SOME EXTREMAL PROBLEMS IN CONFORMAL AND QUASICONFORMAL MAPPING

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INTRODUCTION

This paper centers around extremal problems for which the extremal functions are univalent mappings of multiply connected domains in the extended complex plane onto domains bounded by arcs of generalized lemniscates, arcs where

$$\prod_{j=1}^{n} |w - w_j|^{x_j}$$

is constant for some complex constants $w_1, w_2, \ldots, w_n$ and nonzero real numbers $x_1, x_2, \ldots, x_n$. We formulate an extremal problem for conformal mappings, apply variational methods, and prove that the solution of this extremal problem provides a new proof of a canonical mapping theorem. We modify the extremal problem so it applies to general classes of quasiconformal mappings in order to obtain a new representation of the dielectric Green's function for multiply connected domains.

The theory of lemniscates has always played a significant role in the theory of conformal mapping. On one hand, Hilbert has shown that one can approximate quite general sets of continua by lemniscates. On the other hand, the Green's function with pole at ∞ is obviously elementary in the case of a domain whose boundary is an entire lemniscate.

Julia [4, Chapter 5] extending the work of de la Vallée Poussin, proved

THEOREM 1. *To each n-tuply connected domain* Δ *in the complex plane there corresponds a polynomial* $P(z) = \prod_{j=1}^{n-1} (z - z_j)$ *such that* Δ *can be mapped conformally onto a domain with the property that* $|P(z)|$ *is constant on each component of the boundary.*

Walsh [21] proved theorems about conformal mappings onto regions bounded by all of one or two generalized lemniscates. Jenkins [3], Landau [7], and Pirl [10] have given alternate proofs. In [22] Walsh and Landau considered a limiting case of Walsh's original theorems in which different boundary continua come together. De la Vallée Poussin, Julia, Walsh, and Jenkins all relied on uniformization theorems for their proofs. Landau noted that the lemniscate mappings transform certain harmonic domain functions into functions which are extendible harmonically

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to the entire plane with the exception of a finite number of points. He constructs a sequence of mappings converging to the desired lemniscate mapping. Pirl used the Koebe continuity method.

In this paper we prove a theorem equivalent to results of Grunsky [1] which generalize results of Julia and Walsh:

THEOREM 2. *Let $\mathscr{D}$ be an n-tuply connected domain in the extended z-plane, which contains ∞, and whose complement consists of proper continua, $\mathscr{E}_1, \mathscr{E}_2, \dots \mathscr{E}_n$. Let $x_1, x_2, \dots x_m$ be real numbers, not all zero, with $m > n$, and $\sum_{j=1}^{m} x_j = 0$. Let $z_{n+1}, z_{n+2}, \dots, z_m = \infty$ be distinct points in $\mathscr{D}$. There is a conformal mapping f of $\mathscr{D}$ onto a domain D with complement consisting of components $E_1, E_2, \dots E_n$, such that $f(\infty) = \infty$, and all points w on ∂E_k satisfy*

$$c_k + \sum_{j=1}^{m-1} x_j \log |w - w_j| = 0 \quad k = 1, 2, \dots n,$$

where $w_j \in E_j$ for $j = 1, 2, \dots n$; $w_j = f(z_j)$ for $j = n + 1, n + 2, \dots m$, and the constants c_j are determined up to an additive constant by

$$(0.1) \qquad -2\pi x_k + \sum_{j=1}^{n} c_j \mathscr{P}_{jk} - 2\pi \sum_{j=n+1}^{m} x_j \Omega_j(z_j) = 0, \; k = 1, 2, \dots n,$$

where Ω_j is the harmonic measure of $\partial \mathscr{E}_j$ with respect to $\mathscr{D}$, and $\{\mathscr{P}_{jk}\}_{j,k=1}^{n}$ is the matrix of periods of the harmonic conjugates of $\Omega_1, \Omega_2, \dots \Omega_n$.

Grunsky used uniformization to obtain his results. Kühnau [5] used variational methods to prove the existence of a sequence of solutions to extremal problems, whose limit is the mapping in Grunsky's theorem, but he did not show that the limit function itself is the solution of an extremal problem.

In the first section we define a functional χ on the class Σ of conformal mapping functions of a multiply connected domain $\mathscr{D}$ containing ∞, which hold the point at ∞ fixed. It follows that there is a function f in Σ maximizing χ. To characterize f by variational methods [14, 15] we define varied mapping functions $T \circ f$ where $T(w)$ is a conformal mapping of $f(\mathscr{D})$ of the form

$$T(w) = w + \frac{A\rho^2}{w - w_0} + O(\rho^3) \quad \text{for } |w - w_0| > \rho.$$

We derive an asymptotic formula

$$0 \geq \chi[T \circ f] - \chi[f] \geq \rho^2 \operatorname{Re}\{As(w_0)\} + O(\rho^3).$$

By considering the function $s(w_0)$ appearing in this relation, we show that any extremal function satisfies the conditions in Theorem 2. We calculate $\chi[f]$ and consequently obtain a distortion theorem for conformal mappings and isoperimetric inequalities for multiply connected domains.

