ORTHOGONAL PAIRINGS OF EUCLIDEAN SPACES

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I. INTRODUCTION

1.1. By $\mathbf{R}^n$ we denote n-dimensional Euclidean space. An *orthogonal pairing* is a bilinear map $\mu:\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^p$, such that $\|\mu(x,y)\| = \|x\|\,\|y\|$. We say that μ has type $[m,n,p]$ or is an $[m,n,p]$-pairing. An orthogonal pairing is non-degenerate in the sense that $\mu(x,y) = 0$ implies either $x = 0$ or $y = 0$. In this paper we study the problems of existence and classification of $[m,n,p]$-pairings for some positive integers m, n, p.

1.2. There are many mathematical problems where orthogonal pairings appear.

a) Orthogonal pairings are the main tool for the construction of vector fields on spheres and projective spaces (see Section 2).

b) The T-algebras of rank 3 constructed by E. B. Vinberg for studying homogeneous cones are reduced to orthogonal pairings [15].

c) Any quadratic mapping from a sphere to a sphere is homotopic to the quadratic mapping constructed in a standard way from an orthogonal pairing [16].

d) Any orthogonal (and even any non-degenerate) $[m,m,p]$-pairing generates an elliptic linear system of m differential equations of first order with m unknown functions of p variables [14, p. 273].

1.3. A. Hurwitz [7] and J. Radon [13] gave a complete answer on the existence question for $[m,n,n]$-orthogonal pairing: it exists if and only if $m \le \rho(n)$, where $\rho(n)$ is the Hurwitz-Radon number equal to $2^{c(n)} + 8d(n)$, where

$$n = 2^{c(n)}16^{d(n)}n_1, \qquad 0 \le c(n) \le 3, \quad d(n) \ge 0, \quad n_1 \text{ odd}.$$

The result was proved also by Eckmann [5] with the help of representations of finite groups and by Atiyah, Bott, Shapiro [2] with the help of classification of Clifford modules. A classification (trivial) of $[m,n,n]$-orthogonal pairing is given in these papers too.

There are only partial results for the general case. Beherend [3] and H. Hopf [6] have proved in two different ways necessity of the following condition on m, n, p for existence an orthogonal (or even non-degenerate) $[m,n,p]$-pairing: $C_p^k \equiv 0 \pmod 2), p - m < k < n$. If either m or n is ≤ 8 the condition is sufficient. A lot of papers [10], [11], [4] are devoted to necessary conditions for existence and to constructions of non-degenerate pairings (V. S. Pjasetzky has some unpublished results on orthogonalization of these pairings).

1.4. The plan of the paper is the following. In Section 2 we give a necessary

Received March 5, 1980. Revision received May 22, 1980.

Michigan Math. J. 28 (1981).

condition for the existence of such a pairing independent of the Beherend-Hopf condition and generalizing the Hurwitz-Radon condition. For that we use a standard procedure, well known in topology. The main subject of Section 3 is algebraic: we study addition of subsets in the dyadic group as a method for the construction of pairings. The Beherend-Hopf condition appears unexpectedly again as the necessary and sufficient condition on cardinalities of subsets. We give also examples of existence and nonexistence of pairings of some types. In Section 4 we define an equivalence relation of pairings, find some invariants of the equivalence, introduce algebras connected with pairing, and classify pairings of low dimensions.

2. NECESSARY CONDITIONS FROM FIBRE BUNDLE THEORY

Let $\mu : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^p$ be an orthogonal pairing. Let us fix in $\mathbf{R}^m$ an orthogonal basis $\{e_0, e_1, \ldots, e_{m-1}\}$ and put $A_i x = \mu(e_i, x)$, $i = 0, 1, \ldots, m-1$, $x \in \mathbf{R}^n$. Let us denote by S^{p-1} the unit sphere of $\mathbf{R}^p$ and by S^{n-1} the image of the unit sphere of $\mathbf{R}^n$ by action of A_0. Since the operator A_0 is orthogonal S^{n-1} is embedded linearly into S^{p-1}. The operators $A_i A_0^{-1}$, $i = 1, \ldots, m-1$ are defined on S^{n-1} and generate there $m-1$ mutually orthogonal vector fields tangential to S^{p-1}. Since the fields are skew they generate fields tangential to $\mathbf{R}P^{p-1}$ defined on $\mathbf{R}P^{n-1}$ (which is linearly embedded into $\mathbf{R}P^{p-1}$).

Let us denote by ξ_k the canonical (Hopf) vector bundle on $\mathbf{R}P^k$. Using a standard procedure (see for example [12]) we obtain the following theorem.

THEOREM 1. *If there exists an orthogonal $[m,n,p]$-pairing then there exists a trivial m-dimensional subbundle (with the base $\mathbf{R}P^{n-1}$) in $P\xi_{n-1}$.*

Now for any positive integer k let us denote by $e(k)$ the number of all integers l such that $0 < l \le k$ and $l \equiv 0, 1, 2, 4 \pmod 8$. Then we have (*see* [1], [12]).

COROLLARY. *Under the conditions of Theorem* 1,

$$C_p^k \equiv 0 \pmod{2^{e(n-1)-k+1}}, \qquad p - m < k \le e(n-1) \tag{2.1}$$

Remark 1. In order to prove (2.1), Grothendieck operations in K-theory were used in [1, 12]. Using in the same way the Stiefel-Whitney classes one can obtain another proof of the Beherend-Hopf condition.

Remark 2. Since the definition of pairing is symmetric in m and n we can get a new necessary condition from (2.1) by interchanging m and n:

$$C_p^k \equiv 0 \pmod{2^{e(m-1)-k+1}}, \qquad p - n < k \le e(m-1). \tag{2.2}$$

Remark 3. Generally speaking conditions (2.1) and (2.2) are independent of the Beherend-Hopf condition. If $n = p$ then condition (2.2) is equivalent to the Hurwitz-Radon condition. Therefore it is stronger in that case than the Beherend-Hopf condition and is sufficient.

Example. For the triplet (10,12,16), the Beherend-Hopf condition, (2.1) and (2.2) are all true. But even non-degenerate [10,12,16]-pairings do not exist (see Table II and Remark 3.5 in [11]).

