DIFFERENTIAL SUBORDINATIONS AND UNIVALENT FUNCTIONS

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1. INTRODUCTION

Let $p(z)$ be regular in the unit disc U and let $\psi(r,s,t)$ be a complex function defined in a domain of $\mathbf{C}^3$. With some simple conditions on ψ the present authors in [6] determined a class of functions Ψ for which

(1) $|\psi(p(z),zp'(z),z^2p''(z))| < 1, \quad \text{for } z \in U \quad \Rightarrow \quad |p(z)| < 1, \quad \text{for } z \in U.$

In addition, they determined a different class of functions Ψ for which

(2) $\operatorname{Re}\psi(p(z),zp'(z),z^2p''(z)) > 0, \quad \text{for } z \in U \quad \Rightarrow \quad \operatorname{Re} p(z) > 0, \quad \text{for } z \in U.$

If Δ represents the unit disc in (1) and the right-half complex plane in (2) then both results can be written in the form

(3) $\{\psi(p(z),zp'(z),z^2p''(z)) : z \in U\} \subset \Delta \quad \Rightarrow \quad \{p(z) : z \in U\} \subset \Delta.$

Note that in both cases Δ is simply connected and has a smooth boundary. In this paper we will show that if Δ is any simply connected domain with a "nice boundary," then there is a class of functions Ψ for which (3) is true. Actually we will prove a more general result; if Ω is a domain and Δ is a simply connected domain with a "nice boundary" we will determine a class of functions Ψ for which

(4) $\{\psi(p(z),zp'(z),z^2p''(z)) : z \in U\} \subset \Omega \quad \Rightarrow \quad \{p(z) : z \in U\} \subset \Delta.$

This basic result and applications of it in the theory of differential equations are given in section 2. In section 3 we show that the result has many important applications, especially in the theory of univalent functions; it provides elegantly short proofs for some well-known results and enables us to obtain several new results.

Since many of the results in this paper can be expressed in terms of subordination, we repeat here the definition of subordination between two functions $g(z)$ and $G(z)$ regular in U. We say $g(z)$ *is subordinate to* $G(z)$, written $g(z) \prec G(z)$, if $G(z)$ is univalent, $g(0) = G(0)$ and $g(U) \subset G(U)$.

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If $\psi(r,s,t)$ is holomorphic and $q(z)$ is a conformal mapping of U onto Δ such that $\psi(p(0),0,0) = q(0) = p(0)$, then (3) can be expressed as

$$(3') \qquad \psi(p(z),zp'(z),z^2p''(z)) \prec q(z) \quad \Rightarrow \quad p(z) \prec q(z).$$

Similarly, if $q(z)$ is a conformal mapping of U onto Δ such that $q(0) = p(0)$, then (4) can be expressed as

$$(4') \qquad \{\psi(p(z),zp'(z),z^2p''(z)) : z \in U\} \subset \Omega \quad \Rightarrow \quad p(z) \prec q(z).$$

In the particular case when Ω is simply connected, $\psi(r,s,t)$ is holomorphic and $h(z)$ is a conformal mapping of U onto Ω with $h(0) = \psi(p(0),0,0)$, then (4′) can be expressed as

$$(5) \qquad \psi(p(z),zp'(z),z^2p''(z)) \prec h(z) \quad \Rightarrow \quad p(z) \prec q(z).$$

It is this differential subordination result which is the basis of most of the applications in section 3.

In section 4 we consider the problem of dominating the solutions of the differential subordination

$$(6) \qquad \psi(p(z), zp'(z), z^2p''(z)) \prec h(z).$$

Namely, if $p(z)$ satisfies (6), does there exist a function $q(z)$ such that $p(z) \prec q(z)$? We also consider the problem of finding the "smallest" such $q(z)$.

2. DIFFERENTIAL SUBORDINATIONS

We need to first specify those simply connected domains with the "nice boundary" referred to in section 1. We do this in terms of a mapping q from U onto Δ as follows:

Definition 1. We say $q \in Q$ if $q(z)$ is regular and univalent on $\bar{U}$ except for those points $\zeta \in \partial U$ for which $\lim\limits_{\substack{z\to\zeta \\ z\in U}} q(z) = \infty$.

The domain $\Delta = q(U)$ will be simply connected and its boundary will consist of either a simple closed regular curve or the union (possibly infinite) of pairwise disjoint simple regular curves each of which converges to ∞ in both directions. The functions $q_1(z) = z$ and $q_2(z) = (1 + z)/(1 - z)$ are examples of these two cases. Note that $q_1(U)$ and $q_2(U)$ correspond to the domains used in (1) and (2) respectively.

We will need the following lemma, a version of which is given in [6, Lemma B], to prove our main result.

LEMMA 1. *Let $q \in Q$ with $q(0) = a$, and let $p(z) = a + p_nz^n + p_{n+1}z^{n+1} + \ldots$ be regular in U with $p(z) \not\equiv a$ and $n \geq 1$. If there exists a point $z_0 \in U$ such that $p(z_0) \in q(\partial U)$ and $p(|z| < |z_0|) \subset q(U)$, then*

(7) $$z_0p'(z_0) = m\,\zeta_0 q'(\zeta_0), \qquad \text{and}$$

(8) $$\operatorname{Re}\left[1 + \frac{z_0p''(z_0)}{p'(z_0)}\right] \geq m \operatorname{Re}\left[1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right],$$

where $q^{-1}(p(z_0)) = \zeta_0 = e^{i\theta_0}$ *and* $m \geq n \geq 1$.

