ON FUCHSIAN GROUPS OF DIVERGENCE TYPE

Ch. Pommerenke

1. INTRODUCTION

Let Γ be a Fuchsian group in the unit disk $\mathbf{D}$ with identity ι. We assume throughout that 0 is not an elliptic fixed point and denote by ρ the radius of the maximal disk $\{|z| < \rho\}$ that does not contain Γ-equivalent points. We also assume that Γ does not have a compact fundamental domain, so that $\mathbf{D}/\Gamma$ is an open Riemann surface.

If Γ is of convergence type, that is if $\Sigma_{\gamma\in\Gamma}(1 - |\gamma(0)|) < \infty$, then the Blaschke product

$$(1.1) \qquad g(z) = z \prod_{\gamma\in\Gamma,\gamma\neq\iota} \frac{|\gamma(0)|}{\gamma(0)}\gamma(z) \qquad (z \in \mathbf{D})$$

is called the *Green's function* of Γ with respect to 0; the positive harmonic function $-\log|g(z)|$ corresponds to the Green's function on $\mathbf{D}/\Gamma$.

Let now Γ be of divergence type; in the terminology of classification theory this means that $\mathbf{D}/\Gamma \in O_G$. We consider analogues of Green's functions; their harmonic counterparts on $\mathbf{D}/\Gamma$ are, for instance, the Evans function [15, p. 350] and Tsuji's modified Green's function [16, p. 455].

Let $\mathfrak{G}(\Gamma)$ denote the class of all functions $f(z) = z + a_2z^2 + \dots$ analytic in $\mathbf{D}$ with

$$(1.2) \qquad |f(\gamma(z))| = |f(z)| \qquad \text{for} \qquad \gamma \in \Gamma, \quad z \in \mathbf{D}$$

such that $|f(z)|$ is bounded away from 0 in $\mathbf{D}\setminus \bigcup_{\gamma\in\Gamma} \gamma(D_0)$ for a suitable disk D_0 around 0.

We shall show that the bounds in the last condition are actually only dependent on ρ (see Theorem 3). Every function $f \in \mathfrak{G}(\Gamma)$ is normal, and there is a natural fundamental domain associated with it (see Theorem 2). Perhaps the main result (Theorem 4) is that the functions $f \in \mathfrak{G}(\Gamma)$ that remain bounded at the parabolic vertices satisfy the best possible estimate

$$(1.3) \qquad \log^+|f(z)| = o\left(\frac{1}{1-|z|}\right) \qquad \text{as} \qquad |z| \to 1 - 0.$$

Let $L_0(\Gamma)$ denote the set of all $\zeta \in \partial\mathbf{D}$ for which there exist $\gamma_k \in \Gamma$ with

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(1.4) $$\gamma_k(0) \in \Delta \quad (k = 1,2,\ldots), \qquad \gamma_k(0) \to \zeta \quad (k \to \infty)$$

for some Stolz angle Δ at ζ (*angular limit points,* or points of approximation). Constantinescu [3, Section 52] and Tsuji [16, p. 535] have shown that

(1.5) $$\Gamma \text{ is of divergence type} \Leftrightarrow \operatorname{mes} L_0(\Gamma) = 2\pi.$$

Beardon and Maskit [1, Theorem 2] have proved that, for finitely generated groups of the first kind, every point on $\partial\mathbf{D}$ belongs to $L_0(\Gamma)$ except for the countably many parabolic fixed points. We shall deduce from (1.3) that $\partial\mathbf{D} \setminus L_0(\Gamma)$ is uncountably dense on $\partial\mathbf{D}$ if Γ is not finitely generated of the first kind (Theorem 5).

2. LEVEL SETS AND FUNDAMENTAL DOMAINS

Let $f \in \mathfrak{G}(\Gamma)$. It follows from (1.2) that, for $\gamma \in \Gamma$,

(2.1) $$(1 - |\gamma(z)|^2)|f'(\gamma(z))| = (1 - |z|^2)|\gamma'(z)f'(\gamma(z))| = (1 - |z|^2)|f'(z)|.$$

Since $(1 - |z|^2)|f'(z)|$ is clearly bounded in D_0 and thus in $\bigcup_{\gamma\in\Gamma} \gamma(D_0)$ and since $|f(z)|$ is bounded away from zero in the complement, it follows [9, Lemma] that f is normal in $\mathbf{D}$; this means that [7]

$$\sup_{z\in\mathbf{D}} (1 - |z|^2)|f'(z)|/(1 + |f(z)|^2) < \infty.$$

Now let f be an analytic function satisfying (1.2). Then the level set

(2.2) $$H_R = \{z \in \mathbf{D} : |f(z)| < R\} \qquad (0 < R < +\infty)$$

is invariant under Γ. Let G_R be the simply connected component of H_R that contains 0 and let ϕ_R be the univalent function that maps $\mathbf{D}$ onto G_R such that $\phi_R(0) = 0$, $\phi_R'(0) > 0$. Then

(2.3) $$\Gamma_R = \{\phi_R^{-1} \circ \gamma \circ \phi_R : \gamma \in \Gamma, \quad \gamma(G_R) = G_R\}$$

is a group of Möbius transformation of $\mathbf{D}$ onto $\mathbf{D}$, and it follows from Schwarz's lemma that $\{|z| < \rho\}$ does not contain Γ_R-equivalent points. Hence Γ_R is a Fuchsian group in $\mathbf{D}$.