3. MONOMIAL PAIRING. ADDITION OF SUBSETS IN THE DYADIC GROUP

3.1. Let $\mu : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^p$ be an orthogonal pairing. We say it is *monomial* if there exist orthogonal bases $A = \{a_i\}_{i=1}^m$, $B = \{b_j\}_{j=1}^n$, $C = \{c_k\}_{k=1}^p$ in the spaces $\mathbf{R}^m$, $\mathbf{R}^n$, $\mathbf{R}^p$ such that $\mu(A \times B) \subset C \cup (-C)$. Such a triplet of bases will be called *multiplicative.*

Remark 4. If $p = n$ all pairings are monomial.

3.2. If (A,B,C) is a multiplicative triplet of bases for a monomial pairing μ then two maps are defined:

$$\varphi : A \times B \to C, \qquad \epsilon : A \times B \to \{1,-1\},$$

where $\mu(a,b) = \epsilon(a,b)\varphi(a,b)$, $(a,b) \in A \times B$. The pair (φ,ϵ) is said to be a *basic pairing* for μ.

THEOREM 2. *Let A, B, C be finite sets, $\varphi : A \times B \to C$, $\epsilon : A \times B \to \{1,-1\}$ be maps. The pair (φ,ϵ) is a basic pairing for a monomial pairing if and only if the following three conditions are true:*

3.1) *for any $a \in A$ ($b \in B$) the map $\varphi|_{a \times B}$ ($\varphi|_{A \times b}$)* is *injective,*

3.2) $\varphi(a_1,b_1) = \varphi(a_2,b_2)$ ($a_i \in A$, $b_i \in B$, $i = 1,2$) *implies* $\varphi(a_1,b_2) = \varphi(a_2,b_1)$,

3.3) $\varphi(a_1,b_1) = \varphi(a_2,b_2)$ *implies* $\epsilon(a_1,b_1) \cdot \epsilon(a_2,b_2) = -\epsilon(a_1,b_2)\,\epsilon(a_2,b_1)$.

The proof of the theorem is a direct verification (compare [17, Lemma 1).

3.3. It is natural to begin the construction of monomial pairings with a consideration of functions φ satisfying conditions 3.1 and 3.2. One can get a large class of examples of such functions from addition of finite subsets of the dyadic group D, that is, the Abelian group with a countable set of generators of order two. More precisely, the next lemma is obvious.

LEMMA 1. *Let A, B, C be finite subsets of D, $A + B \subset C$. Then the map $\varphi : A \times B \to C$ defined by the formula*

$$\varphi(a,b) = a + b, \qquad a \in A, \quad b \in B$$

satisfies the conditions 3.1 *and* 3.2.

Remark 5. As a matter of fact, addition of subsets of D generates a multiplication of linear subspaces in the group algebra of D over any field.

Now we will give a complete solution of a question about the sizes of the sets A, B, C.

Definition. A triplet (m,n,p) of nonnegative integers is said to be *pure* if either $mn = 0$ or $m, n \le p$ and the Beherend-Hopf condition is true, that is $C_p^k \equiv 0 \pmod 2$, $p - m < k < n$.

THEOREM 3. *For a triplet (m,n,p) of nonnegative integers the following conditions are equivalent:*

(1) *(m,n,p) is pure,*

(2) *there exist finite subsets A, B, C of cardinalities m, n, p in D such that $A + B \subset C$.*

First we will prove some lemmas. The proofs of Lemmas 2, 3, 4, 6 are straightforward and use only simple properties of binomial coefficients; Lemma 8 is the main one.

LEMMA 2. *If (m,n,p) is pure then (n,m,p) is pure.*

LEMMA 3. *If (m,n,p) is pure, $m' \le m$, $n' \le n$, $p' \ge p$ then (m',n',p') is pure.*

LEMMA 4. *(m,n,p) is pure if and only if $(2m,2n,2p)$ is pure.*

LEMMA 5. *If (m,n,p) is pure and t is a positive integer such that $m, n, p, \le 2^t$ then $(m,2^t - p,2^t - n)$ is pure.*

Proof. Let us assume to the contrary that $p < 2^t$ and that an integer r does exist such that $p - n < r < m$ and $C^r_{2^t - n} \equiv 1 \pmod 2$ (if $m > 2^t - n$ one can put $r = 2^t - n$). By a lemma of Beherend (see [3]), r is a partial sum for $2^t - n$ i.e. if $r = \sum_{i=0}^{t} \alpha_i 2^i$, $2^t - n = \sum_{i=0}^{t} \beta_i 2^i$ where $\alpha_i, \beta_i = 0$ or 1 then $\beta_i = 0$ implies $\alpha_i = 0$. It follows that r is a partial sum for $r + n - 1$ hence $C^r_{r+n-1} \equiv 1 \pmod 2$. Since $r + n - 1 \le p$ and $p - n < r < m$ we have a contradiction.

LEMMA 6. *If (m,n_1,p_1) and (m,n_2,p_2) are pure then $(m,n_1 + n_2,p_1 + p_2)$ is pure.*

LEMMA 7. *If (m,n,p) is pure, k, t are positive integers such that $2^{t-1} < m \le 2^t$, $n = k2^t + n_1$, $p = k2^t + p_1$ where $0 \le n_1, p_1 < 2^t$ then (m,n_1,p_1) is pure.*

Proof. It is sufficient to consider the case when $n_1 > 0$. Let us prove first that $m \le p_1$. If m were greater than p_1, since p_1 is a partial sum for p and $p_1 > p - n = p_1 - n_1$, we would have a contradiction with the purity of (m,n,p).

Let us assume now that a positive integer x does exist such that

$$p_1 - n_1 = p - n < x < m \qquad \text{and} \qquad C^x_{p_1} \equiv 1 \pmod 2.$$

It is clear that $C^x_p \equiv 1 \pmod 2$, but this again contradicts purity of (m,n,p).

