TORUS KNOTS IN THE COMPLEMENTS OF LINKS AND SURFACES

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Our major result is Theorem 2, which states that if $S \subset S^3$ is any nonsingular, nonseparating orientable surface (possibly disconnected), then there exists a torus knot t with mirror image t' such that $(t \# t') \cap S = \emptyset$ and $S \subset F$, where F is a minimal spanning surface (fiber) for $t \# t'$. The basic construction gives insight into the structure of closed 3-manifolds, and some important known results follow as corollaries.

The notation and techniques are standard (cf. [3], [6], [10]). We work exclusively in the *PL* category. The symbols $\partial(\ldots)$, $N(\ldots)$, and $CL(\ldots)$ denote, respectively, the boundary, regular neighborhood, and closure of the object $(\ldots)$. We shall have several occasions to move links and surfaces by ambient isotopies, making the implicit assumption that each may be reversed when the construction is complete.

We begin by constructing torus knots in the complements of links.

THEOREM 1. *Let $k \subset S^3$ be a tame link. There exists a torus knot $t \subset S^3$ such that $t \cap k = \emptyset$ and $k \subset F$, where F is a minimal spanning surface for t.*

Proof. We say that a link in R^3 is in *square bridge position* with respect to the plane $z = 0$ if the projection onto the plane is regular and if each segment above the plane projects to a horizontal segment and each one below to a vertical segment. Our link $k \subset S^3$ may be represented as a closed braid [1, p. 42], and we may assume k lies in a 3-cell which has been identified with $[0,1] \times [0,1] \times [-1,1] \subset R^3$. It is now a simple matter to make each overcrossing horizontal and each undercrossing vertical, and an additional ambient isotopy (cf. figure 1) will put k into square bridge position with respect to the "plane" $[0,1] \times [0,1] \times \{0\}$. (Note that the minimum bridge number of all such square bridge representations is an integral link invariant; we call it the *square bridge number.*)

Let $T \subset S^3$ be a torus which determines the genus one Heegard Splitting (U,V) of S^3. We may assume $T \cap ([0,1] \times [0,1] \times [-1,1]) = [0,1] \times [0,1] \times \{0\}$ and $[0,1] \times [0,1] \times [0,1] \subset U$. Each disk $[0,1] \times \{b\} \times [0,1]$ thus lies in a properly embedded, nonseparating disk $D \subset U$, so $k \cap U \subset \bigcup_{i=1}^{r} D_i$. Similarly, $k \cap V \subset \bigcup_{i=1}^{s} E_i$, where each $E_i \subset V$ is a properly embedded, nonseparating disk containing $\{a\} \times [0,1] \times [-1,0]$ and each $D_i \cap E_j$ is a singleton. If $(r,s) \neq 1$, add more disks D, with $D \cap k = \emptyset$, until $(r,s) = 1$. The curves ∂D_i and ∂E_i inherit an orientation from the coordinatization of $[0,1] \times [0,1] \times \{0\}$.

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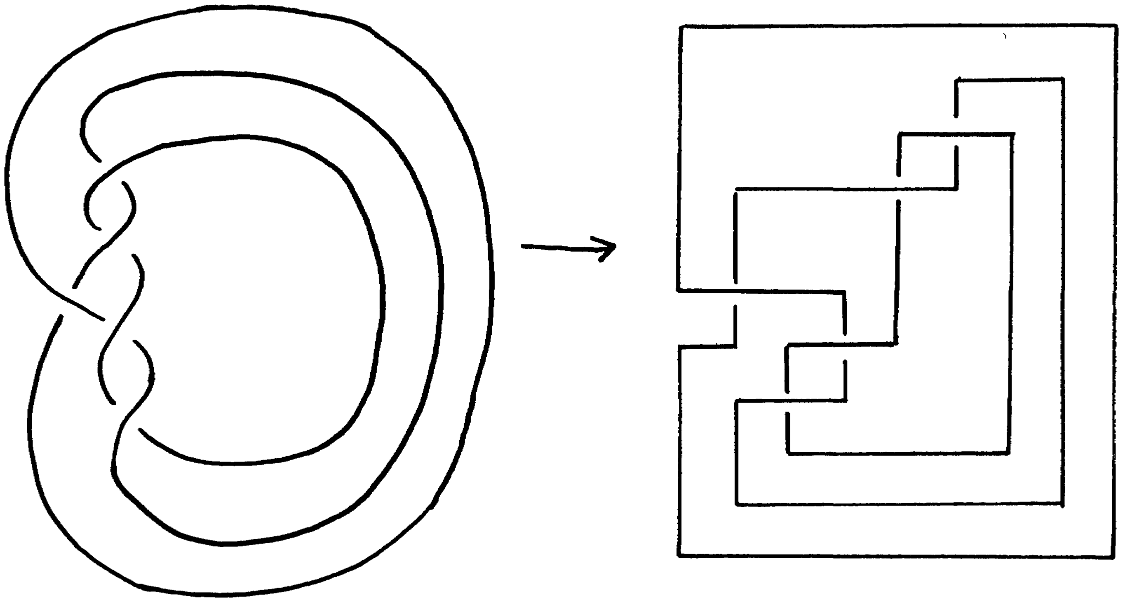