THEOREM 1. *Let Γ be of divergence type and let f be a non-constant normal analytic function with $|f \circ \gamma| = |f|$ for $\gamma \in \Gamma$. Then Γ_R is of convergence type, and*

(2.4) $$g_R(z) = R^{-1} f(\phi_R(z)) \qquad (z \in \mathbf{D})$$

is an inner function. If $f \in \mathfrak{G}(\Gamma)$ then g_R is the Green's function (1.1) *of Γ_R.*

An *inner* function is an analytic function in $\mathbf{D}$ that is bounded by 1 and whose angular limit is of modulus 1 almost everywhere.

Proof. It follows from (2.2) and (2.4) that $|g_R(z)| < 1$ for $z \in \mathbf{D}$. Suppose that g_R is not an inner function. Then there is a set $A \subset \partial\mathbf{D}$ with mes $A > 0$ such that the angular limits $\phi_R(\zeta)$, $g_R(\zeta)$ exist and $|g_R(\zeta)| < 1$ for all $\zeta \in A$.

Let now $\zeta \in A$ and set $S = [0,\zeta)$. Then $\phi_R(S)$ is a curve from 0 to $\omega = \phi_R(\zeta) \in \partial G_R$. We see that $|\omega| = 1$, because $|g_R(\zeta)| < 1$ but

$$|g_R(z)| = R^{-1}|f(\phi_R(z))| = 1 \qquad \text{for} \qquad z \in \mathbf{D} \cap \partial G_R.$$

If $z \in S$, $z \to \zeta$ then $w = \phi_R(z) \in \phi_R(S)$, $w \to \omega$ and $f(w) = Rg_R(z) \to Rg_R(\zeta)$. Since f is normal, it follows from a theorem of Lehto and Virtanen [7], [11, p. 268] that f has the angular limit $Rg_R(\zeta)$ at ω. Since f is non-constant and since $|f(\gamma(z))| = |f(z)|$ for $\gamma \in \Gamma$, we therefore see from (1.4) that $\omega \notin L_0(\Gamma)$.

Hence it follows from (1.5) that mes $\phi_R(A) = 0$. Since $|\phi_R(z)| < 1$ for $|z| < 1$ and $|\phi_R(\zeta)| = 1$ for $\zeta \in A$, an extended form of Löwner's lemma therefore shows that mes $A \le$ mes $\phi_R(A) = 0$, which contradicts our assumption.

It is clear that $|g_R \circ \beta| = |g_R|$ for $\beta \in \Gamma_R$. Since g_R is bounded and non-constant, it follows (for instance from (1.5) and Fatou's theorem) that Γ_R is of convergence type.

Let now $f \in \mathfrak{G}(\Gamma)$ and let g_R^* be the Green's function of Γ_R with respect to 0. Since g_R^* is a Blaschke product with the same zeros as g_R, we see that $h_R = g_R/g_R^*$ is also an inner function. The definition of $\mathfrak{G}(\Gamma)$ shows that

$$|h_R(z)| \ge |g_R(z)| = R^{-1}|f(\phi_R(z))| > c_R > 0$$

if $z \notin \beta \circ \phi_R^{-1}(D_0) = \phi_R^{-1} \circ \gamma(D_0)$ for $\beta \in \Gamma_R$; see (2.3). Hence $|h_R|$ is bounded away from zero in $\mathbf{D}$ because this is trivially true in $\phi_R^{-1}(D_0)$ and therefore in

$$\bigcup_{\beta \in \Gamma_R} \beta \circ \phi_R(D_0).$$

An inner function that is bounded away from zero in $\mathbf{D}$ is a constant of modulus 1. Since, by (2.4),

$$h_R(0) = g_R'(0)/g_R^{*\prime}(0) = R\phi_R'(0)/g_R^{*\prime}(0) > 0$$

it follows that $h_R(z) \equiv z$ and thus that $g_R = g_R^*$.