LEMMA 8. *If (m_1,n_1,p_1), (m_2,n_2,p_1), (m_1,n_2,p_2), (m_2,n_1,p_2) are pure then $(m_1 + m_2, n_1 + n_2, p_1 + p_2)$ is pure.*

Proof. Let us assume for definiteness that $m_1 \le m_2$, $n_1 \le n_2$. If either $p_1 \le p_2$ or $m_1 = m_2$ or $n_1 = n_2$ then the lemma follows from Lemmas 2, 3, 4. Therefore we shall assume that $p_1 > p_2$, $m_1 < m_2$, $n_1 < n_2$. Let us introduce now some notations which we will use only for this proof. N denotes the set of all nonnegative integers. If $a \in N$ then $a(i)$ is the signs of the binary representation of a that is $a = \sum_{i=0}^{l(a)} a(i)2^i$ where $a(i) = 0$ or 1, $i = 0, 1, \ldots, l(a) - 1$, $a_{l(a)} = 1$. We call $l(a)$ the *length* of a, by the *height* of a we mean the sum $h(a) = \sum_{i=0}^{l(a)} a(i)$. For $r, s \in N$, $0 \le s < r \le l(a) + 1$, we put $a_r = \sum_{i=0}^{r-1} a(i)2^i$, $a^{(r)} = a - a_r$, $a(r,s) = a^{(s)} - a^{(r)}$. If $\alpha \subset [s,r) \cap N$ and $a(i) = 1$, $i \in \alpha$, then we put

$$a(r,s,\bar{\alpha}) = \sum_{i \in [r,s) \cap N - \alpha} a(i)2^i.$$

If $b \in N$, $l(b) \le l(a)$ and $b(i) = 1$ implies $a(i) = 1$ then we write $b \subset a$. Finally, if $x \in N$, $0 < x < a$, we put

$$U_a x = \min\{y \in N \mid y > x, y \subset a\}, \qquad D_a x = \max\{y \in N \mid y < x, y \subset a\}.$$

We need the following technical lemma.

LEMMA*. *Let* $a, b, c, d, r \in N, r \leq l(a), a(r) = 1, b \subset a_r, c + d < a_r + b$. *Then* $U_a c + U_a d \leq 2^r + b$.

*Proof of Lemma**. Let us apply induction on $h(b)$. First let $h(b) = 0$, that is $b = 0$, $c + d < a_r$. We denote by s the greatest number from N such that $c_s \geq a_s$ and put

$$\Delta = \{i \in N \mid s + 1 \leq i < r, \quad a(i) = 1, \quad c(i) = 0\},$$
$$\delta = \{i \in N \mid s \leq i < r, \quad a(i) = 1, \quad c(i) = 1\}.$$

Then $U_a c = a(r,s + 1;\bar{\Delta}) + 2^s$, $d < a_r - C \leq a(r,s;\bar{\delta})$ and $U_a d \leq a(r,s;\bar{\delta})$. Hence

$$U_a c + U_a d \leq a(r,s + 1;\bar{\Delta}) + 2^s + a(r,s;\bar{\delta}) = a(r,s) + 2^s \leq 2^r.$$

Now let $h(b) > 0$ and the statement be true for numbers of height less than $h(b)$. For brevity we put $C = U_a c$, $D = U_a d$ and assume $C + D > 2^r + b$.

Let us consider some particular cases.

1) C(or D) $= 2^r$ and consequently $D > b$. Then

$$c \geq D_a C = a_r, \quad d \geq D_a D \geq b \qquad \text{and} \qquad c + d \geq a_r + b.$$

2) C (or D) $> 2^r$. Since $a(r) = 1$, $c \geq D_a C \geq 2^r$. We put $c_1 = c - 2^r$, $s = l(b)$ and note that $U_a c_1 = C - 2^r$, $U_a c_1 + U_a d > 2^s + b_s$. Since $h(b_s) < h(b)$, by induction $c_1 + d \geq a_s + b_s$ and consequently $c + d \geq 2^r + a_s + b_s = 2^r - 2^s + a_s + b \geq a_r + b$.

3) $C, D < 2^r$. We put $t = \max\{i \in N \mid C(i) = D(i) = 1\}$. Since $C + D > 2^r$, $C^{(t)} + D^{(t)} = 2^r$. If C_t(or D_t) $= 0$ then $c \geq D_a C = C^{(t)} - 2^t + a_t$, $d \geq D_a D \geq D^{(t)} + b$ and $c + d \geq 2^r - 2^t + a_t + b \geq a_r + b$. If $C_t \cdot D_t \neq 0$ then we put $c_1 = c - C^{(t)}$, $d_1 = d - D^{(t)}$ and note that $c_1, d_1 \in N$, $U_a c_1 = C_t$, $U_a d_1 = D_t$ and

$$U_a c_1 + U_a d_1 > b = 2^s + b_s.$$

We get $c_1 + d_1 \geq a_s + b_s$ using induction again. The same calculation as in (2) completes the proof of Lemma*.

*Remark**. As is clear from the proof, the condition $a(r) = 1$ is used only in the case (2). It is not required if $c, d < a_r$ (then $C, D \leq a_r < 2^r$ and the case (2) is impossible).

We continue now the proof of Lemma 8. Let us put $m = m_1 + m_2$, $n = n_1 + n_2$, $p = p_1 + p_2$ and assume that there exists a positive integer x such that $p - n < x < m$ and $x \subset p$ (compare the proof of Lemma 5). We put $l = l(x)$ and denote by q_1, q_2 such two elements from N that $q_i \subset p_i$, $i = i, 2$, $q_1 + q_2 = x$ (generally speaking such a pair is not unique and when this is required we will put an extra condition on it). Let us consider some particular cases.

I) $p_{1,l} + p_{2,l} < 2^l$. This means in particular that one of the integers $p_1(l)$, $p_2(l)$ is equal to 0 and the other to 1 and the same is true for $q_1(l)$, $q_2(l)$.

Moreover

$$p_1 - p_2 = p_1^{(l)} - p_2^{(l)} + p_{1,l} - p_{2,l} \le p_1 - n_2 < x < 2^{l+1}$$

from which $p_1^{(l)} - p_2^{(l)} < 2^{l+1} + 2^l$. Consequently $p_1^{(l)} - p_2^{(l)} = 2^l$ and

$$p_1 - p_2 = 2^l + p_{1,l} - p_{2,l}.$$

A) We consider first the case when $p_1(l) = q_1(l) = 1$ and $p_2 - n_1 \ge p_{2,l}$. In that case $p_1 - n_2 < x - p_{2,l} = q_1 + q_2 - p_{2,l} \ge q_1$ and

$$\begin{aligned} p_2 - n_2 &= p_1 - n_2 - (p_1 - p_2) < x_l - p_{1,l} \\ &= q_{1,l} + q_{2,l} - p_{1,l} \le q_{2,l} = q_2. \end{aligned}$$

Since $m_1 + m_2 > x$, either $q_1 < m_2$ or $q_2 < m_1$, which contradicts the purity of either (m_2,n_2,p_1) or (m_1,n_2,p_2).