To discuss Section 2 we need to define $K(z)$-quasiconformal mappings. Let $K(z)\colon \mathscr{D} \to [1,\infty)$ be a measurable function whose essential supremum $\|K\|_\infty$ is finite. A homeomorphism $f\colon \mathscr{D} \to \mathbf{C} \cup \{\infty\}$ is *$K(z)$-quasiconformal* ($K(z)$-q.c.) if f is locally absolutely continuous on a.e. horizontal and vertical line in $\mathscr{D}$ and satisfies the dilatation condition

$$1 \leq \frac{|f_z(z)| + |f_{\bar z}(z)|}{|f_z(z)| - |f_{\bar z}(z)|} \leq K(z) \quad \text{for a.e. } z \in \mathscr{D}.$$

Since $\|K\|_\infty < \infty$, each $K(z)$-quasiconformal mapping is $\|K\|_\infty$-quasiconformal according to the usual definition [8].

After making certain restrictions on $K(z)$, we generalize χ to be defined on Σ_K the class of $K(z)$-quasiconformal mapping functions of $\mathscr{D}$, taking ∞ to ∞. There is a function f which maximizes χ over this class. Using variational methods, we will show

$$(J \circ f)_{\bar z} = -\frac{K(z) - 1}{K(z) + 1}\, \overline{(J \circ f)_z} \text{ a.e. in } \mathscr{D}, \tag{0.2}$$

where $J(w)$ is of the form $-\sum_{j=1}^{m-1} x_j \log(w - w_j)$ and $\operatorname{Re} J(w)$ is constant on each component of the boundary of the image of f. If we define $U(z) = \operatorname{Re}(J \circ f(z))$ and $V(z) = \operatorname{Im}(J \circ f(z))$ locally, then (0.2) is equivalent to the generalized Cauchy-Riemann equations

$$KU_x = V_y \qquad KU_y = -V_x \qquad \text{a.e. in } \mathscr{D}.$$

We prove that consequently U is a weak solution of $\operatorname{div}(K \operatorname{grad} U) = 0$. With properly chosen parameters U plus a constant is a weak fundamental solution of this equation. Thus the dielectric Green's function of $\mathscr{D}$ with respect to $K(z)$ can be represented as an extremal quasiconformal mapping function composed with the real part of a linear combination of logarithmic terms plus a constant. Schiffer and Schober [18] have represented the dielectric Green's function as the solution of a different extremal problem composed with the real part of an analytic function. This analytic function is not necessarily a simple linear combination of logarithmic terms like $J(w)$.

The variational method for quasiconformal mappings [6, 12, 16, 17, 19] involves variations of the form $T(w) = w - \frac{\varepsilon}{\pi} \int_{\mathbf{C}} \frac{a(\zeta)}{\zeta - w}\, d\mu_\zeta + O(\varepsilon^2)$ for appropriately chosen $a(\zeta)$. We use μ to denote two dimensional Lebesgue measure. Once again we calculate an asymptotic expansion involving the extremal function f,

$$0 \geq \chi[T \circ f] - \chi[f] \geq -\frac{\varepsilon}{\pi} \operatorname{Re} \int_{\mathbf{C}} s(\zeta) a(\zeta)\, d\mu_\zeta + O(\varepsilon^2).$$

The references mentioned above allow us to conclude that $J \circ f$ satisfies the differential equation (0.2) with $J(w) = \int \sqrt{s(w)}\, dw$.

When $K(z)$ satisfies additional conditions, we can solve (0.2) sufficiently to compute the maximum of χ over Σ_K. Then we can obtain inequalities for the distortion of various domain-dependent quantities under $K(z)$-*q.c.* mappings.

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1. THE EXTREMAL PROBLEM FOR CONFORMAL MAPPING

Notation. The following notation and assumptions will be used throughout this paper. The symbols $\mathscr{D}$, $\mathscr{E}_j$, x_j, z_j, Ω_j, and $\mathscr{P}_{jk}$ are defined as in Theorem 2. In particular note $\sum_{j=1}^{m} x_j = 0$. We denote the component of the boundary of $\mathscr{D}$ contained in $\mathscr{E}_j$ by $\mathscr{B}_j$. $\mathscr{D}_j$ is the interior of $\mathscr{E}_j$. Σ and Σ_K are the classes of conformal and $K(z)$-quasiconformal mappings of $\mathscr{D}$ holding ∞ fixed. If f is any element of Σ or Σ_K we let $f(\mathscr{D}) = D$, and E_j, B_j, and D_j are defined relative to D as $\mathscr{E}_j$, $\mathscr{B}_j$, and $\mathscr{D}_j$ are to $\mathscr{D}$. In general w_j is a chosen point of E_j for $j = 1, 2, \ldots n$, and $w_j = f(z_j)$ for $j = n+1, \ldots m$.

The Green's function of D, g, and the regular part of the Green's function, h, satisfy the following properties for each fixed $\zeta \in D$:

i) $g(w,\zeta)$ is harmonic in $D - \{\zeta\}$,

ii) $h(w,\zeta) = \begin{cases} g(w,\zeta) + \log|w-\zeta| & \text{if } \zeta \text{ is finite} \\ g(w,\zeta) - \log|w| & \text{if } \zeta = \infty \end{cases}$

is harmonic at ζ,

iii) $g(w,\zeta) \to 0$ as $w \to \partial D$.

We will abbreviate $h(w,w)$ as $h(w)$. The functions g_j and h_j are defined in a corresponding manner for D_j, as $\mathscr{g}$ and $\mathscr{h}$ are for $\mathscr{D}$. We extend $h_j(w)$ to all of E_j by defining $h_j(w) = -\infty$ if $w \in B_j$.