Proof. We will use Lemma A of [6] to prove the result. Although the hypothesis of Lemma A requires $g(z)$ to be regular in U, the proof actually only requires $g(z)$ to be regular in $\{|z| < |z_0|\} \cup \{z_0\}$. If we take $g(z) = q^{-1}(p(z))$ then $g(z)$ will be regular in $\{|z| \leq |z_0|\}$, and we can obtain (7) and (8) by applying Lemma A to this $g(z)$.

Note that (7) is a relation between the outer normals to the curves $p(|z| = |z_0|)$ and $q(\partial U)$ at their point of tangency $p(z_0) = q(\zeta_0)$, and (8) is a relation between the curvatures of these two curves at their point of tangency. These relations form the basis for the following definition.

Definition 2. Let Ω be a domain in $\mathbf{C}$ and $q \in Q$. We define $\Psi_n(\Omega,q)$ to be the class of functions $\psi: \mathbf{C}^3 \to \mathbf{C}$ that satisfy the following:

(a) $\psi(r,s,t)$ is continuous in a domain $D \subset \mathbf{C}^3$,

(b) $(q(0),0,0) \in D$ and $\psi(q(0),0,0) \in \Omega$,

(c) $\psi(r_0,s_0,t_0) \notin \Omega$ when $(r_0,s_0,t_0) \in D$, $r_0 = q(\zeta)$, $s_0 = m\zeta q'(\zeta)$ and

$$\operatorname{Re}[1 + t_0/s_0] \geq m \operatorname{Re}[1 + \zeta q''(\zeta)/q'(\zeta)],$$

where $|\zeta| = 1$, $q(\zeta)$ is finite and $m \geq n \geq 1$.

We write $\Psi_1(\Omega,q)$ as $\Psi(\Omega,q)$.

Remarks. 1. We do not require that Ω be simply-connected or that it have a particularly nice boundary as we do for $q(U)$.

2. If $\Omega \subset \tilde{\Omega}$ then $\Psi_n(\tilde{\Omega},q) \subset \Psi_n(\Omega,q)$, that is, enlarging Ω decreases the class $\Psi_n(\Omega,q)$.

3. Note that $\Psi_n(\Omega,q) \subset \Psi_{n+1}(\Omega,q)$.

We are now prepared to state and prove the principal theorem of this article.

THEOREM 1. *Let* $q(0) = a$ *and let* $\psi \in \Psi_n(\Omega,q)$ *with corresponding domain* D. *Let* $p(z) = a + p_nz^n + p_{n+1}z^{n+1} + \dots$ *be regular in* U *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If* $(p(z), zp'(z), z^2p''(z)) \in D$ *when* $z \in U$ *and*

(9) $$\psi(p(z), zp'(z), z^2p''(z)) \in \Omega \qquad \text{when } z \in U,$$

then $p(z) \prec q(z)$.

Proof. Since $q(z)$ is univalent in U and $p(0) = q(0) = a$, we only need to show that $p(U) \subset q(U)$. Suppose not, and let $z_0 \in U$ be such that

$$p(z_0) \in q(\partial U) \qquad \text{and} \qquad p(|z| < |z_0|) \subset q(U).$$

By Lemma 1, at the point z_0 we must have $p(z_0) = q(\zeta_0)$, $z_0p'(z_0) = m\zeta_0 q'(\zeta_0)$ and

$$\operatorname{Re}\left[1 + \frac{z_0p''(z_0)}{p'(z_0)}\right] \geq m \operatorname{Re}\left[1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right],$$

where $|\zeta_0| = 1$ and $m \geq n \geq 1$. Using part (c) of Definition 2 we obtain

$$\psi(p(z_0), z_0p'(z_0), z_0^2p''(z_0)) \notin \Omega$$

which contradicts (9). Hence $p(U) \subset q(U)$ and $p(z) \prec q(z)$.

The set of functions $p(z)$ satisfying (9) is not a vacuous set. Since Ω is a domain and ψ is continuous at $(a,0,0)$, if we take $p(z) = a + p_nz^n$, then for sufficiently small $|p_n|$

$$\psi(p(z), zp'(z), z^2p''(z)) = \psi(a + p_nz^n, np_nz^n, n(n-1)p_nz^n) \in \Omega$$

when $z \in U$.

On checking the definitions of Q and $\Psi_n(\Omega,q)$ we see that the hypothesis of Theorem 1 requires that $q(z)$ behave very nicely on ∂U. If this is not the case or if the behavior of $q(z)$ on ∂U is not known, it may still be possible to prove that $p(z) \prec q(z)$ by the following limiting procedure.

COROLLARY 1.1. *Let $q(z)$ be univalent in U with $q(0) = a$, and let $q_\rho(z) = q(\rho z)$ for $0 < \rho < 1$. Let $\psi \in \Psi_n(\Omega,q_\rho)$ with domain D, for $0 < \rho < 1$, and let*

$$p(z) = a + p_nz^n + \ldots$$

be regular in U with $p(z) \not\equiv a$ and $n \geq 1$. If $(p(z), zp'(z), z^2p''(z)) \in D$, when $z \in U$, and $\psi(p(z), zp'(z), z^2p''(z)) \in \Omega$, when $z \in U$, then $p(z) \prec q(z)$.

Proof. The function $q_\rho(z)$ will be univalent on $\bar{U}$ for $0 < \rho < 1$. Hence $q_\rho \in Q$ and $\psi \in \Psi_n(\Omega,q_\rho)$ is well defined. If we let $p_\rho(z) = p(\rho z)$, $0 < \rho < 1$, then

$$\psi(p_\rho(z), zp_\rho'(z), z^2p_\rho''(z)) = \psi(p(\rho z), \rho zp'(\rho z), \rho^2z^2p''(\rho z)) \in \Omega$$

when $z \in U$. We apply Theorem 1 to obtain $p_\rho(z) \prec q_\rho(z)$. Hence $p(\rho z) \prec q(\rho z)$, and by letting $\rho \to 1^-$ we obtain $p(z) \prec q(z)$.