B) Let now either $p_2(l) = q_2(l) = 1$ or $p_2 - n_1 < p_{2,l}$. Since

$$\begin{aligned} (p_2 - n_2) + (p_2 - n_1) &= (p - n) - (p_1 - p_2) < x - (p_1 - p_2) \\ &= x_l + p_{2,l} - p_{1,l} \le p_{2,l} + q_{2,l} \end{aligned}$$

the conditions of Lemma* (taking into consideration Remark*) are true for

$$a = p_2,\ b = q_{2,l}, \quad c = p_2 - n_2,\ d = p_2 - n_1, \quad r = l.$$

By that lemma and the purity of (m_1,n_2,p_2), (m_2,n_1,p_2),

$$m_1 + m_2 \le 2^l + q_{2,l} \le 2^l + x_l = x$$

which contradicts the choice of x.

II) $p_{1,l} + p_{2,l} \ge 2^l$. In that case $p_1(l) = p_2(l)$. Since $p_{1,l} + p_{2,l} > x$ and $p_1 - p_2 < x$, then $p_1^{(l)} - p_2^{(l)} < 2^{l+1}$. This implies $p_1^{(l)} = p_2^{(l)}$, $p_1 - p_2 = p_{1,l} - p_{2,l}$. We put $k = \max\{i \in N \mid i < l, p_1(i) = p_2(i) = 1\}$ and put the following extra conditions on q_1, q_2: $p_i(l,k) \subset q_i$, $i = 1, 2$.

At first we consider the case $k < l - 1$. Then $p_{2,l} < 2^{l-1}$ and since

$$m_2 > x/2 \ge 2^{l-1},\ m_2 > p_{2,l}.$$

By purity of (m_2,n_1,p_2), $p_2 - n_1 \ge p_{2,l}$ and therefore

$$p_2 - n_2 < x - p_{2,l} \le q_1, \qquad p_2 - n_1 < x - p_{1,l} \le q_2.$$

The end of the proof is the same as in I, A.

B) Now let $k = l - 1$. If $m_1 \le 2^k$ then $m_2 > x - 2^k \ge \max\{q_1,q_2\}$, and since either $p_1 - n_2 < q_1$ or $p_2 - n_1 < q_2$ this contradicts the purity of either (m_2,n_2,p_1) or (m_2,n_1,p_2). Therefore we can consider the case when $m_1 > 2^k$ and (by the purity of (m_1,n_2,p_2)) $p_2 - n_1 \ge 2^k$. We use induction on $h(x)$. If $h(x) = 1$ (that is $x = 2l$)

then $(p_1 - n_2) + (p_2 - n_1) < 2l$, $p_2 - n_2 < 2^k$ and we get a contradiction. Now let $h(x) > 1$ and the statement be true for all numbers of a smaller height.

We put $p'_i = p_i - 2^k$, $m'_i = m_i - 2^k$ $(i = 1,2)$, $x' = x - 2^l$. It is easy to verify that (m'_1,n_1,p'_1), (m'_2,n_2,p'_2), (m'_1,n_2,p'_2), (m'_2,n_1,p'_2) are pure. On the other hand $x' \subset p' = p'_1 + p'_2$, $p' - n < x' < m' = m'_1 + m'_2$ and $h(x') < h(x)$. By induction we have a contradiction and the proof of Lemma 8 is finished.

Proof of Theorem 3. Necessity. We use induction on p. If $p = 1$ the statement is obvious. Let us assume that it is true for all triplets (m',n',p') where $p' < p$ and let A, B, C be such subsets of D of powers m, n, p that $A + B \subset C$. We can assume that $A \cap B \cap C \ni 0$. Let us denote by $d_1, d_2, \dots$ generators of D and by D_k the subgroup of D generated by $d_1, \dots, d_k$, and put $N = \min\{k \mid D_k \supset A,B,C\}$. Then we put

$$B_1 = B \cap D_{N-1}, \quad A_1 = A \cap D_{N-1}, \quad C_1 = C \cap D_{N-1}, \quad A_2 = A \setminus A_1,$$
$$B_2 = B \setminus B_1, \quad C_2 = C \setminus C_1$$

and denote the cardinalities of A_i, B_i, C_i $(i = 1,2)$ by m_i, n_i, p_i. Since D_{N-1} is a subgroup of index 2 in D_N,

$$A_1 + B_1 \subset C_1, \quad A_2 + B_2 \subset C_1, \quad A_1 + B_2 \subset C_2, \quad A_2 + B_1 \subset C_2.$$

Since $C \ni 0$, $p_1, p_2 < p$ and by induction the triplets (m_1,n_1,p_1), (m_2,n_2,p_1), (m_1,n_2,p_2), (m_2,n_1,p_2) are pure. By Lemma 8 the triplet (m,n,p) is pure.

Sufficiency. Now we use induction on m. If $m = 1$ the statement is obvious. Let us assume that $m > 1$ and for any pure triplet (m',n',p') where $m' < m$ and $m', n', p' \le 2^s$ there exist sets A', B', C' of cardinalities m', n', p' in D_s such that $A' + B' \subset C'$. Let now (m,n,p) be a pure triplet and t the integer such that $2^{t-1} < m \le 2^t$.

1) We assume first that $n < 2^t$, $p < 2^t$ and put $m_1 = 2^t - p$, $n_1 = m$, $p_1 = 2^t - n$. By Lemma 5, the triplet (m_1,n_1,p_1) is pure and since $m_1 \le 2^{t-1} < m$, there exist subsets A_1, B_1, C_1 of cardinalities m_1, n_1, p_1 in D_t such that $A_1 + B_1 \subset C_1$. The sets $A = B_1$, $B = D_t \setminus C_1$, $C = D_t \setminus A_1$ satisfy the statement and lie in D_t.