The domain F in $\mathbf{D}$ is called a *fundamental domain* of Γ if $F \cap \gamma(F) = \emptyset$ for $\gamma \in \Gamma$, $\gamma \ne \iota$ and if

$$\text{area}\left(\mathbf{D} \setminus \bigcup_{\gamma \in \Gamma} \gamma(F)\right) = 0. \tag{2.5}$$

As the referee kindly pointed out this condition does not, in general, imply that every point of $\mathbf{D}$ belongs to $\gamma(\bar{F})$ for some $\gamma \in \Gamma$. Therefore our definition does not quite agree with the usual one; the same remark applies to [12, Theorem 1].

THEOREM 2. *Let* Γ *be of divergence type and let* $f \in \mathfrak{G}(\Gamma)$. *Then there exists a fundamental domain* F *of* Γ *with*

$$\{|z| < \rho/6\} \subset F \tag{2.6}$$

such that f *is univalent in* F *and the image domain* $f(F)$ *is starlike with*

$$\operatorname{area}(\mathbf{C} \setminus f(F)) = 0. \tag{2.7}$$

The domain F is the analogue of Green's fundamental domain [12, Theorem 1] for groups of convergence type which is essentially due to Brelot and Choquet [2].

Proof. Let F be the union of all halfopen arcs that begin at 0 and along which

$$\arg f(z) = \text{const}. \tag{2.8}$$

It is easy to see that F is a domain with $F \cap \gamma(F) = \emptyset$ for $\gamma \in \Gamma$, $\gamma \neq \iota$ and that f maps F one-to-one onto a starlike domain.

By Theorem 1, the function $g_R = R^{-1} f \circ \phi_R$ is the Green's function of Γ_R. By (2.2) and (2.8), $F \cap G_R$ is the union of all arcs from 0 with $\arg f(z) = \text{const}$ and $0 \leq |f(z)| < R$. Hence $\phi_R^{-1}(F \cap G_R)$ is the union of all arcs from 0 with $\arg g_R(z) = \text{const}$. But this is, by definition, the Green's fundamental domain F_R of Γ_R. Hence we have

$$F \cap G_R = \phi_R(F_R) \qquad (0 < R < \infty). \tag{2.9}$$

Using (2.3) we therefore see that

$$\bigcup_{\gamma \in \Gamma} \gamma(F) \supset \bigcup_{\gamma \in \Gamma} \gamma(F \cap G_R) \supset \bigcup_{\beta \in \Gamma_R} \phi_R \circ \beta(F_R) = \phi_R(\mathbf{D} \setminus E)$$

where area $E = 0$. Since $G_R = \phi_R(\mathbf{D})$ it follows that

$$\operatorname{area}\left(\mathbf{D} \setminus \bigcup_{\gamma \in \Gamma} \gamma(F)\right) \leq \operatorname{area}(\mathbf{D} \setminus G_R),$$

and this tends $\to 0$ as $R \to \infty$ because f is analytic in $\mathbf{D}$. Hence (2.5) is satisfied.

We obtain from (2.9) that, for $0 < R < \infty$,

$$\operatorname{area}(\{|w| < R\} \setminus f(F)) = R^2 \operatorname{area}(\mathbf{D} \setminus g_R(F_R)),$$

and this is $= 0$ because F_R is the Green's fundamental domain of Γ_R. Hence (2.7) follows. We postpone the proof of (2.6) to the next section.

3. ESTIMATES

For $z \in \mathbf{D}$, let $\rho(z)$ denote the largest number such that

$$\left\{ \zeta \in \mathbf{D} : \left| \frac{\zeta - z}{1 - \bar{z}\zeta} \right| < \rho(z) \right\} \tag{3.1}$$

contains no Γ-equivalent points; we set $\rho(z) = 0$ if z is an elliptic fixed point. In particular we have $\rho(0) = \rho$.

THEOREM 3. *Let Γ be a group of divergence type and let $f \in \mathfrak{G}(\Gamma)$. Then*

$$|f(z)| \geq \frac{1}{4} \min (\rho, \min_{\gamma \in \Gamma} |\gamma(z)|), \tag{3.2}$$

$$(1 - |z|^2)|f'(z)| \leq 2 \max (|f(z)|,1)\left(\log^+ |f(z)| + \frac{29}{\rho} \right) \tag{3.3}$$

for $z \in \mathbf{D}$ and also

$$(1 - |z|^2)|f'(z)| \leq \frac{12}{\rho(z)} \max (|f(z)|,1). \tag{3.4}$$

Remarks. 1. The above results hold also if Γ is of convergence type and $f(z) = g(z)/g'(0)$ $(z \in \mathbf{D})$ where g denotes the Green's function (1.1); compare [10].

2. We conclude at once from (3.4) and (3.2) that $\mathfrak{G}(\Gamma)$ is compact with respect to locally uniform convergence.