We consider next the case when Ω in $\Psi_n(\Omega,q)$ is a simply-connected domain.

Definition 3. Let h be a conformal mapping of U onto Ω and $q \in Q$. We will denote by $\Psi_n(h,q)$ the class of functions $\psi \in \Psi_n(\Omega,q) = \Psi_n(h(U),q)$ which are holomorphic in their corresponding domains D and satisfy $\psi(q(0),0,0) = h(0)$. We write $\Psi_1(h,q)$ as $\Psi(h,q)$.

The following theorem and corollary are immediately obtained from Theorem 1 and Corollary 1.1.

THEOREM 2. *Let $\psi \in \Psi_n(h,q)$ with corresponding domain D and with $q(0) = a$. Let $p(z) = a + p_nz^n + \ldots$ be regular in U with $p(z) \not\equiv a$ and $n \geq 1$. If*

$$(p(z), zp'(z), z^2p''(z)) \in D$$

when $z \in U$ *then*

(10) $$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z) \quad \Rightarrow \quad p(z) \prec q(z).$$

COROLLARY 2.1. *Let* $h(z)$ *and* $q(z)$ *be univalent in* U *with* $q(0) = a$, *and let* $h_\rho(z) = h(\rho z), q_\rho(z) = q(\rho z)$, *for* $0 < \rho < 1$. *Let* $\psi \in \Psi_n(h_\rho, q_\rho)$ *with domain* D, *for* $0 < \rho < 1$, *and let* $p(z) = a + p_n z^n + \dots$ *be regular in* U *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If* $(p(z), zp'(z), z^2p''(z)) \in D$ *when* $z \in U$, *then*

$$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z) \quad \Rightarrow \quad p(z) \prec q(z).$$

Remarks. 1. In the special case when $h(z) = q(z)$, if $\psi \in \Psi_n(q(z), q(z))$ we obtain

(11) $$\psi(p(z), zp'(z), z^2p''(z)) \prec q(z) \quad \Rightarrow \quad p(z) \prec q(z).$$

For the class of functions $\Psi_n((1+z)/(1-z), (1+z)/(1-z))$, relation (11) reduces to (2). Examples of these classes and applications of (11) are given in [6].

2. The implication in (10) is sharp over the class $\Psi_n(h,q)$, in the sense that there exists $\psi \in \Psi_n(h,q)$ such that (10) will be satisfied when $p(z) = q(z)$. To show this, take $\psi(r,s,t) = h(q^{-1}(r))$. On checking the conditions of Definition 3 we see that ψ is holomorphic in $q(U) \times \mathbf{C} \times \mathbf{C}$,

$$\psi(q(0),0,0) = h(0) \in h(U) \quad \text{and} \quad \psi(r_0, s_0, t_0) = h(\zeta_0) \notin h(U)$$

when $r_0 = q(\zeta_0)$, $|\zeta_0| = 1$. Hence $\psi \in \Psi_n(h(U), q)$ and by Definition 3 we have $\psi \in \Psi_n(h,q)$. Moreover $\psi(q(z), zq'(z), z^2q''(z)) = h(z)$, and so by (10) we obtain the sharp result $q(z) \prec q(z)$.

3. For a particular $\psi \in \Psi_n(h,q)$ it may be possible to improve (10) by finding a function $\tilde{q}(z)$ subordinate to $q(z)$ such that (10) can be replaced by

(12) $$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z) \quad \Rightarrow \quad p(z) \prec \tilde{q}(z) \ (\prec q(z)).$$

The problem of finding the "smallest" $\tilde{q}(z)$ satisfying (12) will be discussed in section 4.

Examples. We will consider the class of functions $\Psi_n(U, (1+z)/(1-z))$. Substituting $\Omega = U$ and $q(z) = (1+z)/(1-z)$ in Definition 2 we see that

$$q(\zeta) = r_2 i \ (r_2 \text{ real}), \qquad \zeta q'(\zeta) = -(1 + r_2^2)/2$$

and $\operatorname{Re}[1 + \zeta q''(\zeta)/q'(\zeta)] = 0$ when $|\zeta| = 1$. Hence the class $\Psi_n(U, (1+z)/(1-z))$ consists of those functions ψ that are continuous in a domain $D \subset \mathbf{C}^3$, with $(1,0,0) \in D$ and $|\psi(1,0,0)| < 1$, and that satisfy $|\psi(r_2 i, s_1, t_1 + t_2 i)| \geq 1$ if $(r_2 i, s_1, t_1 + t_2 i) \in D$, $s_1 \leq -n(1 + r_2^2)/2$ and $s_1 + t_1 \leq 0$, when $n \geq 1$. If in addition ψ is holomorphic in D and $\psi(1,0,0) = 0$, then by Definition 3 we would also have $\psi \in \Psi_n(z, (1+z)/(1-z))$.

If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ is regular in U, with $p(z) \not\equiv 1$ and $n \geq 1$, and if $\psi \in \Psi_n(U,(1+z)/(1-z))$, then by Theorem 1

$$\psi(p(z), zp'(z), z^2p''(z)) \in U \quad \Rightarrow \quad p(z) \prec \frac{1+z}{1-z}, \tag{13}$$

whereas if $\psi \in \Psi_n(z,(1+z)/(1-z))$ then by Theorem 2

$$\psi(p(z), zp'(z), z^2p''(z)) \prec z \quad \Rightarrow \quad p(z) \prec \frac{(1+z)}{(1-z)}. \tag{14}$$

We will now consider some specific cases.

