ORTHOGONAL POLYNOMIALS ASSOCIATED WITH AN INFINITE INTERVAL

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SECTION 1

1.1. *Introduction.* A problem of great interest in approximation theory is the asymptotic behavior of the largest zero of orthogonal polynomials. What we show in this paper, in Theorem 1, is that when such knowledge is available, we can then get further information about the asymptotic behavior of the norms of the orthogonal polynomials, assumed to be monic, and in some instances, we also get the asymptotic distribution of all of the zeros of the orthogonal polynomials. Since the largest zero behavior for the class of weight functions

$$W_{p,m}(x) = |x|^p \exp(-|x|^m), \quad p > -1, \quad m > 0$$

has been determined for $m = 2, 4, 6$, p arbitrary, in [3] and [4], Theorem 1 can be applied to these cases, formulated as Theorem 2, and we are led to the explicit determination of the zero distributions.

Thus, aside from the intrinsic interest in this class of weights, Theorem 1 provides a general method for converting largest zero asymptotics to the asymptotic behavior of other important parameters of orthogonal polynomials.

Added in revision. After this paper was submitted Professor Paul Nevai informed me of an alternate approach to this problem which will appear in "On Asymptotic Average Properties of Zeros of Orthogonal Polynomials," by Paul G. Nevai and Jesus S. Dehesa.

1.2. Let $W(x)$ be a non-negative continuous function defined for $x \in R = (-\infty,\infty)$ and satisfying $0 < \int |t|^n W(t)\,dt < \infty$, $n = 0, 1, \ldots$. Such a function is called an admissible weight function. There are unique polynomials, $p_n(x) = \gamma_n X^n + \ldots$, $n = 0, 1, \ldots$, which satisfy

$$\int p_m(x) p_n(x)\, W(x)\, dx = \delta_{m,n}, \quad n, m = 0, 1, \ldots.$$

These are the orthonormal polynomials associated with the weight function $W(x)$. Let $N_n(W) = 1/\gamma_n$, and refer to $N_n(W)$ as the norm of the monic orthogonal polynomial $P_n(x) = p_n(x)/\gamma_n$. We assume that $W(x)$ is positive for values of x

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of arbitrarily great modulus. Under these circumstances the zeros of $P_n(x)$ are real and simple and so can be designated as

$$\{x_{i,n}\}, \qquad x_{1,n} < x_{2,n} < \ldots < x_{n,n}.$$

We also have the fact that $x_{n,n} - x_{1,n}$ tends to infinity as n tends to infinity.

1.3. We define contracted zero distribution measures as follows. There is a unique linear function which maps $x_{1,n}$ to -1 and $x_{n,n}$ to 1. Let

$$-1 = y_{1,n} < y_{2,n} < \ldots < y_{n,n} = 1$$

be the images of $\{x_{i,n}\}$ by this mapping. Let $\nu_n{}^*$ be the unit measure with mass $1/n$ at the $y_{i,n}$. We call ν_n^* the contracted zero measure of $p_n(x)$. If the sequence $\{\nu_{k_n}^*\}$ converges weakly to ν, for an increasing sequence $\{k_n\}$, we call ν the contracted zero distribution measure for the sequence $\{p_{k_n}(x)\}$. The measure defined on Borel sets $\{B\}$ of $I = [-1,1]$ by

$$\mu(B) = \frac{1}{\pi} \int_B \frac{dx}{\sqrt{1 - x^2}}$$

is called the arcsine measure, and we denote this measure by μ_I.

We have the result in [1] that

$$\overline{\lim_{n\to\infty}} \frac{N_n^{1/n}(W)}{x_{n+1,n+1} - x_{1,n+1}} \leqq \frac{1}{4},$$

and if for any increasing sequence $\{k_n\}$,

$$\lim_{n\to\infty} \frac{N_{k_n}^{1/k_n}(W)}{x_{k_n+1,k_n+1} - x_{1,k_n+1}} = \frac{1}{4},$$

then μ_I is the contracted zero distribution measure for the sequence $\{p_{k_n}(x)\}$. We refer to this result as an upper bound result for norm behavior, and we formulate and prove a lower bound result for norm behavior.