The proof is based on the theory of (circumferentially) mean univalent functions [6, Chapter 5]. The analytic function h is called *mean univalent* in the domain H if

$$\frac{1}{2\pi} \int_0^{2\pi} n(\mathrm{Re}^{i\theta},H)\, d\theta \leq 1 \qquad (0 < R < +\infty) \tag{3.5}$$

where $n(w,H)$ denotes the number of zeros of $h(z) - w$ with $z \in H$.

LEMMA 1. *Every function $f \in \mathfrak{G}(\Gamma)$ is mean univalent in every domain $H \subset \mathbf{D}$ that does not contain Γ-equivalent points.*

Proof. We consider the fundamental domain F of Theorem 2 and set

$$H_0 = H \cap \bigcup_{\gamma \in \Gamma} \gamma(F). \tag{3.6}$$

Then $\{z \in H_0 : |f(z)| = R\}$ is the disjoint union of open sets $\gamma_k(C_k)$ with $C_k \subset F$ and distinct $\gamma_k \in \Gamma$. The sets C_k are disjoint because H does not contain Γ-equivalent points. Hence we obtain from (1.2) that

$$\int_0^{2\pi} n(\mathrm{Re}^{i\theta}, H_0)\,d\theta = \sum_k \int_0^{2\pi} n(\mathrm{Re}^{i\theta}, C_k)\,d\theta$$
$$\leq \int_0^{2\pi} n(\mathrm{Re}^{i\theta}, F)\,d\theta \leq 2\pi$$

because f is (strictly) univalent in F. It follows from (2.7) and (3.6) that

$$\int_0^{2\pi} n(\mathrm{Re}^{i\theta}, H \setminus H_0)\,d\theta = 0.$$

Hence (3.5) is satisfied.

We need the following results of W. K. Hayman for functions h mean univalent in $\mathbf{D}$: If $h(s) = s + \dots$ then [6, p. 99]

$$\frac{|s|}{(1+|s|)^2} \leq |h(s)| \leq \frac{|s|}{(1-|s|)^2}, \tag{3.7}$$

$$|h'(s)| \leq \frac{1+|s|}{(1-|s|)^3}. \tag{3.8}$$

If $h(s) \neq 0$ for $|s| < 1$ then [6, p. 95]

$$|h'(s)| \leq 4\,|h(0)|\,\frac{1+|s|}{(1-|s|)^3}. \tag{3.9}$$

Proof of Theorem 3. (a) Since $\{|z| < \rho\}$ does not contain Γ-equivalent points, the function

$$h(s) = \rho^{-1} f(\rho s) = s + \dots \qquad (|s| < 1) \tag{3.10}$$

is mean univalent in $\mathbf{D}$, by Lemma 1. Hence we obtain from (3.7) that, with $z = \rho s$,

$$|f(z)| = \rho |h(s)| \geq \frac{\rho|s|}{(1+|s|)^2} \geq \frac{|z|}{4} \qquad \text{if } |z| < \rho. \tag{3.11}$$

Therefore it follows from (1.2) that, for $\gamma \in \Gamma$,

$$|f(z)| \geq (1/4)\,|\gamma(z)| \qquad \text{if} \qquad |\gamma(z)| < \rho. \tag{3.12}$$

The function $1/f$ is analytic and bounded in

$$G_1 = \{z \in \mathbf{D} : |\gamma(z)| > \rho \qquad \text{for} \qquad \gamma \in \Gamma\} \tag{3.13}$$

by the definition of $\mathfrak{G}(\Gamma)$. Furthermore $|1/f(z)| \leq 4/\rho$ for $z \in \mathbf{D} \cap \partial G$ by (3.12).

Since $\partial\mathbf{D}$ has zero harmonic measure [16, p. 530] with respect to G_1, it follows that $|1/f(z)| \le 4/\rho$ for $z \in G_1$. Together with (3.12), this proves (3.2).

(b) Applying (3.8) to the function (3.10) we obtain that

$$(1-|z|^2)|f'(z)| \le |f'(z)| = |h'(z/\rho)| \le 12$$

for $|z| \le \rho/2$. Hence it follows from (2.1) that

$$(3.14) \qquad (1-|z|^2)|f'(z)| \le 12 \quad \text{for} \quad z \notin G_2$$

where (compare (3.13))

$$(3.15) \qquad G_2 = \{z \in \mathbf{D} : |\gamma(z)| > \rho/2 \quad \text{for} \quad \gamma \in \Gamma\}.$$

We obtain from (3.11) as in (a) that $|f(z)| \ge 2\rho/9$ for $z \in G_2$. Hence we conclude from (3.14) and [9, Lemma] that

$$\begin{aligned}(1-|z|^2)|f'(z)| &\le 2|f(z)|\left(\log\left|\frac{9f(z)}{2\rho}\right| + \frac{27}{\rho}\right)\\ &\le 2|f(z)|\left(\log|f(z)| + \frac{29}{\rho}\right)\end{aligned}$$

for $z \in G_2$. Together with (3.14), this proves (3.3).