(a) If $\psi_1(r,s,t) = t + 3s - r^2 + 1$ then it is easy to show that

$$\psi_1 \notin \Psi_1(z,(1+z)/(1-z)),$$

but $\psi_1 \in \Psi_n(z,(1+z)/(1-z))$ for $n \geq 2$. If $p(z) = 1 + p_n z^n + \dots$ is regular in U, with $p(z) \not\equiv 1$ and $n \geq 2$, then from (14) we obtain

$$z^2p''(z) + 3zp'(z) - p^2(z) + 1 \prec z \quad \Rightarrow \quad p(z) \prec \frac{1+z}{1-z}.$$

(b) If $\psi_2(r,s,t) = \alpha r + \beta s$, with $-1 < \alpha < 1$ and $|\mathrm{Re}\,\beta| \geq 2$, then

$$\psi_2 \in \Psi_n(U,(1+z)/(1-z))$$

for $n \geq 1$. If $p(z) = 1 + p_n z^n + \dots$ is regular in U, with $p(z) \not\equiv 1$ and $n \geq 1$, then from (13) we obtain

$$|\alpha p(z) + \beta zp'(z)| < 1 \quad \Rightarrow \quad p(z) \prec \frac{(1+z)}{(1-z)}.$$

Theorem 1 can be used to show that the solution of certain second order differential equations are contained in a specific domain. For simplicity we take $n = 1$. The proof of the following theorem follows directly from Theorem 1.

THEOREM 3. *Let $\psi \in \Psi(\Omega,q)$ and let $w(z)$ be a regular function satisfying $w(U) \subset \Omega$. If the differential equation $\psi(p(z), zp'(z), z^2p''(z)) = w(z)$ has a solution $p(z)$ regular in U with $p(0) = q(0)$ then $p(z) \prec q(z)$.*

As an example, if we take $\psi_2 \in \Psi(U,(1+z)/(1-z))$ as in example (b) above, we obtain the differential equation

$$\alpha p(z) + \beta zp'(z) = w(z), \tag{15}$$

where $-1 < \alpha < 1$ and $|\mathrm{Re}\,\beta| \geq 2$. If $w(z)$ is regular then it is easy to show that (15) has a regular solution. If in addition $|w(z)| < 1$ then by Theorem 3 the solution $p(z) = \int_0^z t^{\alpha/\beta - 1} w(t)dt/\beta z^{\alpha/\beta}$ must satisfy $\mathrm{Re}\,p(z) > 0$.

Note that by letting $p(z) = f'(z)$ in (15) we've proved that the differential equation $\alpha f'(z) + \beta z f''(z) = w(z)$, with $w(z)$ regular and $|w(z)| < 1$, has a univalent solution.

3. APPLICATIONS OF DIFFERENTIAL SUBORDINATIONS

In this section we will present several different applications of Theorem 1, mainly in the field of univalent functions, both to obtain new results and to provide very simple proofs for some well-known results.

We first give some definitions that will be used in this section. If

$$f(z) = z + a_2 z^2 + \dots$$

is regular in U and $\operatorname{Re}[zf''(z)/f'(z) + 1] > \alpha$, for $z \in U$, then $f(z)$ is called a *convex function of order* α, while if $\operatorname{Re} zf'(z)/f(z) > \alpha$, for $z \in U$, then $f(z)$ is called a *starlike function of order* α. These classes of univalent functions will be denoted respectively by $C(\alpha)$ and $S^*(\alpha)$. If $0 \leq \alpha < 1$ then $C(\alpha) \subset C(0) = C$, the *class of convex functions,* while $S^*(\alpha) \subset S^*(0) = S^*$, the *class of starlike functions.*

As our first application we obtain a subordination result for the Libera transform. In [4], R. Libera showed that if $f \in S^*$ then $F = L(f)$ given by

$$F(z) = \frac{2}{z} \int_0^z f(t)\,dt \tag{16}$$

is also in S^*.

We shall denote by $K(z)$ the function obtained from (16) when $f(t)$ is the Koebe function $t/(1+t)^2$, that is,

$$K(z) = \frac{2}{z} \int_0^z \frac{t}{(1+t)^2}\,dt = \frac{2}{z}\left[\ln(1+z) - \frac{z}{1+z}\right]. \tag{17}$$

The order of starlikeness of $F(z)$ has remained an open problem. The following theorem essentially solves this problem and also provides a simple proof of Libera's result.

THEOREM 4. *If $f \in S^*$ and F and K are defined by* (16) *and* (17) *respectively, then*

$$\frac{zF'(z)}{F(z)} \prec \frac{zK'(z)}{K(z)} \prec \frac{1-z}{1+z}.$$

Proof. If we let $p(z) = zF'(z)/F(z)$ and $q(z) = zK'(z)/K(z)$ then we obtain

$$p(z) + \frac{zp'(z)}{p(z)+1} = \frac{zf'(z)}{f(z)} \prec \frac{1-z}{1+z} = h(z)$$

and

$$q(z) + \frac{zq'(z)}{q(z)+1} = \frac{1-z}{1+z} = h(z). \tag{18}$$

If we let $\psi(r,s) = r + s/(r+1)$ then we have

$$\psi(p(z), zp'(z)) \prec h(z) \quad \text{and} \quad \psi(q(z), zq'(z)) = h(z). \tag{19}$$

We will first show that $\psi \in \Psi(h,h)$. The function ψ is holomorphic in $(\mathbf{C} - \{-1\}) \times \mathbf{C}$ and satisfies $\psi(h(0),0) = h(0)$. In addition

$$\operatorname{Re} \psi(h(\zeta), m\zeta h'(\zeta)) = \operatorname{Re}\left[h(\zeta) + \frac{m\zeta h'(\zeta)}{h(\zeta)+1} \right] = -\frac{m}{2} < 0,$$

when $|\zeta| = 1$ and $m \geq 1$. Hence $\psi(h(\zeta), m\zeta h'(\zeta)) \notin h(U)$, and by Definition 3 we have $\psi \in \Psi(h,h)$. From (19) and Theorem 2 we obtain

$$p(z) \prec h(z) \quad \text{and} \quad q(z) \prec h(z), \tag{20}$$

or equivalently,

$$\frac{zF'(z)}{F(z)} \prec \frac{1-z}{1+z} \quad \text{and} \quad \frac{zK'(z)}{K(z)} \prec \frac{1-z}{1+z}.$$

(This provides a simple proof of Libera's result that $F \in S^*$.)