Let $\omega_j(w)$ be the harmonic measure of B_j with respect to D. Let P_{jk} be the period of the harmonic conjugate of ω_j about B_k.

We interpret the product of 0 and $-\infty$ as 0 and $f'(\infty)$ as $\lim_{z\to\infty} f(z)/z$.

Let $f \in \Sigma$. We define

$$(1.1) \qquad \chi[f] = \sup_{\substack{w_j \in E_j \\ j=1,2,\ldots n}} \left[\sum_{\substack{j,k=1 \\ j \neq k}}^{m-1} x_j x_k \log|w_j - w_k| + \sum_{j=1}^{n} x_j^2 h_j(w_j) + \sum_{j=n+1}^{m-1} x_j^2 \log|f'(z_j)| - x_m^2 \log|f'(\infty)| \right]$$

For a given $f \in \Sigma$ a set of points $\{w_j\}_{j=1}^n$ with $w_j \in E_j$, such that the supremum in (1.1) is attained, will be called *a set of optimal foci for f.*

LEMMA 1. *Each function f in Σ has a set of optimal foci.*

Proof. We show that the extension of $h_j(w)$ to be $-\infty$ on B_j makes $h_j(w)$ continuous on E_j, and thus $\chi[f]$ is the supremum of a continuous function on the compact set $\bigtimes_{j=1}^{n} E_j$. If $w_j \in D_j$, let F_j be a function which maps the unit disk $\{|z| < 1\}$ conformally onto that component of D_j containing w_j so that $F_j(0) = w_j$. Using the conformal invariance of the Green's function, the fact that the Green's function for the unit disk with pole at 0 is $-\log|z|$, and the Koebe 1/4 theorem we obtain

$$h_j(w_j) = \log|F_j'(0)| \leq \log(4 \inf_{w \in \partial Dj}|w - w_j|). \tag{1.2}$$

Consequently $h_j(w_j) \to -\infty$ as $w_j \to \partial D_j$.

LEMMA 2. *There exists $f \in \Sigma$ such that $\chi[f] = \sup_{\Sigma} \chi$.*

Proof. Let $\{f_\nu\}_{\nu=1}^{\infty}$ be a sequence of functions in Σ such that $\chi[f_\nu] \neq -\infty$, and

$$\lim_{\nu \to \infty} \chi[f_\nu] = \sup_{\Sigma} \chi. \tag{1.3}$$

Let z_0 and z_1 be distinct points in $\mathscr{D}$. Since it is easily seen that $\chi[af + b] = \chi[f]$ for all $a, b \in \mathbf{C}$, $a \neq 0$, we can choose the f_ν so that $f_\nu(z_0) = z_0$, $f_\nu(z_1) = z_1$, $f_\nu(\infty) = \infty$ for each ν. By [8] there is a subsequence of $\{f_\nu\}$, which we will relabel $\{f_\nu\}$, such that for some $f \in \Sigma$, $f_\nu \to f$ uniformly on compact subsets of $\mathscr{D}$. We will derive the necessary inequalities to show that f is an extremal function.

Let $\{w_{j\nu}\}_{j=1}^n$ be a set of optimal foci for f_ν. Let $D_{j\nu}$ be associated with f_ν as D_j is with f. Let $I = \{j \in \{1, 2, \ldots n\}$ such that $x_j \neq 0\}$. Since $\chi[f_\nu] \neq -\infty$, it follows that for each $j \in I$ and $\nu = 1, 2, 3, \ldots, D_{j\nu} \neq \emptyset$, and there is a function $F_{j\nu}$ mapping the unit disk Δ conformally onto that component of $D_{j\nu}$ containing $w_{j\nu}$ so that $F_{j\nu}(0) = w_{j\nu}$. Since the sets $D_{j\nu}$ are bounded, $\{F_{j\nu}\}_{\nu=1}^{\infty}$ is a normal family for each $j \in I$. We can again take a subsequence of $\{f_\nu\}$, which we relabel $\{f_\nu\}$ again, so that for each $j \in I$ there is a function F_j such that $F_{j\nu}(z) \to F_j(z)$ uniformly on compact subsets of Δ as $\nu \to \infty$. Let $h_{j\nu}$ be the regular part of the Green's function of $D_{j\nu}$. Using the first part of (1.2) and the extremality of $\{f_\nu\}$ we obtain

$$\log|F_j'(0)| = \lim_{\nu \to \infty} \log|F_{j\nu}'(0)| = \lim_{\nu \to \infty} h_{j\nu}(w_{j\nu}) > -\infty, \qquad j \in I. \tag{1.4}$$

Consequently F_j is not a constant function, and $F_j(\Delta)$ is an open set.