(c) We fix now $z \in G_2$ and set $r = \rho(z)/3$. The definition of $\rho(z)$ shows that $|\gamma(z)| + \rho \ge \rho(z)$ for $\gamma \in \Gamma$. Therefore we obtain from (3.15) that

$$\left|\frac{z-\gamma^{-1}(0)}{1-\bar{z}\gamma^{-1}(0)}\right| = |\gamma(z)| \ge \frac{|\gamma(z)|+\rho}{3} \ge \frac{\rho(z)}{3} = r$$

for $\gamma \in \Gamma$. It follows that

$$h(s) = f\left(\frac{z+rs}{1+\bar{z}rs}\right) \ne 0 \quad \text{for} \quad s \in \mathbf{D}.$$

This function is mean univalent in $\mathbf{D}$ by the definition of $\rho(z)$ and by Lemma 1. Hence we obtain from (3.9) that

$$(1-|z|^2)|f'(z)| = \frac{1}{r}|h'(0)| \le \frac{4}{r}|h(0)| = \frac{12}{\rho(z)}|f(z)|,$$

and, together with (3.14), this proves (3.4).

Proof of (2.6). Suppose that $D_1 = \{|z| < \rho/6\}$ does not completely lie in the fundamental domain F. Then D_1 intersects $\gamma(F)$ for some $\gamma \in \Gamma \setminus \{\iota\}$, say at z_1. We consider the curve $C \subset \gamma(F)$ through z_1 along which $\arg f(z) = \text{const}$. Since C begins at $\gamma(0)$ and since $|\gamma(0)| > \rho$, there exists $z_2 \in C$ with

$$|z_2| = \rho, \qquad |f(z_2)| < |f(z_1)|. \tag{3.16}$$

Applying (3.7) to the function (3.10) we obtain from $|z_1| < \rho/6$ that

$$|f(z_1)| \le \frac{|z_1|}{(1 - |z_1|/\rho)^2} < \frac{6\rho}{25} < \frac{\rho}{4}$$

whereas (3.11) shows that $|f(z_2)| \ge \rho/4$, in contradiction to (3.16).

4. ESTIMATES FOR THE GROWTH

Every normal analytic function f satisfies

$$\log^+ |f(z)| = O\left(\frac{1}{1 - |z|}\right) \quad \text{as } |z| \to 1 - 0 \tag{4.1}$$

as Hayman [5] has shown. The problem whether (4.1) can be improved for functions in $\mathfrak{G}(\Gamma)$ depends on their behaviour at the parabolic fixed points.

Let $f \in \mathfrak{G}(\Gamma)$ and let $\gamma \in \Gamma$ be parabolic with fixed point ζ. It follows from (1.2) and (4.1) by standard arguments that

$$f(z) = w(z)^{-a} \sum_{k=0}^{\infty} c_k w(z)^k, \qquad w(z) = \exp\left(-b \frac{\zeta + z}{\zeta - z}\right) \tag{4.2}$$

with $c_0 \neq 0$ for suitable $b > 0$ and $a \ge 0$; the case $a < 0$ is excluded by (3.2). If $a > 0$ then

$$\limsup_{|z| \to 1-0} (1 - |z|) \log^+ |f(z)| \ge 2ab > 0$$

so that (4.1) cannot be improved. If $a = 0$ then f has the finite angular limit c_0 at ζ.

Let $\mathfrak{G}_0(\Gamma)$ denote the class of $f \in \mathfrak{G}(\Gamma)$ that have a finite angular limit at each parabolic fixed point of Γ. The above paragraph shows that $\mathfrak{G}_0(\Gamma) = \emptyset$ if Γ is a finitely generated group of divergence type.

THEOREM 4. *Let Γ be an infinitely generated group of divergence type. Then $\mathfrak{G}_0(\Gamma) \neq \emptyset$, and if $f \in \mathfrak{G}_0(\Gamma)$ then*

$$\log^+ |f(z)| = o\left(\frac{1}{1 - |z|}\right) \qquad (|z| \to 1 - 0). \tag{4.3}$$

This estimate is best possible: For every function η with $\eta(r) \to +0$ $(r \to 1 - 0)$, there exists a group Γ such that, for all $f \in \mathfrak{G}(\Gamma)$, (4.4) $$\log^+|f(z)| \neq O\left(\frac{\eta(|z|)}{1-|z|}\right) \qquad (|z| \to 1-0).$$

The estimate (4.3) can be improved if we make further assumptions about Γ. For instance, if $\rho(z) \geq \rho_0 > 0$ for $z \in \mathbf{D}$, then we obtain by integration from (3.4) that

(4.5) $$\log^+|f(z)| = O\left(\log\frac{1}{1-|z|}\right) \qquad (|z| \to 1-0).$$

We need two lemmas in order to prove Theorem 4.