We will use (20) and Corollary 2.1 to show that $p(z) \prec q(z)$. We first show that $q(z)$ is univalent. From (18) and (20) we obtain

$$\operatorname{Re} \frac{\zeta q'(\zeta)}{q(\zeta)+1} = \operatorname{Re}\,[h(\zeta) - q(\zeta)] = -\operatorname{Re} q(\zeta) \leq 0, \tag{21}$$

when $|\zeta| = 1$. Hence $q(|\zeta| = 1)$ is starlike with respect to -1, and consequently $q(z)$ is univalent.

We now show that $\psi \in \Psi(h_\rho, q_\rho)$, for $0 < \rho < 1$. Since $\psi(q_\rho(0),0) = h_\rho(0) = 1$, we only need to show that $\psi(q_\rho(\zeta), m\zeta q_\rho'(\zeta)) \notin h_\rho(U)$ for $0 < \rho < 1$, when $|\zeta| = 1$ and $m \geq 1$. From (20) we obtain $q_\rho(z) \prec h_\rho(z)$. Combining this with (21) and the fact that $h_\rho(U)$ is a disc we obtain

$$\psi(q_\rho(\zeta), m\zeta q_\rho'(\zeta)) = q_\rho(\zeta) + \frac{m\zeta q_\rho'(\zeta)}{q_\rho(\zeta)+1} = q_\rho(\zeta) + m\,[h_\rho(\zeta) - q_\rho(\zeta)] \notin h_\rho(U).$$

Hence by (19) and Corollary 2.1 we have $p(z) \prec q(z)$, and consequently

$$zF'/F \prec zK'/K.$$

Note that as a consequence of this theorem the order of starlikeness of $L(S^*)$, the class of Libera transforms of starlike functions, is given by $\inf_{|z|<1} \operatorname{Re} zK'(z)/K(z)$. Recently P. Mocanu, M. Reade and D. Ripianu have evaluated this infimum [8].

We next use Theorem 2 to provide a new and shorter proof of a recent result of T. MacGregor [5], concerning a subordination of convex functions of order α.

THEOREM 5. *If* $f \in C(\alpha)$, $0 \le \alpha < 1$, *then* $zf'(z)/f(z) \prec q(z)$, *where* $q(z)$ *is the univalent function defined by*

$$q(z) + \frac{zq'(z)}{q(z)} = \frac{1 + (1 - 2\alpha)z}{1 - z} \qquad (q(0) = 1). \tag{22}$$

Proof. If we let $p(z) = zf'(z)/f(z)$ and $\psi(r,s) = r + s/r$, then since $f \in C(\alpha)$ we obtain

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)} + 1 \prec \frac{1 + (1 - 2\alpha)z}{1 - z} = h(z),$$

that is

$$\psi(p(z), zp'(z)) \prec h(z). \tag{23}$$

The function $q(z)$ defined by (22) is regular in U and satisfies

$$\psi(q(z), zq'(z)) = h(z). \tag{24}$$

We will first show that $\psi \in \Psi(h,h)$. The function ψ is holomorphic in $(\mathbf{C} - \{0\}) \times \mathbf{C}$ and satisfies $\psi(h(0),0) = h(0)$. In addition, a simple calculation shows that

$$\operatorname{Re} \psi(h(\zeta), m\zeta h'(\zeta)) = \operatorname{Re}\left[h(\zeta) + \frac{m\zeta h'(\zeta)}{h(\zeta)} \right] \le \operatorname{Re} h(\zeta) = \alpha,$$

when $|\zeta| = 1$ and $m \ge 1$. Hence $\psi(h(\zeta), m\zeta h'(\zeta)) \notin h(U)$, and by Definition 3 we have $\psi \in \Psi(h,h)$. From (23), (24) and Theorem 2 we obtain

$$p(z) \prec h(z) \qquad \text{and} \qquad q(z) \prec h(z). \tag{25}$$

We shall use this result and Corollary 2.1 to show that $p(z) \prec q(z) \prec h(z)$. In order to do so, we need first show that $q(z)$ is univalent. From (22) and (25) we obtain

$$\operatorname{Re} \frac{\zeta q'(\zeta)}{q(\zeta)} = \operatorname{Re}\,[h(\zeta) - q(\zeta)] = \alpha - \operatorname{Re} q(\zeta) \le 0, \tag{26}$$

when $|\zeta| = 1$. Hence $q(|\zeta| = 1)$ is starlike with respect to the origin and consequently $q(z)$ is univalent.