SECTION 2

2.1 *A lower bound result for norm behavior.*

THEOREM 1. *Let $W(x)$ be an even, positive admissible weight function and let $X_n(W)$, $N_n(W)$ be the largest zero and the norm of the associated orthogonal polynomial $P_n(x)$, respectively.*

Assume that

$$\lim_{n\to\infty} (W(X_n(W)x))^{1/2n} = w(x), \quad x \in I, \tag{2.1}$$

where $w(x)$ is continuous and positive for $x \in I$, and also assume that

$$\lim_{n\to\infty} \int_{-1}^{1} \log(W(X_n(W)s))^{1/2n}\, ds = \int_{-1}^{1} \log w(s)\, ds. \tag{2.2}$$

Let

$$G(w) = \exp\left(1/\pi \int_{-1}^{1} \log w(s) \frac{ds}{\sqrt{1-s^2}}\right). \tag{2.3}$$

(a) It is then true that

$$\varliminf_{n\to\infty} \frac{N_n^{1/n}(W)}{2X_n(W)} \geqq \frac{G(w)}{4}. \tag{2.4}$$

Assume that there is a unit measure μ defined on the Borel subsets of I which satisfies

$$\exp \int \log |x-t|\, d\mu = k/w(x), \tag{2.5}$$

$x \in I$, for some value of k.

(b) If for some increasing sequence $\{k_n\}$

$$\lim_{n\to\infty} \frac{N_{k_n}^{1/k_n}(W)}{2X_{k_n}(W)} = \frac{G(w)}{4}, \tag{2.6}$$

then the measure arising in (2.5) is the contracted zero distribution measure for the sequence $\{p_{k_n}(x)\}$.

2.2. *Proof of Theorem* 1, (a).

Proof of (2.4). We start with the definition

$$N_n^2(W) = \int_{-\infty}^{\infty} P_n^2(t)\, W(t)\, dt.$$

Let $t = X_n(W)s$ to obtain

$$\begin{aligned} N_n^2(W) &= X_n(W) \int_{-\infty}^{\infty} P_n^2(X_n(W)s)\, W(X_n(W)s)\, ds \\ &\geqq \int_{-1}^{1} P_n^2(X_n(W)s)\, W(X_n(W)s)\, ds. \end{aligned}$$

The second inequality holds as soon as $X_n(W) > 1$. We then obtain

$$\frac{N_n^2(W)}{X_n^{2n}(W)} \geqq \int_{-1}^{1} q_n^2(s)\, W(X_n(W)s)\, ds, \tag{2.7}$$

where $q_n(s)$ is the monic polynomial $P_n(X_n(W)s)/X_n^n(W)$, and has n simple zeros on I.

Next, we invoke the arithmetic-geometric inequality to obtain

$$\begin{aligned}\int_{-1}^{1} q_n^2(s)\, W(X_n(W)s)\, ds &= \frac{1}{\pi}\int_{-1}^{1} q_n^2(s)\,[\pi\sqrt{1-s^2}\, W(X_n(W)s]\, \frac{ds}{\sqrt{1-s^2}} \\ &\geqq \exp\frac{1}{\pi}\int_{-1}^{1} (\log q_n^2(s))\frac{ds}{\sqrt{1-s^2}} \\ &\quad\cdot \exp\frac{1}{\pi}\int_{-1}^{1} \frac{\log \pi\sqrt{1-s^2}}{\sqrt{1-s^2}}\, ds \\ &\quad\cdot \exp\frac{1}{\pi}\int_{-1}^{1} \log W(X_n(W)s)\frac{ds}{\sqrt{1-s^2}}.\end{aligned}$$

We next use the fact that

$$\frac{1}{\pi}\int \log|x-s|\frac{ds}{\sqrt{1-s^2}} = \log\frac{1}{2}, \quad x\in I, \tag{2.8}$$

the definition (2.3) and the assumptions (2.1) and (2.2) to obtain

$$\varliminf_{n\to\infty} \frac{N_n^{1/n}(W)}{2\,X_n(W)} \geqq \frac{G(w)}{4}.$$

this completes the proof of Theorem 1, (a).