LEMMA 2. *Let $f \in \mathfrak{G}(\Gamma)$. Suppose that $h = \log f$ is analytic and univalent in some domain $H \subset \partial\mathbf{D}$ with $\partial H \cap \partial\mathbf{D} = \{\zeta\}$ and that*

(4.6) $$h(H) = \{w: \operatorname{Re} w > u_1, \quad v_1 < \operatorname{Im} w < v_1 + \lambda_1\}, \qquad \lambda_1 > 2\pi.$$

Then ζ is a parabolic fixed point for Γ and f has the angular limit ∞ at ζ.

Proof. Let A be an analytic arc in H such that

$$h(A) = \{\operatorname{Re} w = u_0, v_0 \leq \operatorname{Im} w \leq v_0 + \lambda_0\} \subset h(H), \qquad \lambda_0 > 2\pi.$$

Let F be the fundamental domain of Γ constructed in Theorem 2 and let H_0 be defined by (3.6). Then $\operatorname{mes}[A \cap (H \setminus H_0)] = 0$ by Theorem 2, and $A \cap H_0$ is the disjoint union of open sets $\gamma_k(C_k)$ with $C_k \subset F$ and distinct $\gamma_k \in \Gamma$. It follows that

(4.7) $$\lambda_0 = \operatorname{Im} \sum_k \int_{\gamma_k(C_k)} h'(z)\,dz = \operatorname{Im} \sum_k \int_{C_k} \frac{f'(z)}{f(z)}\,dz$$

because $h'(z) = f'(z)/f(z) = \gamma_k'(z) f'(\gamma_k(z))/f(\gamma_k(z))$ by (1.2).

Since $|f(z)| = e^{u_0}$ for $z \in C_k$ and since f is univalent in F, we conclude from (4.7) that the sets $C_k \subset F$ cannot be disjoint because $\lambda_0 > 2\pi$. Hence there exists $z_0 \in C_k \cap C_l$ with $k \neq l$. The points $z_1 = \gamma_k(z_0)$ and $z_2 = \gamma_l(z_0)$ lie in A, and

(4.8) $$z_2 = \gamma_l \circ \gamma_k^{-1}(z_1) = \gamma(z_1), \qquad \gamma = \gamma_l \circ \gamma_k^{-1} \in \Gamma \setminus \{\iota\}.$$

Let $C_j (j = 1,2)$ be the curves in H from z_j to ζ defined by

(4.9) $$h(C_j) = \{u_0 \leq \operatorname{Re} w < +\infty, \quad \operatorname{Im} w = \operatorname{Im} h(z_j)\}.$$

It follows from (1.2) that $h(\gamma(z)) = h(z) + ib$ for some $b \in \mathbf{R}$. Hence

$$h(\gamma(C_1)) = h(C_1) + ib = h(C_2),$$

by (4.8) and (4.9). Since h is univalent in H we conclude that $C_2 = \gamma(C_1)$. It follows that $\zeta = \gamma(\zeta)$ because C_1 and C_2 both end at ζ.

Since $f(z) \to \infty$ as $z \to \zeta$, $z \in C_1$ and since f is normal, the theorem of Lehto and Virtanen [7] shows that f has the angular limit ∞ at ζ. Hence ζ cannot be a hyperbolic fixed point so that γ is parabolic.

LEMMA 3. *Let Γ be of divergence type with Ford fundamental domain F_0. We denote by $l(r)$ the total length of $L(r) = F_0 \cap \{|z| = r\}$ $(0 < r < 1)$. Let $f \in \mathfrak{G}(\Gamma)$ and set*

$$M(r) = \max_{|z|=r} |f(z)| \tag{4.10}$$

for $0 < r < 1$. Then

$$2\pi \int_\rho^r \frac{dt}{l(t)} \leq \log \frac{4}{\rho} + \log M(r). \tag{4.11}$$

Proof. Since Γ-equivalent boundary points of F_0 have the same distance from 0, we see that

$$\int_{L(r)} \frac{f'(z)}{f(z)}\, dz = 2\pi i \qquad (0 < r < 1).$$

Hence we obtain from Schwarz's inequality that

$$4\pi^2 \leq l(r) \int_{L(r)} \left| \frac{f'(z)}{f(z)} \right|^2 |dz|$$

and therefore

$$4\pi^2 \int_\rho^r \frac{dt}{l(t)} = \iint_{F \cap \{\rho < |z| < r\}} \left| \frac{f'(z)}{f(z)} \right|^2 dxdy.$$

By Lemma 1, (3.2) and (4.10), this is bounded by

$$\iint_{\rho/4 < |w| < M(r)} \frac{dudv}{|w|^2} = 2\pi \left(\log M(r) + \log \frac{4}{\rho} \right).$$

Proof of Theorem 4. (a) Let $\Gamma = \{\gamma_\nu : \nu \in \mathbf{N}\}$ and let Γ_n be the group generated by $\gamma_1, \ldots, \gamma_n$. Since the Ford fundamental domain of Γ_n contains that of Γ, it has infinite non-euclidean area (because Γ is infinitely generated). Hence Γ_n is of convergence type. Let g_n denote its Green's function (1.1).