Figure 1.

Within a small neighborhood, in T, of each component of $\partial D_i \cap \partial E_j$, $1 \leq i \leq r$, $1 \leq j \leq s$, add to $\left(\bigcup_{i=1}^{r} D_i\right) \cup \left(\bigcup_{i=1}^{s} E_i\right)$ a pair of triangular 2-cells as in Figure 2.

The result is a 2-manifold F, $k \subset F$, and ∂F is the torus knot t of type (r,s). Since we have constructed F to have genus $(1/2)(r-1)(s-1)$, it must be a minimal spanning surface for t.

If M is a 3-manifold fibered over S^1, then a polygonal simple closed curve $s \subset M$ is said to be *perpendicular* to the fibering if s is transverse to each fiber, and thus intersects each fiber in the same finite number of points.

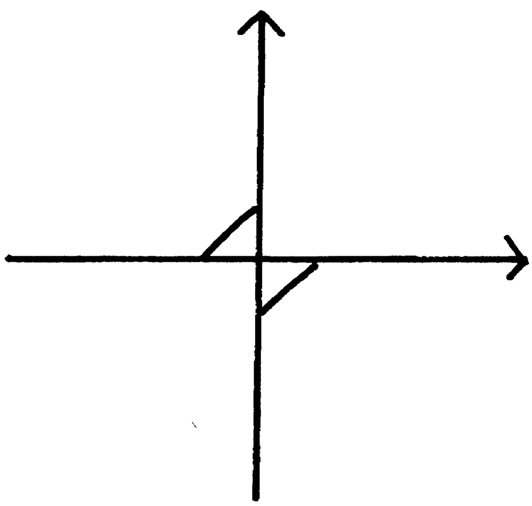


Figure 2.

COROLLARY 1. *Let* $\ell = \bigcup_{i=1}^{n} \ell_i$ *be a link of n components, and assign to each component* ℓ_i *an integer* s_i. *There exists a torus knot* $t \subset S^3 - \ell$ *such that* $lk(t,\ell_i) = s_i$, $1 \leq i \leq n$. If $s_i \neq 0$, ℓ_i *is perpendicular to the fibering of*

$$CL(S^3 - N(t)).$$

Proof. Construct t and F as in the theorem, so that $\ell \subset F$. If $s_i = 0$, we are finished with ℓ_i, so we may assume $s_i \neq 0$. Note from our construction that we can draw an arc $\alpha_i \subset F$ from ℓ_i to t such that $(\alpha_i \cap \ell) \subset \partial\alpha_i$. The torus $\partial N(t)$ inherits an $S^1 \times S^1$ product structure from the fibering of $CL(S^3 - N(t))$, and $(N(\alpha_i) \cap F) \cap \partial N(t)$ is an arc δ_i in $\partial N(t)$. Let $A_i = \delta_i \times S^1 \subset \partial N(t)$ be an annulus which is contractible in $N(t)$. Let $\beta_i \subset F$ be a nonsingular arc which begins in ∂A_i, travels in $\partial N(\alpha_i)$ to ℓ_i, proceeds in a positive direction along $\ell_i - N(\alpha_i)$, and then returns to ∂A_i along $\partial N(\alpha_i)$, and assume β_i has been parameterized accordingly as $\beta_i(x)$, $0 \leq x \leq 1$. Let δ_i be parameterized as $\delta_i(x)$, $1 \leq x \leq 2$, with $\beta_i(1) = \delta_i(1)$ and $\beta_i(0) = \delta_i(2)$, so that β_i and δ_i together represent a new simple closed curve in F which is isotopic, in $S^3 - \bigcup_{j \neq i} \ell_j$, to ℓ_i. Without loss of generality we may assume that $s_i > 0$ and that each simple closed curve of the form $\{a\} \times S^1 \subset \delta_i \times S^1 = A_i$ has linking number $+1$ with t. Since $CL(S^3 - N(t))$ is fibered over S^1, it contains a product of the form