Because $f_\nu \to f$ uniformly on compact subsets of $\mathscr{D}$,

$$\emptyset \neq F_j(\Delta) \subset D_j \quad j \in I. \tag{1.5}$$

For each $j \in I$ let

$$w_j = \lim_{\nu \to \infty} w_{j\nu}, \tag{1.6}$$

and let $\hat{h}_j(w_j)$ be the regular part of the Green's function for $F_j(\Delta)$ at w_j. From (1.4), (1.5), and the maximum principle we obtain

$$\lim_{\nu\to\infty} h_{j\nu}(w_{j\nu}) = \log|F_j'(0)| = \hat{h}_j(w_j) \le h_j(w_j). \tag{1.7}$$

We can use (1.6), (1.7), and the uniform convergence of f_ν to f to prove the final relations

$$\begin{aligned}\sup_\Sigma \chi = \lim_{\nu\to\infty} \chi[f_\nu] &\le \sum_{\substack{j,k=1\\ j\ne k}}^{m-1} x_j x_k \log|w_j - w_k| + \sum_{j=1}^{n} x_j^2 h_j(w_j)\\ &+ \sum_{j=n+1}^{m-1} x_j^2 \log|f'(z_j)| - x_m^2 \log|f'(\infty)|\\ &\le \chi[f] \le \sup_\Sigma \chi.\end{aligned} \tag{1.8}$$

Now that we have proved that an extremal function exists, we shall characterize all such functions by variational methods.

Proof Of Theorem 2. Let $f \in \Sigma$ be an extremal function for χ with optimal foci $\{w_j\}_{j=1}^n$. Let Δ_j be the component of D_j containing w_j for $j \in I$. Let θ be a real number, $\ell \in I$, and $w_o \in \Delta_\ell - \{w_\ell\}$. Let $\rho > 0$ be small enough so that $T(w) = w + (\rho^2 e^{i\theta}/(w - w_o))$ in schlicht in D. Then $f^* = T\circ f$ is in Σ. Let g_j^* be the Green's function for the domain bounded by $T(\partial D_j)$, and let $w^* = T(w)$. For $j \in I - \{\ell\}$ we have the statement of conformal invariance

$$g_j^*(w^*, w_j^*) = g_j(w, w_j). \tag{1.9}$$

For g_ℓ^* we have the asymptotic expansion of Schiffer [15, pp. 299–300].

$$g_\ell^*(w^*, w_\ell^*) - g_\ell(w, w_\ell) = \rho^2 \operatorname{Re}\{e^{i\theta} G_\ell'(w, w_o) G_\ell'(w_\ell, w_o)\} + O(\rho^3) \tag{1.10}$$

where $G_\ell(w, \zeta)$ is analytic in w and $g_\ell(w, \zeta)$ is its real part.

After a long but straightforward calculation using (1.9) and (1.10) we find that

$$\begin{aligned}0 \ge \chi[f^*] - \chi[f] \ge \rho^2 \operatorname{Re}\Bigg\{ e^{i\theta} \Bigg[&(x_\ell G_\ell'(w_o, w_\ell))^2\\ &- \left(\sum_{j=1}^{m-1} \frac{x_j}{w_o - w_j}\right)^2 \Bigg]\Bigg\} + O(\rho^3).\end{aligned} \tag{1.11}$$

The left-hand inequality is due to the extremal property of f. After dividing by ρ^2 and letting ρ tend to 0, we obtain

$$0 \ge \operatorname{Re}\left\{ e^{i\theta}\left[(x_\ell G_\ell'(w_o, w_\ell))^2 - \left(\sum_{j=1}^{m-1} \frac{x_j}{w_o - w_j}\right)^2\right]\right\}.$$

Since this is true for any real number θ,

$$0 = (x_\ell (G'_\ell (w_o, w_\ell))^2 - \left(\sum_{j=1}^{m-1} \frac{x_j}{w_o - w_j} \right)^2 .$$

Since w_o can be any point in $\Delta_\ell - \{w_\ell\}$ we will replace it with w and find

$$x_\ell G'_\ell (w, w_\ell) = - \sum_{j=1}^{m-1} \frac{x_j}{w - w_j} \text{ for } w \in \Delta_\ell - \{w_\ell\}. \tag{1.12}$$

The sign of the right-hand side was chosen so that the singularities at w_ℓ match.

Integrating and taking real parts, we obtain

$$x_\ell g_\ell (w, w_\ell) = - \left(\sum_{j=1}^{m-1} x_j \log |w - w_j| \right) - c_\ell , \tag{1.13}$$

where c_ℓ is some real constant. On the boundary of Δ_ℓ, $g_\ell (w, w_\ell) = 0$ so

$$- \sum_{j=1}^{m-1} x_j \log |w - w_j| = c_\ell \quad \text{for } w \in \partial\Delta_\ell . \tag{1.14}$$

The complement of D may contain more than $\bigcup_{j \in I} \bar{\Delta}_j$. If so, let E be a proper connected continuum contained in $\mathbf{C} - \bigcup_{j \in I} \bar{\Delta}_j \cup D$.

Schiffer has shown [14, 15] that for any $w_o \in E$ there exist functions of the form

$$T(w) = w + (A\rho^2 / (w - w_o)) + O(\rho^3) \tag{1.15}$$

which are analytic and univalent in the complement of E and where $O(\rho^3)$ can be estimated uniformly for $|w - w_o| > \rho$. Let $f^* = T \circ f$. After a calculation similar to (1.11) we obtain

$$0 \geq \chi [f^*] - \chi [f] \geq \rho^2 \operatorname{Re} \left\{ -A \left(\sum_{j=1}^{m-1} \frac{x_j}{w_o - w_j} \right)^2 \right\} + O(\rho^3). \tag{1.16}$$

Schiffer has also shown [14] that if (1.16) is true for all $T(w)$ of the form (1.15), and if E contains no points where

$$s(w) = \left(\sum_{j=1}^{m-1} (x_j / (w - w_j)) \right)^2 = 0, \tag{1.17}$$

then E is an analytic arc satisfying $s(w)\, dw^2 < 0$. Thus $\operatorname{Re} (\sqrt{s}\, dw) = 0$. χ was defined so $s(w)$ would be a perfect square. Therefore we may integrate simply and find that $\sum_{j=1}^{m-1} x_j \log |w - w_j|$ is constant on E. Consequently, for $k = 1, 2, \dots n$,

$D_k \neq \emptyset$ precisely when $x_k \neq 0$. Also ∂E_k lies on a generalized lemniscate given by $0 = \left(\sum_{j=1}^{m-1} x_j \log |w - w_j| \right) + c_k$ where c_k is some constant.