2) In general case there exist nonnegative integers k, l such that $n = k2^t + n_0$, $p = l2^t + p_0$ where $0 \le n_0$, $p_0 < 2^t$. Let first $l > k$. We denote by A, B_0 subsets of cardinalities m, n_0 in D_t, by $a_1, \dots, a_{k+1}$ elements of D_s where $2^{s-1} < k + 1 \le 2^s$, we represent D_{t+s} as a direct sum $D_t \oplus D_s$ and put $B = \bigcup_{i=1}^{k} (D_t \oplus a_i) \cup (B_0 \oplus a_{k+1})$. The cardinality of B is n. Identifying D_t with the subgroup $D_t \oplus 0$ of D_{t+s} we get the set $C' = A + B$ of size less than $(k + 1)2^t$ in D_{t+s}. If u is the integer such that $2^{u-1} < l + 1 \le 2^u$ if $p_0 > 0$, $2^{u-1} < l \le 2^u$ if $p_0 = 0$ and C is a set of cardinality p such that $C' \subset C \subset D_{t+u}$ we get the desired sets A, B, C.

3) Finally let $l = k$. By Lemma 7 the triplet (m,n_0,p_0) is pure and as was proved in (1) there exist sets A, B_0, C_0 of cardinalities m, n_0, p_0 in D_t such that $A + B_0 \subset C_0$. The proof can be easily completed by using a direct sum construction (compare (2)).

3.4. *Conjecture.* Let A, B, C be finite sets of cardinalities m, n, p and φ be a map $A \times B \to C$ satisfying the conditions (1) and (2) of Theorem 2. Then the triplet (m,n,p) is pure.

The conjecture can be formulated in terms of either graph theory or matrix theory. It is proved in a lot of special cases but a complete proof is unknown to me.

3.5. Let $A, B, C \subset D_N$, $A + B \subset C$, $\varphi(a,b) = a + b$, $a \in A$, $b \in B$. We consider now the question of existence of a map $\epsilon: A \times B \to \{-1,1\}$ satisfying the condition (3) of Theorem 2. For any pair of different elements x, y from D_N we denote by $H_{x,y}$ the subgroup of the group $G = D_N \oplus D_N$ which consists of the four elements: (0,0), $(x,0)$, $(0,y)$, (x,y). We denote by α the set of all cosets in G of all subgroups $H_{x,y}$ $(x,y \in D_N, x \neq y)$. Finally, for any finite collection of subsets $X_1, \ldots, X_k$ of G we denote by $\mathop{\Delta}\limits_{i=1}^{k} X_i$ their symmetric difference, that is, the set of all elements of G which belong to an odd number of sets from the collection $X_1, \ldots, X_k$.

THEOREM 4. *The following are equivalent:*

i) *there exists a map* $\epsilon: A \times B \to \{1,-1\}$ *satisfying the condition* (3) *of Theorem* 2,

ii) *any finite collection* $\{X_1,\ldots,X_k\}$ *of subsets of* G, *such that*

$$X_i \subset A \times B, \quad X_i \in \alpha \quad \text{and} \quad \mathop{\Delta}\limits_{i=1}^{k} X_i = \emptyset,$$

contains an even number of sets.

Sketch of Proof. Using additive notation we can formulate condition 2) of Theorem 2 as follows

$$\sum_{x \in H} \epsilon'(x) = 1 \tag{3.4}$$

where $\epsilon': A \times B \to \{0,1\}$, $H \subset A \times B$, $H \in \alpha$. Hence i) is equivalent to the solvability of system (3.4) of linear equations over field F_2. The statement directly follows from standard criteria of solvability.

3.6. Checking condition ii) implies as a rule technical difficulties. We shall formulate now some new results which nevertheless have been obtained using that condition.

a) If $n \equiv 0 \pmod 4$ then there exists an $[2n + 2, 2^{n-1} + 2^{n-3}, 2^n]$-pairing (e.g. [10,10,16]).

b) If $n \equiv 1 \pmod 4$, $n \geq 5$ then there exists an $[2n + 2, 2^n - 4 \cdot C_{n-2}^{(n-3)/2}, 2^n]$-pairing (e.g. [12,20,32]).

c) No monomial [10,11,16]-pairing exists.

4. CLASSIFICATION OF ORTHOGONAL PAIRINGS

4.1. Let $\mu, \mu' : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^p$ be two orthogonal pairings. They are said to be *equivalent* if there exist orthogonal operators $S \in O(m)$, $T \in O(n)$, $U \in O(p)$ such that

$$\mu'(Sx,Ty) = U\mu(x,y), \qquad x \in \mathbf{R}^m, \quad y \in \mathbf{R}^n.$$

We consider some invariants of a pairing with respect to this equivalence.

4.2. Let μ_1, μ_2 be pairings of types $[m_1,n,p_1]$, $[m_2,n,p_2]$. The $[m_1+m_2,n,p_1+p_2]$-pairing $\mu_1 \overset{l}{\oplus} \mu_2$ which is defined by formula

$$(\mu_1 \overset{l}{\oplus} \mu_2)((x_1,x_2),y) = (\mu_1(x_1,y),\mu_2(x_2,y)), \quad x_i \in \mathbf{R}^{m_i}, \quad i = 1, 2, y \in \mathbf{R}^n,$$

is said to be their left direct sum. It is called trivial if either m_1 or m_2 equals 0.

The right direct sum $\mu_1 \overset{r}{\oplus} \mu_2$ (and trivial one) of pairings μ_1, μ_2 of types $[m,n_1,p_1]$ and $[m,n_2,p_2]$ is defined similarly.

A pairing is called left (right) irreducible if it is not equivalent to any nontrivial left (right) direct sum and irreducible if it is left and right irreducible.

It is obvious that irreducibility (left, right) is an invariant of a pairing.

Remark 6. Generally speaking the direct sums of equivalent pairings are not equivalent (see 4.7).

4.3. Let μ be $[m,n,p]$-pairing. We fix an orthonormal basis $\{e_1,\ldots,e_m\}$ in $\mathbf{R}^m$ and put $A_i x = \mu(e_i,x)$, $x \in \mathbf{R}^n$, $p_{ij} = A_i A_j^*$, $i, j = 1, \ldots, m$. One can easily prove the following;

a) the operators A_i are orthogonal, that is $A_i^* A_i = \mathbf{1}$,

b) if $i \neq j$ then $A_i^* A_j + A_j^* A_i = 0$ (compare [8]),

c) P_{ij} is a partially isometrical operator the initial domain of which is $\mathrm{Im} A_j$ and the final one is $\mathrm{Im} A_i$,

d) P_{ii} is the orthogonal projection at $\mathrm{Im} A_i$.