Since $\psi(q(0),0) = h(0) = 1$, in order to show that $\psi \in \Psi(h_\rho,q_\rho)$ we only need to show that $\psi(q_\rho(\zeta), m\zeta q'_\rho(\zeta)) \notin h_\rho(U)$ when $|\zeta| = 1$, $m \geq 1$ and $0 < \rho < 1$. From (25) we obtain $q_\rho(z) \prec h_\rho(z)$. Since $h_\rho(U)$ is a disc, from (26) we obtain

$$\psi(q_\rho(\zeta), m\zeta q'_\rho(\zeta)) = q_\rho(\zeta) + \frac{m\zeta q'_\rho(\zeta)}{q_\rho(\zeta)} = q_\rho(\zeta) + m\,[h_\rho(\zeta) - q_\rho(\zeta)] \notin h_\rho(U).$$

Hence by (23) and Corollary 2.1 we have $p(z) \prec q(z)$, which completes the proof of the theorem.

Note that by proving $p(z) \prec h(z)$ in (25) we have actually obtained a simple proof of $C(\alpha) \subset S^*(\alpha)$.

In the previous applications we have used Theorem 2 and have dealt with functions ψ which were holomorphic. In the following application the function ψ is only continuous; we will have to use the more general Theorem 1.

THEOREM 6. *Let* $\alpha \geq 0$, $h(z) = 2z/(1 - z^2)$ *and* $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ *be regular in U. If*

$$\left\{\frac{zp'(z)}{p(z)} - \alpha p(z)\,[zp'(z) + \operatorname{Re} z^2 p''(z)] : z \in U\right\} \subset h(U) \tag{27}$$

then $p(z) \prec (1 + z)/(1 - z)$.

Proof. Let $q(z) = (1 + z)/(1 - z)$ and $\psi(r,s,t) = s/r - \alpha r(s + \operatorname{Re} t)$. The conclusion will follow from Theorem 1 if we can show $\psi \in \Psi(h(U),q)$. The function ψ is continuous on $(\mathbf{C} - \{0\}) \times \mathbf{C}^2$ and satisfies $\psi(q(0),0,0) = h(0) = 0$. Note that the domain $h(U)$ consists of the complex plane with the slits $\{ib : b \geq 1, b \leq -1\}$ removed. To complete the proof we only need to show that $\psi(r_0, s_0, t_0) \notin h(U)$ if $|\zeta| = 1$, $r_0 = q(\zeta) = ai$, $s_0 = m\zeta q'(\zeta) = -m(1 + a^2)/2$ and

$$\operatorname{Re}[1 + t_0/s_0] \geq m \operatorname{Re}[1 + \zeta q''(\zeta)/q'(\zeta)] = 0,$$

when $m \geq 1$. Since $|(1 + a^2)/2a| \geq 1$ if $a \neq 0$, we obtain

$$\begin{aligned}\psi(r_0,s_0,t_0) &= -m\frac{1 + a^2}{2ai} - \alpha\, ai(s_0 + \operatorname{Re} t_0)\\ &= i\left[m\frac{1 + a^2}{2a} - \alpha\, as_0 \operatorname{Re}(1 + t_0/s_0)\right] \equiv ib \notin h(U),\end{aligned}$$

where $|b| \geq 1$. Hence $\psi \in \Psi(h(U),q)$ and $p(z) \prec (1 + z)/(1 - z)$.

In the special case when $\alpha = 0$, the function ψ is holomorphic. In this case (27) simplifies to

$$\frac{zp'(z)}{p(z)} \prec \frac{2z}{1 - z^2} \quad\Rightarrow\quad p(z) \prec \frac{1 + z}{1 - z}.$$

If we take $p(z) = f'(z)$ in this differential subordination we obtain the following sufficiency condition for univalence:

$$\frac{zf''(z)}{f'(z)} \prec \frac{2z}{1-z^2} \quad \Rightarrow \quad \operatorname{Re} f'(z) > 0.$$

Our final application deals with a result originally proved by D. Hallenbeck and S. Ruscheweyh [2, p. 192]. Their proof required the use of the Hadamard convolution and the Herglotz integral formula. We present a much simpler proof using differential subordinations.

THEOREM 7. *Let $h(z)$ be a convex univalent function and $\gamma \neq 0$ with* $\operatorname{Re} \gamma \geq 0$. *If $p(z)$ is regular in U and $p(0) = h(0)$ then $p(z) + zp'(z)/\gamma \prec h(z) \Rightarrow p(z) \prec h(z)$.*

Proof. Let $\psi(r,s) = r + s/\gamma$ and $h_\rho(z) = h(\rho z)$ for $0 < \rho < 1$. The conclusion of the theorem will follow from Corollary 2.1 if we show that $\psi \in \Psi(h_\rho, h_\rho)$ for $0 < \rho < 1$. The function ψ is holomorphic in $\mathbf{C}^2$ and satisfies $\psi(h(0),0) = h(0)$. We only need to show that

$$\psi(h(\rho\zeta), m\rho\zeta h'(\rho\zeta)) = h(\rho\zeta) + m\gamma^{-1}\rho\zeta h'(\rho\zeta) \notin h_\rho(U),$$

when $|\zeta| = 1$ and $m \geq 1$. But this follows immediately since $h_\rho(U)$ is convex, $h(\rho\zeta) \in h_\rho(\partial U)$, $\rho\zeta h'(\rho\zeta)$ is the outer normal to $h_\rho(\partial U)$ and

$$\arg[m\gamma^{-1}\rho\zeta h'(\rho\zeta)] = \arg[\rho\zeta h'(\rho\zeta)] + \arg \gamma^{-1}$$

where $|\arg \gamma^{-1}| \leq \pi/2$.

4. DOMINANTS OF DIFFERENTIAL SUBORDINATIONS

Let $\psi: \mathbf{C}^3 \to \mathbf{C}$ be holomorphic in a domain D and let $h(z)$ be univalent in U. Suppose $p(z)$ is regular in U, $(p(z), zp'(z), z^2p''(z)) \in D$ when $z \in U$, and $p(z)$ satisfies the differential subordination

$$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z). \tag{28}$$

In this section we will be concerned with dominating the solutions of this differential subordination. We first define our terms.