To characterize the constants c_k, we note that

$$\sum_{j=1}^{m-1} x_j \log |w - w_j| + \sum_{j=1}^{n} c_j \omega_j(w) + \sum_{j=n+1}^{m} x_j g(w, w_j) = 0 \tag{1.18}$$

identically for $w \in \bar{D}$ since the left-hand side is a regular harmonic function in D, which is zero everywhere on the boundary. If we consider (1.18) on a smooth cycle γ_k homologous to B_k, take the normal derivative, and integrate around γ_k, we obtain

$$-2\pi x_k + \sum_{j=1}^{n} c_j P_{jk} - 2\pi \sum_{j=n+1}^{m} x_j \omega_k(w_j) = 0 \qquad \text{for } k = 1, 2, \ldots n. \tag{1.19}$$

Both the period matrix and harmonic measures are conformal invariants so (1.19) is equivalent to (0.1) in Theorem 2. The null space of the matrix $\mathscr{P}$ consists of n-vectors whose components are equal, so the n equations (0.1) determine the c_j's up to an additive constant.

Now we can calculate $\chi[f]$ for the extremal function to prove

THEOREM 3. *With notation as in the beginning of this section, let* $f \in \Sigma$ *let* $w_j \in E_j$ *for* $j = 1, 2, \ldots n$, *and define*

$$\mathscr{R} = (\mathscr{R}_{jk})_{j,k=1}^{n-1} \text{ such that } \mathscr{R}^{-1} = \frac{1}{2\pi} (\mathscr{P}_{jk})_{j,k=1}^{n-1}; \tag{1.20}$$

then

$$\begin{aligned} &\sum_{\substack{j,k=1 \\ j \neq k}}^{m-1} x_j x_k \log |w_j - w_k| + \sum_{j=1}^{n} x_j^2 h_j(w_j) \\ &\quad + \sum_{j=n+1}^{m-1} x_j^2 \log |f'(z_j)| - x_m^2 \log |f'(\infty)| \\ &\quad \leq - \sum_{j,k=1}^{n-1} \mathscr{R}_{jk} \left(x_j + \sum_{\ell=n+1}^{m} x_\ell \Omega_j(z_\ell) \right) \left(x_k + \sum_{\ell=n+1}^{m} x_\ell \Omega_k(z_\ell) \right) \\ &\quad - \sum_{\substack{j,k=n+1 \\ j \neq k}}^{m} x_j x_k \mathscr{g}(z_j, z_k) - \sum_{j=n+1}^{m} x_j^2 \mathscr{h}(z_j). \end{aligned} \tag{1.21}$$

There is equality in (1.21) *if and only if* B_k *lies on a generalized lemniscate given by* $\sum_{j=1}^{m-1} x_j \log |w - w_j| + c_k = 0$ *for each* $k = 1, 2, \ldots n$.

Grunsky [2] proved an equivalent result for domains with smooth boundaries using a potential theoretic inequality. He extended his result to an arbitrary domain by exhausting the domain with a sequence of smooth domains. In our proof of Theorem 3 we have avoided such approximations by smooth domains by consideration of the functional χ.

Proof of Theorem 3. After using (1.13) to find $h_j(w_j)$ for the extremal function, we can use (1.18) and the transformation properties of g, ω_j, and h under conformal mapping to show that

$$\sup_{\Sigma} \chi = -\sum_{j=1}^{n} c_j\left(x_j + \sum_{k=n+1}^{m} x_k \Omega_j(z_k)\right) - \sum_{\substack{j,k=n+1 \\ j\neq k}}^{m} x_j x_k \mathscr{g}(z_j, z_k) - \sum_{j=n+1}^{m} x_j^2 \mathscr{h}(z_j). \tag{1.22}$$

To restate $\sup_{\Sigma} \chi$ in such a way that the constants c_j are eliminated, we can use the fact that $\sum_{j=1}^{n} \mathscr{P}_{jk} = 0$ to transform the linear system (0.1) into

$$c_j - c_n = \sum_{k=1}^{n-1} \mathscr{R}_{jk}\left(x_k + \sum_{\ell=n+1}^{m} x_\ell \Omega_k(z_\ell)\right) \text{ for } j = 1, 2, \ldots, n-1. \tag{1.23}$$

Thus $\sup_{\Sigma} \chi$ equals the right hand side of (1.21), an expression purely in terms of the parameters x_j and classical quantities associated with the domain $\mathscr{D}$. Because we have calculated the maximum value for χ, the inequality (1.21) holds for all functions in Σ.