2.3. *Proof of Theorem* 1, (b). We have

$$\begin{aligned}\frac{N_{k_n}^{1/k_n}(W)}{2X_{k_n}(W)} &= \frac{\left(\int_{-\infty}^{\infty} P_{k_n}^2(t)\, W(t)\, dt\right)^{1/2k_n}}{2X_{k_n}(W)} \\ &= \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{P_{k_n}^2(X_{k_n}(W)s)\, W(X_{k_n}(W)s)X_{k_n}(W)}{X_{k_n}^{2k_n}(W)}\, ds\right)^{1/2k_n} \\ &= \frac{1}{2}X_{k_n}^{1/2k_n}(W)\left(\int_{-\infty}^{\infty} q_{k_n}^2(s)\, W(X_{k_n}(W)s)\, ds\right)^{1/2k_n},\end{aligned} \tag{2.9}$$

where $q_{k_n}(s)$ has been defined following display (2.7).

For sufficiently large n, $X_n(W) > 1$, so that from (2.6) and (2.9) we have

$$\overline{\lim}\frac{1}{2}\left(\int_{-1}^{1} q_{k_n}^2(s)\, W(X_{k_n}(W)s)\, ds\right)^{1/2k_n} \leqq \frac{G(w)}{4}. \tag{2.10}$$

Let $\nu_{k_n}^*$ be the contracted zero measure of $p_{k_n}(x)$, and hence the zero measure of $q_{k_n}(x)$. Let $\{t_n\}$ be an increasing subsequence of $\{k_n\}$ for which $\nu_{t_n}^*$ converges weakly, say to ν. We will show that ν is the measure μ of (2.5). Thus it follows that the sequence $\{\nu_{k_n}^*\}$ converges weakly to μ, so that μ is the contracted zero distribution measure for $\{p_{k_n}(X)\}$. This will then complete the proof of Theorem 1, (b).

By the application of Lemma 5.3, [7, p. 139], (see displays (5.10) and (5.11) of that reference), it follows from (2.10) that for some increasing subsequence $\{p_n\}$ of $\{t_n\}$

$$\overline{\lim_{n\to\infty}} |q_{p_n}(x)|^{1/p_n} | W(X_{p_n}(W)x)|^{1/2p_n} \leqq \frac{G(w)}{2}, \quad \text{for } x \in I,$$

except for a Borel set of measure zero, say Z_1. Now by assumption (2.1) we have

$$\overline{\lim_{n\to\infty}} |q_{p_n}(x)|^{1/p_n} \leqq \frac{G(w)}{2w(x)}, \; x \in I \setminus Z_1. \tag{2.11}$$

The left side of (2.11) can be written

$$\begin{aligned}\overline{\lim_{n\to\infty}} \exp \int \log |x-t|\, d\nu_{p_n}^* &= \exp\left(-\frac{\lim}{n\to\infty}\int \log\frac{1}{|x-t|}\, d\nu_{p_n}^*\right)\\ &= \exp\left(-\int \log\frac{1}{|x-t|}\, d\nu\right),\end{aligned}$$

where the last equality holds for $x \in I$ except for a set Z_2 of capacity zero by the lower envelope theorem of potential theory, ([7], Lemma 4.3, p. 133). We have at this stage

$$\exp \int \log |x-t|\, d\nu \leqq \frac{G(w)}{2w(x)}, \quad x \in I \setminus Z_3, \tag{2.12}$$

where $Z_3 = Z_1 \cup Z_2$, and hence is a Borel set of measure zero. We use the fact that a linear set of capacity zero has measure zero, ([5], p. 286).