Let us denote by $\mathfrak{P}$ the matrix of order $m \times m$ with components P_{ij} and call it the pairing-matrix of μ with respect to the basis $\{e_1,\ldots,e_m\}$. Of course we can multiply $\mathfrak{P}$ by any scalar matrix of corresponding order. If R is an invertible operator on $\mathbf{R}^p$ then we denote by $\mathfrak{P}^R$ the matrix with components $RP_{ij}R^{-1}$. The following lemmas are obvious.

LEMMA 9. *If $\mathfrak{P}_1$ is the pairing-matrix of μ in an orthogonal basis $\{e_1',\ldots,e_m'\}$ and R is the change-of-basis matrix then $\mathfrak{P}_1 = R^{-1}\mathfrak{P}R$.*

LEMMA 10. *If μ_1 is an equivalent pairing to μ, S, U are the same as in* *the definition of equivalence and* $\mathfrak{P}_1$ *is the pairing-matrix of* μ_1 *in the basis* $\{Se_1,\ldots,Se_m\}$ *then* $\mathfrak{P}_1 = \mathfrak{P}^U$.

It follows from these lemmas that the classification of pairings is reduced to the classification of their matrices with respect to the action of group $O(m) \times O(p)$ described in the lemmas. We get in particular another invariant of a pairing.

THEOREM 5. *If pairings* μ *and* μ' *are equivalent then the trace-operators* $\sum_{i=1}^{m} P_{ii}$ *and* $\sum_{i=1}^{m} P'_{ii}$ *are orthogonally equivalent.*

Remark 7. Since the operator $\sum_{i=1}^{m} P_{ii}$ is self-conjugated, Theorem 5 gives non-ordered row of p numbers which is the same for equivalent pairings. The trace-operator $\sum_{i=1}^{m} P_{ii}$ does not change on interchanging m and n.

Remark 8. The matrix $\mathfrak{P}$ defines a self-conjugated operator Π on $\mathbf{R}^{mp}$ where a structure of tensor product $\mathbf{R}^m \otimes \mathbf{R}^p$ is fixed. The problem is to classify the operators with respect to the conjugation by the operators from the subgroup $O(m) \otimes O(p)$ of $O(mp)$. I do not know a complete system of invariants for the conjugation. Some new invariants will be described in 4.7. It is easily to prove that the orthogonal type of the operator Π is defined by its trace, i.e. by the trace of the trace-operator $\sum_{i=1}^{m} P_{ii}$.

4.4. We consider now a case when a complete study is possible: $m = 2$.

We fix an orthogonal basis in $\mathbf{R}^2$ and put $V = \mathrm{Im} A_1$, $W = \mathbf{R}^p \ominus V$. For any basis $\{g_1,\ldots,g_n\}$ in $\mathbf{R}^n$ we shall always take a basis in $\mathbf{R}^p$ such that its first n vectors are $A_1 g_1, \ldots, A_1 g_n$ and we shall always denote a matrix of an operator by the same symbol as this operator. We can write then

$$A_1 = \begin{pmatrix} E_n \\ 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} A \\ B \end{pmatrix} \tag{4.1}$$

where E_n is the identity matrix of order $n \times n$, $A = (a_{ij})$, $B = (b_{ij})$ are matrices of orders $n \times n$ and $(p - n) \times n$. By (b) of 4.3, the operator A is skew-symmetrical and by choosing a suitable basis in $\mathbf{R}^n$ we can have its matrix in the following form:

$$\begin{cases} a_{2i-1,2i} = -a_{2i,2i-1} = a_i, & a = 1, \ldots, k, \qquad 0 < a_i < 1, \\ a_{2i-1,2i} = -a_{2i,2i-1} = 1, & i = k+1, \ldots, k+l, \qquad 2(k+l) \leq n, \\ a_{ij} = 0 \ \text{otherwise.} & \end{cases} \tag{4.2}$$

By orthogonality of the operator A_2 we can choose a basis in W such that the matrix B will have the following form:

$$\begin{cases} b_{2i-1,2i-1} = b_{2i,2i} = \sqrt{1 - a_i^2}, & i = 1, \ldots, k, \\ b_{2k+i,2(k+l)+i} = 1, \ i = 1, \ldots, n - 2(k+l), & p - n \geq n - 2l, \\ b_{ij} = 0 \quad \text{otherwise.} & \end{cases} \tag{4.3}$$

It is easily to verify that the orthogonal type of $P_{11} + P_{22}$ is defined by the unordered row of numbers $a_1, \ldots, a_k$, $0 < a_i < 1$, $i = 1, \ldots, k$, and nonnegative integer l. We will say that the row $\{a_1,\ldots,a_k\}$ is the *characteristic* of μ and l is the *link coefficient* of μ.

THEOREM 6. 1) *Let n, p be positive integers, $n \leq p$. For any unordered row α of positive integers $a_1, \ldots, a_k$, $0 < a_i < 1$, $i = 1, \ldots, k$, and nonnegative integer l such that $2(k + l) \leq n$, $p - n \geq n - 2l$, there exists a $[2,n,p]$-pairing with the characteristic α and the link coefficient l.*

2) *For the equivalence of two pairings with $m = 2$ it is necessary and sufficient that their characteristics and link coefficients be the same.*

Proof. 1) The conditions on k and l give the opportunity to define A, B by the equalities (4.2), (4.3) and then A_1, A_2 by the equalities (4.1). It is easy to verify that $A_i (i = 1,2)$ satisfies the conditions (a) and (b) of 4.3 and therefore the pairing μ defined by formula

$$\mu(\lambda_1 e_1 + \lambda_2 e_2, y) = (\lambda_1 A_1 + \lambda_2 A_2)(y), \qquad \lambda_1, \lambda_2 \in \mathbf{R}, \quad y \in \mathbf{R}^n,$$

is the desired one (compare [17]).

2) The necessity is a corollary of Theorem 5. The sufficiency follows from the possibility of transforming A_1 and A_2 to a canonical form (4.1), (4.2), (4.3) by a choice of bases in $\mathbf{R}^n$ and $\mathbf{R}^p$.