$$(F \cap CL(S^3 - N(t))) \times [0,\varepsilon],$$

where $F \cap CL(S^3 - N(t))) = (F \cap CL(S^3 - N(t))) \times \{0\}$ and the increasing second coordinate determines the positive direction on A_i. Replace β_i by

$$\hat{\beta}_i = (\beta_i(x), \varepsilon x),$$

so

$$\hat{\beta}_i \subset (F \cap CL(S^3 - N(t))) \times [0,\varepsilon], \quad 0 \leq x \leq 1,$$

and replace δ_i by $\hat{\delta}_i = (\delta_i(x), (2\pi s_i - \varepsilon)(x - 1) + \varepsilon) \subset A_i$, where $1 \leq x \leq 2$ and the second coordinate is taken modulo 2π. Then $\hat{\ell}_i = \hat{\beta}_i \cup \hat{\delta}_i$ is a simple closed curve isotopic to ℓ_i in $S^3 - \bigcup_{j \neq i} \ell_j$, $lk(\hat{\ell}_i, t) = s_i$, and $\hat{\ell}_i$ is perpendicular to the fibering of $CL(S^3 - N(t))$. Similar constructions applied to each component of ℓ complete the proof.

COROLLARY 2. ([2], [9], [10, p. 340]) *Every closed orientable 3-manifold M has an open book decomposition.*

Proof. By Hilden [4] and Montesinos [8] M is an irregular 3-fold branched covering of S^3, branched over a knot k. Letting $\ell = k$ and $s_1 \neq 0$, we see from Corollary 1 that there exists a torus knot $t \subset S^3 - k$ such that $CL(S^3 - N(t))$ is fibered. Since k is perpendicular to the fibering, the fibering lifts to

$CL\,(M - N(p^{-1}(t)))$, where p is the covering map. Hence M has the desired structure. (Note that a stronger result is proved in Corollary 4.)

COROLLARY 3. *If $\ell \subset S^3$ is any link, then there exists a torus knot $t \subset S^3 - \ell$ such that $CL\,(S^3 - N(\ell \cup t))$ is fibered over S^1.*

Proof. Follow the construction with $\prod_{i=1}^{n} s_i \neq 0$.

THEOREM 2. *Let $S \subset S^3$ be an orientable, nonseparating and nonsingular surface. There exists a torus knot t with mirror image t' such that*

$$(t \# t') \cap S = \emptyset, \qquad S \subset F,$$

where F is a fiber for $t \# t'$, $F - S$ is connected, and S is incompressible in $S^3 - (Bd(S) \cup (t \# t'))$.

Proof. Since S is nonseparating, each component S_i of S, $1 \leq i \leq n$, must have boundary. Hence, by standard arguments [7, pp. 43 ff.]), we may assume $S_i = D_i \cup \left(\bigcup_j b_{ij}\right)$, where D_i is a disk and the b_{ij} are bands attached to ∂D_i. Let c_{ij} be a core of b_{ij}, so $c_{ij} \subset b_{ij}$ is an arc and $c_{ij} \cap \partial b_{ij} = c_{ij} \cap D_i = \partial c_{ij}$ meets each component of $b_{ij} \cap D_i$. Using the notation of the proof of Theorem 1, we move S by an ambient isotopy until $S \subset [0,1] \times [0,1] \times [-1,1]$,

$$S \cap \partial([0,1] \times [0,1] \times [-1,1]) = \bigcup_{i=1}^{n} D_i, \qquad D_i \subset [(i-1)/n, i/n] \times \{0\} \times [0,1],$$