We next turn to (2.5), and show that $k = G(w)/2$. Indeed, if μ is a solution of (2.5), we obtain

$$\int \log |x-t|\, d\mu = \log k - \log w(x).$$

Hence

$$\frac{1}{\pi}\int_{-1}^{1}\left(\int \log|x-t|\,d\mu\right)\frac{dx}{\sqrt{1-x^2}} = \log k - \frac{1}{\pi}\int_{-1}^{1}\log w(x)\frac{dx}{\sqrt{1-x^2}}.$$

Use Fubini's theorem on the left side, and (2.8) to obtain

$$\log(1/2) = \log k - \frac{1}{\pi}\int_{-1}^{1}\log w(x)\frac{dx}{\sqrt{1-x^2}}.$$

Hence by (2.3) we have $k = G(w)/2$. Using (2.5) and (2.11), we then have

$$\exp\int \log|x-t|\,d\nu \leqq \exp\int \log|x-t|\,d\mu \tag{2.13}$$

for almost all $x \in I$, and it remains to show that (2.13) implies $\nu = \mu$.

It is shown in ([6], p. 259–262) that if ω is a unit measure defined on the Borel sets of I, and if $v(z) = \int \log|z-t|\,d\omega$, then for $z \not\subset I$

$$v(z) = -\log|2\zeta| + \int_0^{2\pi} P(r,\theta,\phi)(v(\cos\phi) + \log 2)\,d\phi, \tag{2.14}$$

where $\zeta = re^{i\theta}$, $z = (\zeta + \zeta^{-1})/2$, $0 \leqq r < 1$ and $P(r,\theta,\phi)$ is the Poisson kernel $(1/2\pi)(1-r^2)/(1+r^2-2r\cos(\theta-\phi))$. Thus if we let $a(z) = \int \log|z-t|\,d\nu$ and $b(z) = \int \log|z-t|\,d\mu$ then from (2.13) and (2.14) we find for $z \notin I$

$$a(z) - b(z) = \int_0^{2\pi} P(r,\theta,\phi,)(a(\cos\phi) - b(\cos\phi))\,d\phi.$$

Since by (2.13) $a(\cos\phi) - b(\cos\phi) \leqq 0$ for almost all ϕ in $[0, 2\pi]$, it follows that $a(z) - b(z) \leqq 0$ for $z \not\subset I$. Since $a(z) - b(z)$ is harmonic for $z \notin I$ and regular at infinity where it has the value 0, it follows from the maximum principle for harmonic functions that $a(z) - b(z) \equiv 0$ for $z \notin I$. It then follows from a property of potentials ([7], Lemma 4.4, p. 134) that $\mu = \nu$. This completes the proof of Theorem 1, (b).

SECTION 3

3.1. *The contracted zero distribution measure for a class of weight functions.*

THEOREM 2. *Let*

$$W_{p,m} = |x|^p \exp(-|x|^m), \quad p > -1, \quad m > 0.$$

The contracted zero distribution measures of the orthogonal polynomials associated with $W_{p,m}$ are independent of p for $m = 2$, 4 and 6, and are absolutely continuous measures. If they are designated as ν_2, ν_4 and ν_6, we have the formulas

$$\begin{aligned} d\nu_2 &= \frac{2}{\pi}\sqrt{1-x^2}\,dx, \\ d\nu_4 &= \frac{2}{\pi}\sqrt{1-x^2}\left(\frac{2}{3}\right)(1+2x^2)\,dx, \qquad (3.1) \\ d\nu_6 &= \frac{2}{\pi}\sqrt{1-x^2}\left(\frac{1}{5}\right)(3+4x^2+8x^4)\,dx. \end{aligned}$$

3.2. *Proof of Theorem 2.* We first cite results concerning certain parameters of the orthogonal polynomials associated with the weight functions $W_{p,m}$, and then use those results in conjunction with Theorem 1 to arrive at the stated conclusions.

Let $X_r(W_{p,m})$, $N_n(W_{p,m})$ designate the largest zero and the norm of the monic orthogonal polynomials $P_n(x) = P_n(x)/\gamma_n$ associated with the weight function $W_{p,m}$. It is shown in [2], [3] and [4] that if $c_{n-1/2} = \dfrac{\gamma_{n-1}}{\gamma_n}$, then for $m = 2$, 4 and 6, $p > -1$, we have

$$\lim_{n\to\infty} n^{-1/m} c_{n-1/2} = \left(\frac{\Gamma(m+1)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+1\right)}\right)^{-1/m} = d_m \qquad (3.2)$$

and

$$\lim_{n\to\infty} \frac{X_n(W_{p,m})}{c_{n-1/2}} = 2. \qquad (3.3)$$

It is conjectured in [3] that (3.2) is true for all positive, even values of m. If this is true, then (3.3) would be true by the method of [4] and the method that follows would yield the contracted zero distribution measures for all such values of m.