Remark 9. It follows from Theorem 6 that the set of all equivalence classes of $[2,n,p]$-pairings can be realized as a simplex of dimension $[(p + 1)/2] - [(n + 1)/2]$ and the vertices of the simplex correspond to the classes of the monomial pairings.

In particular the classes of $[2,2,4]$-pairings correspond to the number of the interval $[0,1]$ (compare 4.6, $m = 2$). The existence of a continum of these classes was noted in [15] and [9].

4.5. Here we will define a correspondence between pairings and representations of algebras.

Let m be a positive integer. We denote by $\Gamma(m)$ the $*$-algebra over $\mathbf{R}$ with generators p_{ij}, $i, j = 1, \ldots, m$, and the following relations: 1) $p_{ij}p_{jk} = p_{ik}$, 2) $p_{ij}p_{lk} = -p_{il}p_{jk}$, 3) $p_{ij}^* = p_{ji}$, i, j, k, $l = 1, \ldots, m, j \neq l$. If φ is a p-dimensional symmetrical representation of $\Gamma(m)$ over $\mathbf{R}$ and $P_{ij} = \varphi(p_{ij})$ then by the relations (1) and (3) P_{ii} is an orthogonal projection at a space $V_i (i = 1,\ldots,m)$, P_{ij} is a partially isometrical operator with the initial domain V_i and the final domain $V_j (i, j = 1,\ldots,m)$ and hence all spaces V_j have the same dimension, say n. If we fix now an orthonormal basis $e_1, \ldots, e_m$ in $\mathbf{R}^m$, put $\mu(e_i, y) = P_{i1}y$, $i = 1, \ldots, m, y \in V_1$ and extend μ by linearity on $\mathbf{R}^m$ then the relation (2) implies that μ is an orthogonal $[m,n,p]$-pairing. The construction of 4.3 shows that any orthogonal pairing can be obtained in that way from a symmetrical representation of algebra $\Gamma(m)$.

All algebras $\Gamma(m)$ with $m \geq 2$ have infinite dimension. A classification of symmetrical representations of $\Gamma(2)$ is, as a matter of fact described, in 4.4. For $m \geq 3$ a complete classification is absent.

The problem becomes more perspective if one considers only the monomial pairings. Let us denote by $\Gamma_0(m)$ *-algebra with generators p_{ij}, $i, j = 1, \ldots, m$, relations (1)-(3) and an extra relation:

4) for any two elements a, b of the semigroup S spanned at $p_{ij}(i,j = 1,\ldots,m)$ the elements aa^* and bb^* commute.

Relations (1)-(4) imply that all self-conjugated idempotents of $\Gamma_0(m)$ mutually commute and for any $\alpha \in S$ the element aa^* is a self-conjugated idempotent (induction on "length" of a).

THEOREM 7. *Let* μ *be a monomial* $[m,n,p]$*-pairing,* $e = \{e_1,\ldots,e_m\}$ *be a basis in* $\mathbf{R}^m$ *belonging to a multiplicative triplet of bases for* μ, P_{ij} *be the elements of the pairing matrix of* μ *with respect to e. Then the map* φ *defined on the generators by formula*

$$\varphi(p_{ij}) = P_{ij}, \qquad i,j = 1, \ldots, m,$$

extends to a p-dimensional symmetrical representation of the algebra $\Gamma_0(m)$. *If* μ *is right irreducible then* φ *is irreducible. If two representations are equivalent then their generating pairings are equivalent too.*

Proof. It is sufficient to prove that for any two operators A, B from the semigroup σ spanned by $P_{ij}(i,j = 1,\ldots,m)$ the operators AA^*, BB^* commute. Let $C = \{c_1,\ldots,c_p\}$ be a basis in $\mathbf{R}^p$ belonging to a multiplicative triplet of bases for μ together with e and Σ be the semigroup of all partially isometrical operators on $\mathbf{R}^p$ whose matrices with respect to C have only elements $1, -1, 0$. Since $P_{ij} \in \Sigma$, $i, j = 1, \ldots, m$, $\sigma \subset \Sigma$ and for any $A \in \sigma$ the operator AA^* is the orthogonal projection at the subspace on $\mathbf{R}^p$ spanned by a subset of C. This implies the statement.

Remark 10. The converse statement is true also: the operators of any symmetrical representation of the algebra $\Gamma_0(m)$ give a monomial pairing. It is obvious to every concrete case below but the general proof which is known to me is very cumbersome.

4.6. Let us consider cases of small m. We denote by $\mathbf{C}$ and $\mathbf{H}$ the real *-algebras of the complex numbers and the quaternions with the usual conjugation and for any *-algebra F over $\mathbf{R}$ we denote $F(k)$ the complete matrix algebra order k with elements from F with the usual conjugation.

$m = 2$. The algebra $\Gamma_0(2)$ has the dimension 6 and is a direct sum of two symmetrical simple ideals isomorphic $\mathbf{C}$ and $\mathbf{R}(2)$. The ideals are generated by the idempotents $p_{11}p_{22}$, $p_{11} + p_{22} - 2p_{11}p_{22}$. They give two symmetrical irreducible representations of $\Gamma_0(2)$ and therefore two right irreducible monomial pairings: the complex multiplication of type [2,2,2] and the trivial pairing of type [2,1,2]. By Theorem 7 any right irreducible monomial $[2,n,p]$-pairing is equivalent to one of those two.