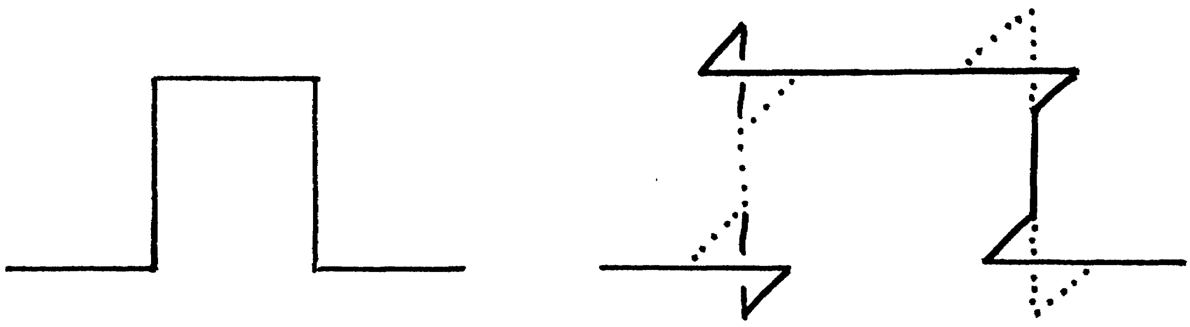
and

$$S \cap ([0,1] \times \{0\} \times \{0\}) = \bigcup_{i,j} (D_i \cap b_{ij}).$$

Temporarily ignoring the bands b_{ij} and concentrating on the c_{ij}, we may, keeping $\bigcup_{i,j} \partial c_{ij}$ fixed, apply the construction of Theorem 1 to obtain a torus knot t with fiber F' such that $\bigcup_{i,j} (D_i \cup c_{ij}) \subset F'$. The surface

$$S' = \left(\bigcup_i D_i\right) \cup \left(F' \cap N\left(\bigcup_{i,j} c_{ij}\right)\right) \subset F'$$

is homeomorphic to S and differs from S only by twists in the bands b_{ij}. Our construction assures us that we may draw an arc in $F' - \bigcup_{i,j} (D_i \cup c_{ij})$ from either side of each c_{ij} to t. Hence neither $\bigcup_{i,j} (D_i \cup c_{ij})$ nor S' separate F'. Moreover,



a. b.

Figure 3.

we can introduce additional right-hand twists in any one of the bands b'_{ij} of S' as follows. Replace a small horizontal segment of the projection of c_{ij} with the projection shown in figure 3(a). The resulting right-hand twist in b_{ij} is indicated in figure 3(b), where the perspective is from above, the solid and dotted lines represent opposite sides of the band, and only the top side is indicated when they lie above one another. Of course, this operation replaces the torus knot t with another.

We encounter difficulty, however, if we attempt to introduce left-hand twists in the bands b'_{ij}. This is due to the inherent right-handedness of the surface F', which is in turn derived from the way the torus knot t was defined. We shall surmount this difficulty by composing t with its mirror image, which has a minimal spanning surface in which all bands have left-hand twists.

Let $\gamma \subset F' - \left(t \cup \left(\bigcup_{i=1}^{n} D_i\right)\right)$ be an arc chosen so that γ meets and is transverse to each c_{ij}. Let $C = N(\gamma)$ and let $L \subset CL(S^3 - C)$ be a properly embedded disk chosen so that $L \cap \partial C = F' \cap \partial C$. Note that each component of

$$G = \left(F' - \bigcup_{i,j} (D_i \cup c_{ij})\right) \cap CL(S^3 - C)$$

must meet ∂C, because if not, $\bigcup_{i,j} (D_i \cup c_{ij})$ would separate F'. Next, staying within the 3-cell $C = N(\gamma)$, we apply our previous construction with the complex

$$\left(\bigcup_{i,j} (c_{ij} \cap C) \cup L\right) \subset S^3$$

replacing $\bigcup_{i,j} (D_i \cup c_{ij})$. This time, however, replace the instructions in figure 2 for the construction of the torus knot with those in figure 4. The result will be a torus knot $t' \subset C$ of the form (r', s'), with $r' > 0$ and $s' < 0$, and with a spanning surface (fiber) F'' such that