We see from Remark 1 after Theorem 3 that the functions

$$f_n(z) = g_n(z)/g_n'(0) = z + \ldots \qquad (z \in \mathbf{D}) \tag{4.12}$$

form a normal sequence. If $f(z) = z + \ldots$ is the limit of a convergent subsequence it follows from (3.2) that $f \in \mathfrak{G}(\Gamma)$.

Let now ζ be a parabolic fixed point and let γ be a generator of the stabilizer of ζ in Γ. Then $\gamma \in \Gamma_n$ for $n > n_0$. Since the function f_n is bounded and $|f_n \circ \gamma| = |f_n|$, we have as in (4.2) that

$$f_n(z) = \sum_{k=0}^{\infty} c_{nk} w(z)^k, \qquad w(z) = \exp\left(-b \frac{\zeta + z}{\zeta - z}\right) \tag{4.13}$$

where $b > 0$ depends only on γ. The boundary of the horocycle

$$H = \{z : |z - \zeta/2| < 1/2\} = \{z : |w(z)| < e^{-b}\}$$

has the form $\bigcup_{k=-\infty}^{+\infty} \gamma^k(A)$ for some arc $A \subset \mathbf{D}$. Hence

$$\sup_{z \in H} |f_n(z)| = \max_{z \in A} |f_n(z)| \qquad (n = 1,2,\ldots)$$

by (4.13). It follows that f is bounded in H. Hence the angular limit is finite and $f \in \mathfrak{G}_0(\Gamma)$.

(b) Suppose that (4.3) is false. Then [13] there exist $\zeta \in \partial\mathbf{D}$ and $c > 0$ such that

$$\log f(z) = c \frac{\zeta + z}{\zeta - z} + o\left(\frac{1}{|\zeta - z|}\right) \qquad (z \to \zeta)$$

in every Stolz angle at ζ. Writing $s = (\zeta + z)/(\zeta - z)$ we deduce that

$$\phi(s) = \log f(z) = cs + o(|s|) \qquad \text{as} \quad s \to \infty, \qquad |\arg s| < \pi/4. \tag{4.14}$$

Hence $\xi^{-1}\phi(\xi w) \to c$ as $\xi \to +\infty$ locally uniformly in $\{\operatorname{Re} w > 0\}$ and it follows by differentiation that $\phi'(\xi w) \to c$. Thus we see that

$$\phi'(s) \to c \qquad \text{as} \qquad s \to \infty, \qquad |\arg s| < \pi/4. \tag{4.15}$$

We obtain from (4.15) by integration that ϕ is univalent in

$$\{\sigma < |s| < \infty, |\arg s| < \pi/4\}$$

for some σ and that the image domain contains some Stolz angle at ∞ and therefore a halfstrip (4.6). Hence it follows from (4.14) and Lemma 2 that ζ is a parabolic fixed point and that $\log f$ has the angular limit ∞ at ζ, in contradiction to the definition of $\mathfrak{G}_0(\Gamma)$.

(c) In order to show that (4.3) is best possible, we construct an increasing sequence of finitely generated Fuchsian groups Γ_n of the second kind. Their Ford fundamental domains F_n are symmetric with respect to $\mathbf{R}$; there are finitely many cusps and the arc $(e^{-i\theta_n}, e^{i\theta_n})$ of $\partial\mathbf{D}$ is the only free side of F_n. We also construct a sequence $r_n \to 1 - 0$ such that, with $l_n(r) = \operatorname{mes}(F_n \cap \{|z| = r\})$,

$$\int_{\rho}^{r_n} \frac{dr}{l_n(r)} > \frac{k\eta(r_k)}{1 - r_k} \qquad \text{for} \quad k = 1, \ldots, n. \tag{4.16}$$