Definition 4. The univalent function $q(z)$ is said to be a dominant of the differential subordination (28) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (28). If $\tilde{q}(z)$ is a dominant of (28) and $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (28) then $\tilde{q}$ is said to be the best dominant of (28).

Remarks. (1) If there are two best dominants q_1 and q_2, then $q_1(z) \prec q_2(z)$ and $q_2(z) \prec q_1(z)$. This implies that $q_1(z) = q_2(e^{i\theta}z)$. Hence the best dominant of (28), if it exists, will be unique up to rotation of U.

(2) If $q(z)$ is a dominant of (28) and also satisfies (28) then $q(z)$ will be the best dominant. Note that this was the case in Theorems 4 and 5 (see (18) and (24) respectively).

The study of differential subordinations of the form (28), or more generally of the form $\psi(p(z), zp'(z), \ldots, z^k p^{(k)}(z)) \prec h(z)$, is a relatively untouched area of research. Our purpose here is to introduce the subject and to give some definitive answers when $k = 2$.

Several special cases of (28) have appeared in the literature. In 1935 G. M. Goluzin [1] showed that if $h(z)$ is convex then $zp'(z) \prec h(z)$ has best dominant $g(z) = \int_0^z h(t)t^{-1}dt$. In 1970 T. Suffridge [10, p. 777] showed that Goluzin's result is true if h is starlike. In 1947 R. Robinson [9, p. 22] showed that if $h(z)$ and $q(z) = z^{-1}\int_0^z h(t)dt$ are univalent then $p(z) + zp'(z) \prec h(z)$ has best dominant $q(z)$, at least for $|z| < 1/5$. In 1975 D. Hallenbeck and S. Ruscheweyh [2, p. 192] showed that if $\gamma \neq 0$, Re $\gamma \geq 0$ and $h(z)$ is convex, then $p(z) + zp'(z)/\gamma \prec h(z)$ has best dominant $q(z) = \gamma z^{-\gamma}\int_0^z h(t)t^{\gamma-1}dt$.

The following theorem and corollary prove existence of the best dominant of (28) for certain ψ; they also provide a method for obtaining the best dominant. The proofs which follow immediately from Theorem 2 and Corollary 2.1. are omitted.

THEOREM 8. *Let $\psi: \mathbf{C}^3 \to \mathbf{C}$ be holomorphic in a domain D and let $h(z)$ be univalent in U. Suppose $p(z) = a + p_n z^n + \ldots$ is regular in U,*

$$p(z) \not\equiv a,\ n \geq 1, \qquad (p(z), zp'(z), z^2p''(z)) \in D$$

when $z \in U$, and $\psi(p(z), zp'(z), z^2p''(z)) \prec h(z)$. If the differential equation $\psi(q(z), zq'(z), z^2q''(z)) = h(z)$ has a solution $q \in Q$, with $q(0) = a$, and if

$$\psi \in \Psi_n(h,q)$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

COROLLARY 8.1. *Let $\psi: \mathbf{C}^3 \to \mathbf{C}$ be holomorphic in a domain D and let $h(z)$ be univalent in U. Suppose $p(z) = a + p_n z^z + \ldots$ is regular in U, $p(z) \not\equiv a$, $n \geq 1$, $(p(z), zp'(z), z^2p''(z)) \in D$ when $z \in U$, and $\psi(p(z), zp'(z), z^2p''(z)) \prec h(z)$. If the differential equation $\psi(q(z), zq'(z), z^2q''(z)) = h(z)$ has a univalent solution q, with $q(0) = a$, and if $\psi \in \Psi_n(h(\rho z), q(\rho z))$ for $0 < \rho < 1$, then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.*

From the above theorem and corollary we see that the problem of finding the best dominant of the differential subordination (28) reduces to solving a differential equation and the algebraic operation of checking whether $\psi \in \Psi_n(h,q)$ or $\psi \in \Psi_n(h(\rho z), q(\rho z))$. Note that the conclusion of the theorem and corollary can be written in the symmetric form

$$\psi(p(z), zp'(z), z^2p''(z)) \prec \psi(q(z), zq'(z), z^2q''(z)) \quad \Rightarrow \quad p(z) \prec q(z).$$

Examples. (a) Consider the differential subordination

(29) $$(p(z))^{1-\alpha}(zp'(z))^{\alpha} \prec h(z),$$

where $h \in S^*$, $0 < \alpha \le 1$ and $p(z)$ is regular in U with $p(0) = 0$. The differential equation

(30) $$(q(z))^{1-\alpha}(zq'(z))^{\alpha} = h(z)$$

has solution $q(z) = (\alpha^{-1} \int_0^z h^{1/\alpha}(t)t^{-1}dt)^{\alpha}$ which is α-convex and is in S^* [7, p. 219]. Let $\psi(r,s,t) = r^{1-\alpha}s^{\alpha}, h_\rho(z) = h(\rho z)$ and $q_\rho(z) = q(\rho z)$. In order to use Corollary 8.1 we only need to show that $\psi \in \Psi(h_\rho, q_\rho)$ for $0 < \rho < 1$. To do this we need to show $\psi(q(\rho\zeta), m\rho\zeta q'(\rho\zeta)) \notin h_\rho(U)$, when $|\zeta| = 1$ and $m \ge 1$. Using (30) together with the fact that $h_\rho(U)$ is a starlike domain we obtain

$$\psi(q(\rho\zeta), m\rho\zeta q'(\rho\zeta)) = (q(\rho\zeta))^{1-\alpha}(m\rho\zeta q'(\rho\zeta)) = m^{\alpha}h(\rho\zeta) \notin h_\rho(U).$$

Hence, by Corollary 8.1, $q(z)$ is the best dominant of (29) and we have the sharp result

$$(p(z))^{1-\alpha}(zp'(z))^{\alpha} \prec h(z) \quad \Rightarrow \quad p(z) \prec \left(\alpha^{-1}\int_0^z h^{1/\alpha}(t)t^{-1}dt\right)^{\alpha}.$$

If $\alpha = 1$ then this reduces to the result of Suffridge [10, p. 777].