If we let $f(z) = z$ and $\mathscr{D} = D$, we can obtain a corollary about the relationship between domain dependent quantities for a given n-tuply connected domain D with no mention of conformal mapping. To help eliminate reference to $\mathscr{D}$ we define the matrix

$$R = (R_{jk})_{j,k=1}^{n-1} \text{ such that } R^{-1} = \frac{1}{2\pi}(P_{jk})_{j,k=1}^{n-1} \tag{1.24}$$

COROLLARY 1. *Let D be an n-tuply connected domain containing ∞, whose complement consists of n proper continua $E_1, E_2, \ldots, E_n$. Let $w_j \in E_j$ for each $j = 1, 2, \ldots, n$, and let $w_{n+1}, w_{n+2}, \ldots, w_m = \infty$ be distinct points in D. Then with notation as in the beginning of this section,*

(1.25)

$$\sum_{\substack{j,k=1 \\ j\neq k}}^{m-1} x_j x_k \log|w_j - w_k| + \sum_{\substack{j,k=n+1 \\ j\neq k}}^{m} x_j x_k g(w_j, w_k) + \sum_{j=1}^{n} x_j^2 h_j(w_j)$$

$$+ \sum_{j=n+1}^{m} x_j^2 h(w_j) + \sum_{j,k=1}^{n-1} R_{jk}\left(x_j + \sum_{\ell=n+1}^{m} x_\ell \omega_j(w_\ell)\right)\left(x_k + \sum_{\ell=n+1}^{m} x_\ell \omega_k(w_\ell)\right) \leq 0.$$

There is equality in (1.25) *if and only if* $\sum_{j=1}^{m-1} x_j \log|w - w_j| + c_k = 0$ *on* B_k *for* $k = 1, 2, \ldots n$.

A very special case of Corollary 1 was proved by G. Polya [11] in 1922. To state his result we need two definitions. Let Δ be a bounded simply connected domain in the w-plane such that $\bar{\Delta}$ is also simply connected. Let $w_o \in \Delta$. Let F be a function which maps Δ conformally onto a disk with center $F(w_o)$ such that $F'(w_o) = 1$. Then the radius of the image circle is the *interior radius of* Δ *at* w_o. Let Λ be the complement of $\bar{\Delta}$ in the extended complex plane. Let H be a function which maps Λ conformally onto the exterior of a closed disk such that $H(w) = w + 0(1)$ at ∞. Then the radius of this disk is the *exterior radius* of Δ. It is easy to show that if $D = \Lambda$ and $D_1 = \Delta$, then in our standard notation the interior radius of Δ at the point w_1 is $e^{h_1(w_1)}$, and the exterior radius of Δ is $e^{-h(\infty)}$.

COROLLARY 2. (G. Polya) *If* Δ *is a bounded simply connected domain in the complex plane such that* $\bar{\Delta}$ *is also simply connected, then the interior radius of* Δ *at any point* w_1 *in* Δ *is less than or equal to the exterior radius of* Δ, *with equality only if* Δ *is a disk with center* w_1.

Proof. Let $D = \Lambda$, $n = 1$, $m = 2$, $x_1 = 1$, $x_2 = -1$, and apply Corollary 1.

We consider one other way to specialize Corollary 1. Let.

$$0 = x_{n+1} = x_{n+2} = \ldots = x_m,$$

and we have

COROLLARY 3. *If* $\sum_{j=1}^{n} x_j = 0$, *then*

$$\sum_{\substack{j,k=1 \\ j \neq k}}^{n} x_j x_k \log|w_j - w_k| + \sum_{j=1}^{n} x_j^2 h_j(w_j) + \sum_{j,k=1}^{n-1} x_j x_k R_{jk} \leq 0. \tag{1.26}$$

Since the parameters $x_1, x_2, \ldots, x_{n-1}$ may be chosen arbitrarily while we set $x_n = -\sum_{j=1}^{n-1} x_j$, (1.26) provides a fairly simple bound on the quadratic form associated with R. If the disks $\{|w - w_j| < r_j\} \subset D_j$ for each j, we can obtain one even simpler bound using (1.26) and the maximum principle.

$$\sum_{j,k=1}^{n-1} x_j x_k R_{jk} \leq -\sum_{j,k=1}^{n} x_j x_k \log|w_j - w_k| - \sum_{j=1}^{n} x_j^2 \log r_j.$$

Another consequence of (1.26) is the domain monotonicity of the quadratic form associated with R:

COROLLARY 4. *Let* $\tilde{\mathscr{D}}$ *be an n-tuply connected domain containing* ∞, *whose complement consists of components* $\tilde{\mathscr{E}}_j$ *where* $\mathscr{E}_j \subset \tilde{\mathscr{E}}_j$ *for each* $j = 1, 2, \ldots, n$. *Let* $\tilde{\mathscr{R}}$ *be the matrix for* $\tilde{\mathscr{D}}$ *corresponding to* $\mathscr{R}$. *Then the matrix* $\mathscr{R} - \tilde{\mathscr{R}}$ *is positive* *semidefinite, and it is positive definite if* $\tilde{\mathscr{D}}_j \cap \mathscr{D}$ *is nonempty for* $j = 1,2,\ldots,n,$ *where* $\tilde{\mathscr{D}}_j$ *is the interior of* $\tilde{\mathscr{E}}_j$.