Suppose the constructions have been carried out up to n. Let C_n and C'_n be the circles orthogonal to $\partial\mathbf{D}$ from 1 to $e^{i\theta_n}$ and from 1 to $e^{-i\theta_n}$ and let γ_n^* be the parabolic transformation that satisfies $\gamma_n^*(C_n) = C'_n$ and for which C_n and C'_n are isometric circles. The group $\langle \Gamma_n, \gamma_n^* \rangle$ generated by Γ_n and γ_n^* is Fuchsian [4, p. 56] and of the first kind. Hence $l_n^*(r) = O((1 - r)^2)$ as $r \to 1 - 0$. Thus we can find r_{n+1} with $(1 + r_n)/2 < r_{n+1} < 1$ such that

$$\int_{\rho}^{r_{n+1}} \frac{dr}{l_n^*(r)} > \frac{(n + 1)\eta(r_{n+1})}{1 - r_{n+1}}. \tag{4.17}$$

Now we replace C_n and C'_n by circles from $e^{i\theta_{n+1}}$ to $e^{i\theta_n}$ and from $e^{-i\theta_{n+1}}$ to $e^{-i\theta_n}$ with $0 < \theta_{n+1} < \theta_n$. Let γ_{n+1} be the hyperbolic transformation for which these circles are isometric and let $\Gamma_{n+1} = \langle \Gamma_n, \gamma_{n+1} \rangle$. If θ_{n+1} is chosen sufficiently small then (see (4.17))

$$\int_{\rho}^{r_{n+1}} \frac{dr}{l_{n+1}(r)} > \frac{(n + 1)\eta(r_{n+1})}{1 - r_{n+1}}.$$

Since $l_{n+1}(r) < l_n(r)$, it follows that the inequalities (4.16) hold with n replaced by $n + 1$. This concludes the construction of (Γ_n) and (r_n).

Now the union Γ of all groups Γ_n has $F_0 = \bigcap F_n$ as its Ford fundamental domain, and since $l(r) \leq l_n(r)$ we obtain from (4.16) that

$$\int_{\rho}^{r_k} \frac{dt}{l(t)} dt > \frac{k\eta(r_k)}{1 - r_k} \qquad (k = 1,2,\ldots). \tag{4.18}$$

Let $f \in \mathfrak{G}(\Gamma)$. Then we obtain from Lemma 3 and (4.18) that

$$\log M(r_k) \geq -\log \frac{4}{\rho} + \frac{2\pi k\eta(r_k)}{1 - r_k} \qquad (k = 1,2,\ldots)$$

so that

$$\log M(r) \neq O\left(\frac{\eta(r)}{1 - r}\right) \qquad (r \to 1 - 0).$$

This proves the final assertion of Theorem 4 because of (4.10).

5. THE ANGULAR LIMIT POINTS OF Γ

Let $L_0(\Gamma)$ be the set of angular limit points of Γ; see (1.4).

THEOREM 5. *Let Γ be a Fuchsian group that is not both finitely generated* *and of the first kind. Then $\partial\mathbf{D}\setminus L_0(\Gamma)$ has uncountably many points on each arc of $\partial\mathbf{D}$.*

Proof. By conjugation, we may assume that 0 is not an elliptic fixed point of Γ and, by (1.5), we may assume that Γ is of divergence type. It follows then that Γ is infinitely generated. By Theorem 4, there exists a function $f \in \mathfrak{G}_0(\Gamma)$ and this function satisfies (4.3).

Since f is normal, it therefore follows from a result of Lohwater and the author [8, Theorem 3] that the set A of points where f has a finite or infinite angular limit has uncountably many points on each arc of $\partial\mathbf{D}$.

Let $\zeta \in L_0(\Gamma)$ and choose $\gamma_k \in \Gamma$ as in (1.4). If $f(z_1) \neq 0$ then

$$|f(\gamma_k(0))| = |f(0)| = 0, \qquad |f(\gamma_k(z_1))| = |f(z_1)| \neq 0 \qquad (k = 1,2,\ldots)$$

by (1.2). Since both $\gamma_k(0)$ and $\gamma_k(z_1)$ lie in some Stolz angle at ζ, we see that f cannot have an angular limit at ζ. Hence $A \cap L_0(\Gamma) = \emptyset$ and the assertion follows.

Remark. The set $\partial\mathbf{D}\setminus L_0(\Gamma)$ is Γ-invariant. Equivalence classes of points in $\partial\mathbf{D}\setminus L_0(\Gamma)$ were called ideal boundary points of $\mathbf{D}/\Gamma$ by Constantinescu [3, p. 49]. It follows from Theorem 5 that, if $\mathbf{D}/\Gamma$ is infinitely connected or of infinite genus, then there are uncountably many ideal boundary points in this sense. It is, however, easy to construct a Riemann surface of infinite genus (a semi-infinite string of tori) that appears to have only one ideal boundary point in an intuitive sense. Thus it seems that Constantinescu's definition is too general.

Purzitsky [14] has studied the unexpected difficulties that arise in connection with ideal boundaries for infinitely generated Fuchsian groups.

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Technische Universitat Berlin
Fachbereich 3-Mathematik
Strasse de 17 Juni 135
1 Berlin 12, WEST GERMANY