3.2. We now turn to showing the applicability of Theorem 1. We first show that (3.2) and (3.3) enable us to make the required computation for (2.1). Indeed, we have

$$\begin{aligned} &(W_{p,m}(X_n(W_{p,m}(x))^{1/2n} \\ &= |X_n(W_{p,m})|^{p/2n}|x|^{p/2n}\exp\left(-\frac{X_n^m(W_{p,m})x^m}{2n}\right) \\ &= |X_n(W_{p,m})|^{p/2n}|x|^{p/2n}\exp\left(\left(-\frac{1}{2}\right)\left(\frac{c_{n-1/2}}{n^{1/m}}\right)^m\left(\frac{X_n(W_{p,m})}{c_{n-1/2}}\right)^m |x|^m\right). \end{aligned}$$

Hence as n tends to infinity this becomes

$$\exp(-2^{m-1} d_m^m X^m) = \exp(-e_m X^m) = w_m(x). \tag{3.4}$$

Note that (3.4) contains the definition of e_m. It is also true that (2.2) holds.

3.3. The computation of $G(w_m)$ for $m = 2$, 4 and 6 proceeds as follows. Since $\log w_m(t) = -e_m t^m$,

$$G(w_m) = \exp\left(-\frac{e_m}{\pi}\int_{-1}^{1}\frac{t^m}{\sqrt{1-t^2}}\,dt\right).$$

This equals

$$\exp\left[-\frac{1}{\pi}2^{m-1}\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+1\right)}{\Gamma(m+1)}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m}{2}+1\right)}\right] = \exp\left(-\frac{1}{m}\right), \tag{3.5}$$

where the duplication formula for the gamma function is used in the last step.

3.4. We next verify that (2.6) holds for the full sequence $\{n\}$. Indeed,

$$\lim_{n\to\infty}\frac{N_n^{1/n}(W_{p,m})}{2X_n(W_{p,m})} = \lim_{n\to\infty}\frac{1}{2}\;\frac{c_{n-1/2}}{X_n(W_{p,m})}\;\frac{n^{1/m}}{c_{n-1/2}}\;\frac{N_n^{1/n}(W_{p,m})}{n^{1/m}}.$$

By (3.2) and (3.3) this can be simplified to

$$\frac{1}{4}\cdot\frac{1}{d_m}\lim_{n\to\infty}\frac{N_n^{1/n}(W_{p,m})}{n^{1/m}}. \tag{3.6}$$

The limit in (3.6) has the same value as

$$\lim_{n\to\infty}\frac{N_{n+1}/(n+1)^{(n+1)/m}}{N_n/n^{n/m}}, \tag{3.7}$$

if this limit exists. In fact it does exist since we can equate (3.7) to

$$\lim_{n\to\infty}\frac{c_{n+1/2}}{(n+1)^{1/m}}\cdot\left(\frac{n}{n+1}\right)^{n/m} = \frac{d_m}{e^{1/m}}.$$

This combined with (3.6) yields

$$\lim_{n\to\infty}\frac{N_n^{1/n}(W_{p,m})}{2X_n(W_{p,m})} = \frac{1}{4}\sqrt[m]{\frac{1}{e}} = \frac{G(w_m)}{4}, \tag{3.8}$$

the desired result.

3.5. We now construct a solution for (2.5) when $w(x)$ is replaced by $w_m(x)$ and k is given the value $G(w_m)/2$, $m = 2$, 4 and 6. Our procedure is to define μ_m by

$$d\mu_m = \frac{q_m(x)}{\pi\sqrt{1-x^2}}\,dx \tag{3.9}$$

and demonstrate that a valid solution is given by

$$\begin{aligned} q_2(x) &= 1 - T_2(x) \\ q_4(x) &= 1 - \frac{1}{12}(4T_4(x) + 8T_2(x)) \\ q_6(x) &= 1 - \frac{1}{60}(6T_6(x) + 2\,4T_4(x) + 30T_2(x)), \end{aligned} \tag{3.10}$$

where $T_k(x)$, $k = 2, 4, \ldots$, is the Tchebycheff polynomial normalized so that $T_k(1) = 1$.