Proof. Let $x_1, x_2, \ldots, x_n$ be real numbers, not all zero, whose sum is zero. Let $x_{n+1} = x_{n+2} = \ldots = x_m = 0$. Let f be an extremal function for χ over Σ with this choice of parameters. Let $\tilde{D}$ be the image of $\tilde{\mathscr{D}}$ under f, and let $\tilde{D}_j$ and $\tilde{h}_j$ correspond to D_j and h_j. Let f have optimal foci $\{w_j\}_{j=1}^{n}$. Since $D_j \subset \tilde{D}_j$, we have by the maximum principle

$$h_j(w_j) \leq \tilde{h}_j(w_j), \quad j = 1,2,\ldots,n, \tag{1.27}$$

and thus

$$\begin{aligned} \sum_{j,k=1}^{n-1} x_j x_k \mathscr{R}_{jk} &= \sum_{\substack{j,k=1 \\ j \neq k}}^{n} x_j x_k \log |w_j - w_k| + \sum_{j=1}^{n} x_j^2 h_j(w_j) \\ &\leq \sum_{\substack{j,k=1 \\ j \neq k}}^{n} x_j x_k \log |w_j - w_k| + \sum_{j=1}^{n} x_j^2 \tilde{h}_j(w_j) \\ &\leq \sum_{j,k=1}^{n-1} x_j x_k \tilde{\mathscr{R}}_{jk}, \end{aligned} \tag{1.28}$$

so

$$0 \leq \sum_{j,k=1}^{n-1} x_j x_k (\mathscr{R}_{jk} - \tilde{\mathscr{R}}_{jk}). \tag{1.29}$$

If $x_j \neq 0$ and $\tilde{\mathscr{D}}_j \cap \mathscr{D}$ is nonempty, then we consider two cases. Either the component of $\tilde{\mathscr{D}}_j$ containing w_j intersects $\mathscr{D}$, or another component does. In the first case the inequality (1.27) and the first inequality in (1.28) are strict. In the second case $\tilde{D}$ cannot be an extremal domain so that second inequality in (1.28) is strict. In either case (1.29) becomes a strict inequality. Therefore, if $\tilde{\mathscr{D}}_j \cap \mathscr{D}$ is nonempty for each j, $\mathscr{R} - \tilde{\mathscr{R}}$ is positive definite.

2. THE EXTREMAL PROBLEM FOR QUASICONFORMAL MAPPINGS

In general the functional χ does not make sense for f in the class Σ_K defined in the introduction because of the terms $f'(z_j)$. With certain restrictions on $K(z)$ Schiffer and Schober [18] have defined a functional Φ on Σ_K which has the same functional derivative as $f'(z_j)$. We will use this functional to generalize χ. Let

$$\lambda(r, z_o, K) = \begin{cases} \operatorname*{ess\,sup}_{|z - z_o| \leq r} |K(z) - K(z_o)| & \text{if } z_o \neq \infty \\ \\ \operatorname*{ess\,sup}_{|z| \geq 1/r} |K(r) - K(z_o)| & \text{if } z_o = \infty, \end{cases} \tag{2.1}$$

and let $A(r, z_o, f)$ be the area of $f(\{|z - z_o| \le r\})$. Schiffer and Schober have shown [18, lemma 1] that if K satisfies

$$(2.2) \qquad \int_0^\delta (\lambda(r, z_o, K)/r)\, dr < \infty \quad \text{for some } \delta > 0, \qquad \text{then}$$

$$(2.3) \qquad \Phi(z_o; K, f) = \begin{cases} \lim\limits_{r \to 0} \dfrac{\sqrt{A(r, z_o, f)/\pi}}{r^{1/K(z_o)}} & \text{for } z_o \ne \infty \\[2ex] \Phi\left(0; K(1/z), \dfrac{1}{f(1/z)}\right) & \text{for } z_o = \infty \end{cases}$$

is well defined and finite for all $f \in \Sigma_K$.

It is easy to show that if $\lim\limits_{z \to z_o} \dfrac{|f(z) - f(z_o)|}{|z - z_o|^{1/K(z_o)}}$ exists, then it equals $\Phi(z_o; K, f)$. In particular, if $K \equiv 1$, then $\Phi(z_o; K, f) = |f'(z_o)|$ for $f \in \Sigma_K = \Sigma$.

Henceforth we will assume K does satisfy (2.2) at $z_{n+1}, z_{n+2}, \dots, z_m$. Then we can use Φ to generalize χ to be defined on Σ_K.

$$(2.4) \qquad \chi[f] = \sup_{\substack{w_j \in E_j \\ j = 1, 2, \dots n}} \left[\sum_{\substack{j,k=1 \\ j \ne k}}^{m-1} x_j x_k \log |w_j - w_k| + \sum_{j=1}^{n} x_j^2 h_j(w_j) + \sum_{j=n+1}^{m} x_j^2 \log \Phi(z_j; K, f) \right].$$

Note that the generalized functional χ depends explicitly on the parameters $K(z_{n+1}), K(z_{n+2}), \dots, K(z_m)$.

We can extend the definition of a set of optimal foci of f to $f \in \Sigma_K$. The proof of Lemma 1 still applies, so any function in Σ_K has a set of optimal foci.

LEMMA 3. *There exists an f in Σ_K such that* $\chi[f] = \sup\limits_{\Sigma} \chi > -\infty$.

