langle u, \partial^\alpha L^s \partial^\alpha v\rangle_0 = \langle u, E^s v\rangle_0$$

where $E^s v = P\left(\sum_{|\alpha|\le s}(-1)^{|\alpha|}\partial^\alpha L^s \partial^\alpha v\right)$. Furthermore, $E^s v \in h^\infty(\bar{D})$ and

$$\|E^s v\|_{-s} = \mathrm{Sup}\{|\langle E^s v, \phi\rangle_0| : \phi \in C_0^\infty(D);\ \|\phi\|_s = 1\}$$
$$= \mathrm{Sup}|\langle E^s v, P\phi\rangle_0| = \mathrm{Sup}|\langle v, P\phi\rangle_s| \le C\|v\|_s$$

because P is bounded from $W^s(D)$ to $W^s(D)$. Hence E^s extends to be a bounded operator from $h^s(D)$ to $h^{-s}(D)$, and since $\langle u, v\rangle_s = \langle L^s u, E^s v\rangle_0$ holds for u and v in $h^\infty(\bar{D})$, it follows that $\langle u, v\rangle_s = \langle L^s u, E^s v\rangle_0 = \langle u, E^s v\rangle_0$ for all u in $h^\infty(\bar{D})$ and v in $h^s(D)$ by completion.

Suppose that T is a continuous linear functional on $h^\infty(\bar{D})$. Then T satisfies an inequality of the form $|Tu| \le C\|u\|_s$ for some positive integer s and constant C. Hence, there is a function g in $h^s(D)$ such that $Tu = \langle u, g\rangle_s = \langle u, E^s g\rangle_0$ for all $u \in h^\infty(\bar{D})$, and the functional T is represented uniquely by $E^s g$.

Now suppose that T is a continuous linear functional on $h^{-\infty}(D)$. Then T is continuous on $h^{-s}(D)$ for all positive integers s. The Hahn-Banach theorem implies that for each positive integer s, there is a function $\phi_s \in W_0^s(D)$ such that $Tv = \langle v, \phi_s\rangle_0$ for all $v \in h^{-s}(D)$. We claim that $P\phi_s = u \in h^\infty(\bar{D})$ for all $s > 0$. Indeed, if $t \ne s$, then $\langle v, P\phi_s\rangle_0 = \langle v, P\phi_t\rangle_0 = Tv$ for all $v \in h^0(D)$ and therefore $P\phi_s = P\phi_t$. Allowing s to be arbitrarily large, we see that $u = P\phi_s \in h^\infty(\bar{D})$ by Sobolev's lemma. We shall now prove that $Tv = \langle v, u\rangle_0$ for all v in $h^{-\infty}(D)$. For each positive integer t, choose a sequence of functions $\{\phi_t^k\}$ in $C_0^\infty(D)$ such that $\phi_t^k \longrightarrow \phi_t$ in $W^t(D)$ as $k \longrightarrow \infty$. Suppose that $v \in h^{-s}(D)$. Let $t = s + N$ where $N = (s/2)(s+3)$. Then

$$Tv = \langle v, \phi_s\rangle_0 = \langle v, \phi_t\rangle_0 = \lim_{k\to\infty}\langle v, \phi_t^k\rangle_0.$$

But $\langle v, \phi_t^k\rangle_0 = \mathrm{Lim}_{\epsilon\to 0}\langle v, L_\epsilon^s P_\epsilon \phi_t^k\rangle_0 = \langle v, L^s P\phi_t^k\rangle_0$. Hence

$$Tv = \lim_{k\to\infty}\langle v, L^s P\phi_t^k\rangle_0 = \langle v, L^s u\rangle_0 = \langle v, u\rangle_0$$

and T is represented by u. This completes the proof of theorem 1. □

REMARKS. A) There is a standard way to extend the operator P to $C^{-\infty}(D)$, the dual space of $C^\infty(\bar{D})$. If we write $\langle f, \bar{\lambda}\rangle_0 = \lambda(f)$ for $\lambda \in C^{-\infty}(D)$ and $f \in C^\infty(\bar{D})$, then $P\lambda$ can be defined via the relation $\langle Pf, \lambda\rangle_0 = \langle f, P\lambda\rangle_0$, i.e., $P\lambda$ is the element of $C^{-\infty}(D)$ which represents the functional $f \mapsto \langle Pf, \lambda\rangle_0$. This extension of P takes on greater meaning in light of theorem 1. Indeed, if $\lambda \in C^{-\infty}(D)$, then $P\lambda \in h^{-\infty}(D)$ and $\overline{P\lambda(z)} = \langle P\theta_z, \lambda\rangle_0$ where θ_z is a radially symmetric function about z in $C_0^\infty(D)$ with $\int \theta_z = 1$. Theorem 1 implies that P maps $C^{-\infty}(D)$ *onto* $h^{-\infty}(D)$.

B) It is a classical fact from potential theory that $h^\infty(\bar{D}) \approx C^\infty(bD)$ via restriction to the boundary. Hence, the dual space of $h^\infty(\bar{D})$ can be identified with $\mathfrak{D}'(bD)$, the space of distributions on bD. If $\phi \in C^\infty(bD)$ and $\lambda \in \mathfrak{D}'(bD)$, let us write

$\langle \phi, \lambda \rangle_b = \lambda(\phi)$. We now define operators T and S via $\langle u, v \rangle_0 = \langle u, Tv \rangle_b$, $\langle u, S\lambda \rangle_0 = \langle u, \lambda \rangle_b$ for all $u \in h^\infty(\bar{D})$ and $v \in h^{-\infty}(D)$. The operator T is an isomorphism of $h^{-\infty}(D)$ onto $\mathfrak{D}'(bD)$ and S is the inverse of T. It is not hard to verify that if $u \in h^\infty(\bar{D})$ and $\psi \in C^\infty(bD)$, then $Tu(\theta) = \langle u, P(\cdot, \theta) \rangle_0$ and $S\psi(x) = \langle \psi, K(\cdot, x) \rangle_b$ where $P(x, \theta)$ is the Poisson kernel function and $K(y, x)$ is the harmonic Bergman kernel function. Hence the Poisson kernel and the Bergman kernel functions can be viewed as defining inverse operators.

C) The main result of this paper has roots which can be traced back to the classical duality between the spaces A^∞ and $A^{-\infty}$ of holomorphic functions on the unit disc in **C** (see [4, 5, 6]). The classical duality is exhibited via an L^2 pairing *on the boundary* and exploits the symmetry of the disc.

D) The techniques used in this paper were motivated by a similar program which has been carried out to show that A^∞ and $A^{-\infty}$ are mutually dual in a smooth bounded strictly pseudoconvex domain contained in $\mathbf{C}^n$ ([2]). This result has applications in the theory of boundary behavior of biholomorphic mappings (see [1]).

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