$m = 3$. The algebra $\Gamma_0(3)$ has the dimension 53 and is a direct sum of six symmetrical ideals $I_i(i = 1,\ldots,6)$ of the following types: $I_1 = \mathbf{H}$, $I_2 = \mathbf{R}(4)$, $I_j = \mathbf{C}(2)$, $j = 3, 4, 5$, $I_6 = \mathbf{R}(3)$. Those ideals are generated by self-conjugated idempotents $q_i(i = 1,\ldots,6)$ which can be expressed in terms of the p_{ij} in the following way:

$$q_1 = p_{12}p_{33}, \; q_2 = q_0 + p_{12}q_0p_{21} + p_{13}q_0p_{31} + p_{23}q_0p_{32},$$

where

$$q_0 = p_{11}p_{22}p_{33} - q_1, \; q_3 = p_{22}p_{33}(e - p_{11}) - p_{23}q_0p_{32}$$
$$+ p_{12}p_{33}(e - p_{11})p_{21} - p_{13}q_0p_{31},$$
$$q_4 = p_{11}p_{33}(e - p_{22}) - p_{13}q_0p_{31} + p_{21}p_{33}(e - p_{22})p_{12} - p_{23}q_0p_{32},$$
$$q_5 = p_{11}p_{22}(e - p_{33}) - p_{12}q_0p_{21} + p_{31}p_{22}(e - p_{33})p_{13} - p_{32}q_0p_{23},$$

where

$$e = p_{11} + p_{22} + p_{33} - p_{11}p_{22} - p_{11}p_{33} - p_{22}p_{33} + p_{11}p_{22}p_{33},$$
$$q_6 = p_{11}(e - p_{22})(e - p_{33}) + p_{22}(e - p_{11})(e - p_{33}) + p_{33}(e - p_{11})(e - p_{22})$$
$$- p_{12}p_{33}(e - p_{11})p_{21} + p_{13}q_0p_{31} - p_{21}p_{33}(e - p_{22})p_{12}$$
$$+ p_{23}q_0p_{32} - p_{31}p_{22}(e - p_{33})p_{13} + p_{32}q_0p_{23}.$$

The irreducible symmetrical representations of the algebra $\Gamma_0(3)$ defined by the ideals I_i generate (modulo equivalence) four right irreducible monomial pairings of types [3,4,4], [3,3,4], [3,2,4], [3,1,3] which are restrictions of the quaternion multiplication $\mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}^4$ (the non-equivalent irreducible representations of $\Gamma_0(3)$ corresponding to I_3, I_4, I_5 give equivalent pairings).

$m = 4$. The algebra $\Gamma_0(4)$ has an infinite dimension and an infinite series of mutually non-equivalent finite-dimensional symmetric representations. Yet all the corresponding pairings are left reducible and some of them are right-reducible. There are (modulo equivalence) only 10 irreducible pairings. They have the following types: [4,3,4], [4,4,4], [4,4,7], [4,4,8], [4,5,8] (two pairings), [4,6,8] (two pairings), [4,7,8], [4,8,8] and all are restrictions of either the quaternion or Cayley multiplications.

Remark 11. For bigger m the problem of the classification of the irreducible monomial pairings is effectively solvable but very cumbersome. Examples (in particular for type [10,10,16]) show that there exist monomial pairings which are not restrictions of any pairings of types $[m,n,n]$.

4.7. Here we consider an example of two right reducible non-equivalent monomial [3,4,8]-pairings whose direct addends are mutual equivalent. There are (explicit and implicit) series of invariants in the proof.

Let us consider first a general situation. For any positive integers m, p we denote by $M(m,p)$ the algebra of all matrices of order $m \times m$ whose elements are operators on $\mathbf{R}^p$. If $A \in M(m,p)$ and $A = (A_{ij})_{i,j=1}^m$ then we put $A^{(0)} = A$, $A^{(1)} = (A_{ji})$, $A^{(2)} = (A_{ij}^*)$. If $i = (i_1, i_2, \ldots, i_k)$ is a sequence of 0, 1, 2 then we put $A^{(i)} = A^{(i_1)}A^{(i_2)} \ldots A^{(i_k)}$. A matrix $A = (A_{ij})$ is said to be pseudoscalar if there exist an operator B and numbers $\alpha_{ij} \in \mathbf{R}$ $(i,j = 1,\ldots,m)$ such that $A_{ij} = \alpha_{ij}B$. The group $O(m) \times O(p)$ is naturally embeded in $M(m,p)$ (the tensor product) and therefore acts on $M(m,p)$ by inner automorphisms. The equivalence of matrices with respect to this action we shall call simply equivalence and denote by sign "$\sim$". The following two lemmas are obvious.

LEMMA 11. *If A_1, $A_2 \in M(m,p)$ and $A_1 \sim A_2$ then for any finite sequence $i = (i_1, i_2, \dots, i_k)$ of numbers* 0, 1, 2, $A_1^{(i)} \sim A_2^{(i)}$.

LEMMA 12. *If A_1, $A_2 \in M(m,p)$, $A_1 \sim A_2$ and A_1 is pseudoscalar then A_2 is pseudoscalar too.*

Now we construct the example. Let $\alpha : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ be complex multiplication, β be a [1,2,2]-pairings, $\mu = \alpha \overset{l}{\bigoplus} \beta$, $\{e_1, e_2, e_3\}$ be an orthonormal basis in $\mathbf{R} = C \oplus \mathbf{R}^1$ such that $e_1 = 1$, $e_2 = i$, S be the operator on $\mathbf{R}^3$ definited by the following equality: $Se_1 = e_1$, $Se_2 = e_3$, $Se_3 = e_2$. We put

$$\mu'(x,y) = \mu(Sx,y), \qquad x \in \mathbf{R}^3, \quad y \in \mathbf{R}^2, \qquad \mu_1 = \mu \overset{r}{\bigoplus} \mu, \ \mu_2 = \mu \overset{r}{\bigoplus} \mu'.$$

THEOREM 8. *The pairings μ_1 and μ_2 are not equivalent.*

Proof. Let $\mathfrak{P}_i = (P_{kji})_{k,j=1}^3$, $i = 1, 2$, be the pairing matrix of μ_i with respect to the basis $\{e_1, e_2, e_3\}$. We put $X_i = (\mathfrak{P}_i \mathfrak{P}_i^{(1)})^{(2)} \mathfrak{P}_i \mathfrak{P}_i^{(1)}$ and denote x_{jk}^i $(j,k = 1,2,3,$ $i = 1,2)$ the elements of X_i. Using the equalities $P_{111} = P_{221}$, $P_{111} \cdot P_{331} = 0$ one can easily find that $x_{33}^1 = P_{331}$, $x_{jk}^1 = 0$ otherwise and therefore the matrix X_1 is pseudoscalar. On the other hand $x_{22}^2 = Q_1$, $x_{33}^2 = Q_2$ where the operators Q_1, Q_2 are different nonzero projections and the matrix X_2 is not pseudoscalar. The proof is completed now by using considerations of 4.3, Lemmas 11 and 12.

Remark 12. The trace-operators of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ coincide.

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