The method which follows shows that if (3.8) holds for any even integer m, and we use definition (3.9), then (2.5) will have a solution μ_m, where

$$q_m(x) = 1 - \frac{e_m}{2^{m-1}} \sum_{0}^{(m/2)-1} (m-2k)\binom{m}{k} T_{m-2k}(x).$$

We now proceed. Making the mentioned substitutions in (2.5) we obtain

$$\exp \int \log|x-t|\,d\mu = \frac{1}{2}\sqrt[m]{\frac{1}{e}}\exp e_m x^m,$$

or

$$\frac{1}{\pi}\int \log|x-t|\frac{q_m(t)}{\sqrt{1-t^2}}\,dt = \log\frac{1}{2} - \frac{1}{m} + e_m x^m. \tag{3.11}$$

To solve (3.11) we use (2.8) and the relationship

$$\frac{1}{\pi}\int \log|x-t|\frac{T_n(t)}{\sqrt{1-t^2}}\,dt = -\frac{1}{n}T_n(x), \quad x \in I. \tag{3.12}$$

The relationship (3.12) can be obtained by contour integration.

In order to use (3.12) we use the identity valid for positive even integer m,

$$X^m = \frac{1}{2^{m-1}}\left(T_m(x) + \binom{m}{1}T_{m-2}(x) + \ldots + \binom{m}{\frac{m}{2}-1}T_2(x) + \frac{1}{2}\binom{m}{\frac{m}{2}}\right). \tag{3.13}$$

With this identity, the right side of (3.11) becomes

$$\log \frac{1}{2} + \frac{e_m}{2^{m-1}} \sum_{0}^{(m/2)-1} \binom{m}{k} T_{m-2k}(x) \tag{3.14}$$

Now we use (2.8) and (3.12) to find

$$q_m(x) = 1 - \frac{e_m}{2^{m-1}} \sum_{0}^{(m/2)-1} \binom{m}{k} (m-2k)\, T_{m-2k}(x). \tag{3.15}$$

Although we now have a solution to (2.5), in order to show that the right side of (3.9) defines a unit measure, we must show that $q_m(x) \geqq 0$, $x \in I$, and that

$$\frac{1}{\pi} \int_{-1}^{1} \frac{q_m(t)}{\sqrt{1-t^2}}\, dt = 1. \tag{3.16}$$

Since $|T_k(x)| \leqq 1$, the non-negativeness of $q_m(x)$ will follow from (3.15) by showing that

$$\frac{e_m}{2^{m-1}} \sum_{0}^{(m/2)-1} (m-2k) \binom{m}{k} = 1.$$

To see this, start with the identity $(m-2k)\binom{m}{k} = m\left(\binom{m-1}{k} - \binom{m-1}{k-1}\right)$, $k = 1, \ldots, m-1$. Then we have

$$\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+1\right)}{\Gamma(m+1)}\, m \left(1 + \sum_{1}^{(m/2)-1} \left(\binom{m-1}{k} - \binom{m-1}{k-1}\right)\right)$$
$$= \frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+1\right)}{\Gamma(m)} \binom{m-1}{(m/2)-1} = 1.$$

We also can verify (3.16) by term by term integration.

3.6. We know that (3.8) holds only for $m = 2, 4, 6$, so μ_m given by (3.9) with (3.15) substituted yields solution of (2.5) only for these values of m. The final step for obtaining (3.10) is to substitute

$$\begin{aligned} T_2(x) &= 2x^2 - 1 \\ T_4(x) &= 8x^4 - 8x^2 + 1 \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \end{aligned} \tag{3.17}$$

into (3.15) for $m = 2, 4, 6$, and place this in (3.9). Since $q_m(x)$ has a factor

$(1 - x^2)$, division gives the final result as found in (3.1). Note that in (3.1) we equate ν_m to μ_m for $m = 2, 4, 6$, as is permitted by Theorem 1.

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