If $f \in S^*$ then the differential equation $(p)^{1-\alpha}(zp')^{\alpha} = f$ has solution

$$p(z) = \left(\alpha^{-1}\int_0^z f^{1/\alpha}(t)t^{-1}dt\right)^{\alpha}.$$

In this case our result yields

(31) $$f \prec h \quad \Rightarrow \quad \left(\alpha^{-1}\int_0^z f^{1/\alpha}(t)t^{-1}dt\right)^{\alpha} \prec \left(\alpha^{-1}\int_0^z h^{1/\alpha}(t)t^{-1}dt\right)^{\alpha},$$

for $0 < \alpha \le 1$.

(b) Consider the differential subordination

(32) $$p(z) + zp'(z)/\gamma \prec h(z),$$

where $h \in C$, $\gamma \ne 0$, Re $\gamma \ge 0$, and $p(z)$ is regular in U with $p(0) = 0$ [2, p. 192]. The differential equation

(33) $$q(z) + zq'(z)/\gamma = h(z)$$

has a univalent solution $q(z) = \gamma z^{-\gamma}\int_0^z h(t)t^{\gamma-1}dt$ (see [3, p. 115]). Taking $\psi(r,s) = r + s/\gamma, h_\rho(z) = h(\rho z)$ and $q_\rho(z) = q(\rho z)$, we need to show that $\psi \in \Psi(h_\rho, q_\rho)$ for $0 < \rho < 1$. To do this we need to show that

$$\psi(q(\rho\zeta), m\rho\zeta q'(\rho\zeta)) \notin h_\rho(U),$$

when $|\zeta| = 1$ and $m \geq 1$. From (33) and Theorem 7 we obtain $q(z) \prec (hz)$. Hence $q(\rho z) \prec h(\rho z)$ and $q(\rho\zeta) \in \overline{h_\rho(U)}$. Using this together with (32) and the fact that $h_\rho(U)$ is a convex domain we obtain

$$\psi(q(\rho\zeta), m\rho\zeta q'(\rho\zeta)) = q(\rho\zeta) + m\,[h(\rho\zeta) - q(\rho\zeta)] \notin h_\rho(U).$$

Hence, by Corollary 8.1, $q(z)$ is the best dominant of (32) and we have the sharp result

$$p(z) + zp'(z)/\gamma \prec h(z) \quad \Rightarrow \quad p(z) \prec \gamma z^{-\gamma} \int_0^z h(t)t^{\gamma-1}dt.$$

In the special case when $p(z) = a + p_n z^n + \ldots$, with $n \geq 2$, a similar analysis, using $\Psi_n(h_\rho, q_\rho)$ with $n \geq 2$, shows that if $p(z) + zp'(z)/\gamma \prec h(z) = q(z) + nzq'(z)/\gamma$ then $p(z) \prec q(z) = \gamma/n z^{-\gamma/n} \int_0^z h(t)t^{\gamma/n-1}dt$ (see also [2]).

(c) Consider the linear differential subordination

$$Az^2p''(z) + Bzp'(z) + Cp(z) \prec h(z) = z,$$

where $p(z) = p_n z^n + \ldots$ is regular in U, $A \geq 0$, $B \geq -A$ and $C > -B$. If we take $\psi(r,s,t) = At + Bs + Cr$, then the differential equation $\psi(q(z), zq'(z), z^2q''(z)) = z$ has univalent solution $q(z) = z/(B + C)$. It is a simple calculation to show that $\psi \in \Psi_n(h,q)$. Hence $p(z) \prec z/(B + C)$ and $z/(B + C)$ is the best dominant.

REFERENCES

1. G. M. Goluzin, *On the majorization principle in function theory* (Russian). Dokl. Akad. Nauk. SSSR 42 (1953), 647–650.

2. D. J. Hallenbeck and S. Ruscheweyh, *Subordination by convex functions.* Proc. Amer. Math. Soc. 52 (1975), 191–195.

3. Z. Lewandowski, S. Miller and E. Złotkiewicz, *Generating functions for some classes of univalent functions.* Proc. Amer. Math. Soc. 56 (1976), 111–117.

4. R. J. Libera, *Some classes of regular univalent functions.* Proc. Amer. Math. Soc. 16 (1965), 755–758.

5. T. H. MacGregor, *A subordination for convex functions of order* α. J. London Math. Soc. (2) 9 (1974/75), 530–536.

6. S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane.* J. Math. Anal. Appl. 65 (1978), no. 2, 289–305.

7. S. S. Miller, P. T. Mocanu and M. O. Reade, *Bazilevič functions and generalized convexity.* Rev. Roumaine Math. Pures Appl. 19 (1974), 213–224.

8. P. T. Mocanu, M. O. Reade and D. Ripianu, *The order of starlikeness of a Libera integral operator.* Mathematica (Cluj), 19(42) (1977/78), no. 1, 67–73.

9. R. M. Robinson, *Univalent majorants.* Trans. Amer. Math. Soc. 61 (1947), 1–35.

10. T. J. Suffridge, *Some remarks on convex maps of the unit disk.* Duke Math. J. 37 (1970), 775–777.

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