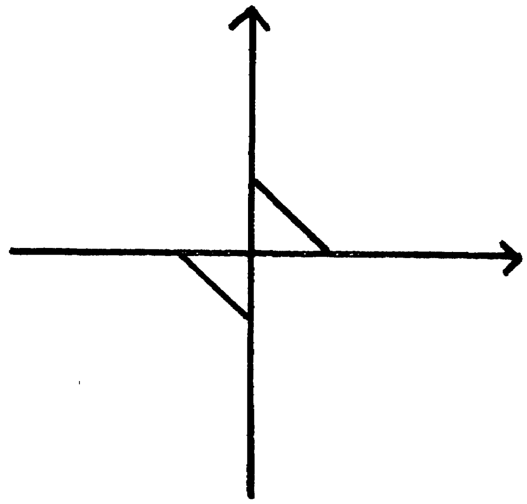


Figure 4.

$$\left(\bigcup_{i,j} (c_{ij} \cap C) \cup L\right) \subset F''$$

and $F'' \cap \partial C = F' \cap \partial C$. As before $\bigcup_{i,j} (c_{ij} \cap C) \cup L$ does not separate F'' and $H = \left(F'' - \bigcup_{i,j} c_{ij}\right) \cap C$ is connected because we can draw an arc in H from either side of each component of $c_{ij} \cap C$ to t'. This time, however, we may presume to have inserted any number of *left-hand* twists in each of the bands $N(c_{ij} \cap C) \cap F''$. Thus, if we let F''' denote the surface

$$(F'' \cap C) \cup (F' \cap CL(S^3 - C)),$$

each band $N(c_{ij}) \cap F'''$ may be twisted an arbitrary number of times in either direction, so we may assume each $b_{ij} \subset F'''$, and hence, $S \subset F'''$. By adding disks again, we may also assume that this construction has been carried out with $r' = r$ and $s' = -s$, so t' is the mirror image of t. Now

$$F''' - \bigcup_{i,j} (D_i \cup c_{ij}) = G \cup H,$$

H is connected, and each component of G meets H, so $\bigcup_{i,j} (D_i \cup c_{ij})$, and hence S, cannot separate F'''. It is thus possible for us to find an arc δ in $F''' - S$ which begins in t, meets ∂C once, and ends in t'. Now $t \subset CL(S^3 - C)$ and $t' \subset C$, so replacing $(t \cup t') \cap N(\delta)$ *by* $F''' \cap \partial(N(\delta))$, and F''' by

$$F = CL(F''' - N(\delta)),$$

we obtain the knot $t \# t' = \partial F$ with fiber F, and $S \subset F$. But δ does not disconnect $F''' - S$, so $F - S$ must be connected. If S were compressible in $S^3 - (Bd(S) \cup (t \# t'))$, there would exist, by standard arguments, a nonsingular disk J such that $J \cap F = J \cap S = \partial J$ would be noncontractible in S. Since F is incompressible, ∂J would have to bound a disk $J' \subset F$, and since S does not separate F, we would have $J' \subset S$, a contradiction. Thus S is incompressible in $S^3 - (Bd(S) \cup (t \# t'))$.

COROLLARY 4. ([2],[9],[10, p. 341]). *Every closed orientable 3-manifold M has an open book decomposition with a connected binding.*

Proof. According to Lickorish ([5], cf. also [10, p. 273]) M may be obtained from S^3 by framed surgery on a link $\ell = \bigcup_{i=1}^{n} \ell_i$, where each ℓ_i is unknotted and has surgery coefficient ± 1. Assume each $\ell_i \subset A_i$, where A_i is an annulus such that each component of $A_i \cap \partial(N(\ell_i))$ is a longitude (that is, is contractible in $S^3 - N(\ell_i)$), ℓ_i is noncontractible in A_i and $A_i \cap A_j = \emptyset$ if $i \neq j$. Apply theorem 2 to $S = \bigcup_{i=1}^{n} A_i$, yielding a knot $t \# t'$ with fiber F and $S \subset F$. But then surgery is merely a twist [10, p. 274] along each A_i, the fibering is maintained, and $t \# t'$ remains the connected binding.

Note added in proof. Related results, using different methods are described in J. R. Stallings, *Constructions of fibered knots and links.* Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976) part 2, pp. 55–60. Proc. Sympos. Pure Math., XXXII. Amer. Math. Soc., Providence, R.I., 1978.

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