Proof. Schiffer and Schober [18, lemma 3] proved that there is a function f_o in Σ_K such that $\Phi(z_j; K, f_o) > 0$ for each $j = n + 1, n + 2, \dots, m$. This function can be composed with a function F univalent and analytic on $f_o(\mathscr{D})$ such that $F \circ f_o$ is in Σ_K and each term $h_j(w_j)$ in $\chi[F \circ f_o]$ can be made finite. It follows from the definition of Φ that

$$(2.5) \qquad \Phi(z_j; K, F \circ f_o) = \Phi(z_j; K, f_o) |F'(f_o(z_j))| > 0$$

and thus $\chi[F \circ f_o] > -\infty$.

Except in two places the remainder of the proof is just like the proof of Lemma 2 if we replace Σ by Σ_K and $|f'(z_j)|$ by $\Phi(z_j; K, f)$. Once again there is a normalized extremal sequence $\{f_y\}$ which converges to a function f uniformly on compact subsets of $\mathscr{D}$. That this function is in Σ_K follows from a compactness result of Strebel [20]. In the proof of Lemma 2 we used the fact that if $f_n \to f$ uniformly on compact sets, and f_n, $f \in \Sigma$, then $f_n'(z_j) \to f'(z_j)$. For the proof of Lemma 3 we substitute a result of Schiffer and Schober [18, lemma 2],

$$\overline{\lim_{n\to\infty}}\, \Phi(z_j; K, f_n) \le \Phi(z_j; K, f) \tag{2.6}$$

if $f_n, f \in \Sigma_K$ and $f_n \to f$ uniformly on compact subsets of $\mathscr{D}$.

THEOREM 4. *Let $\mathscr{D}$ be an n-tuply connected domain in the extended z-plane, which contains ∞, and whose complement consists of proper continua. Let $z_{n+1}, z_{n+2}, \ldots, z_m = \infty$ be distinct points in $\mathscr{D}$. Let $K\colon \mathscr{D} \to [1,\infty)$ be an essentially bounded, measurable function satisfying* (2.2) *at $z_{n+1}, z_{n+2}, \ldots, z_m$. For each real valued m-vector $\vec{x}$ whose coordinates $x_1, x_2, \ldots, x_m$ sum to zero there is a unique function $U(z;\vec{x})$ defined on $\bar{\mathscr{D}}$ satisfying conditions* i)–iv). *We abbreviate $U(z;\vec{x})$ by $U(z)$.*

i) $U(z) = -\sum_{j=1}^{m-1} x_j \log|f(z) - w_j| + C$ *in $\mathscr{D}$, where $f \in \Sigma_K$; w_j is in the jth component of the complement of the image of f for $j = 1, 2, \ldots, n$,* and $w_j = f(z_j)$ for $j = n+1, n+2, \ldots, m$.

ii) $U(z)$ *and* $V(z) = V(z;\vec{x}) = \operatorname{Im}\left(-\sum_{j=1}^{m-1} x_j \log(f(z) - w_j)\right)$ *satisfy the generalized Cauchy-Riemann equations*

$$KU_x = V_y, \quad KU_y = -V_x \text{ a.e. in } \mathscr{D} \tag{2.7}$$

iii)

$$\lim_{z\to z_j}\left(U(z) + \frac{x_j}{K(z_j)}\log|z - z_j|\right) \text{ exists}, j = n+1, n+2, \ldots m-1;$$

(2.8)

$$\lim_{z\to\infty}\left(U(z) - \frac{x_m}{K(\infty)}\log|z|\right) \text{ exists.}$$

iv) *$U(z)$ is continuous in $\bar{\mathscr{D}} - \{z_{n+1}, z_{n+2}, \ldots, z_m\}$; $U(z) = c_j$ on the jth component of the boundary of $\mathscr{D}$ for $j = 1, 2, \ldots, n$, and $c_n = 0$.*

Remarks. To discuss this theorem we need some definitions. A real or complex valued function F defined on a domain Δ has L^2 *derivatives* in Δ if F is locally absolutely continuous on almost every horizontal and vertical line in Δ, and if the partial derivatives of F are locally in L^2.

A real valued function U defined on Δ is a *weak solution* of $\operatorname{div}(K \operatorname{grad} U) = 0$ in Δ if U has L^2 derivatives in Δ and if for any C^2 test function t with compact support in Δ

$$\int_{\Delta} [t_x(KU_x) + t_y(KU_y)]\, d\mu = 0.$$

We claim that the function U in the theorem is a weak solution to

$$\operatorname{div}(K \operatorname{grad} U) = 0$$

in $\mathscr{D}' = \mathscr{D} - \{z_{n+1}, z_{n+2}, \ldots, z_m\}$. It is proved in [8] that any $\|K\|_\infty$-quasiconformal mapping function f has L^2 derivatives. The same holds true for U since it is a harmonic function composed with a $\|K\|_\infty$-quasiconformal mapping. Let t be a C^2 test function with compact support in $\mathscr{D}'$. Then

$$\begin{aligned}\int_{\mathscr{D}'} (t_x(KU_x) + t_y(KU_y))\, d\mu &= \int_{\mathscr{D}'} (t_x V_y - t_y V_x)\, d\mu \\ &= -\int_{\mathscr{D}'} (t_{xy} - t_{yx})\, V d\mu = 0,\end{aligned}$$

so U does satisfy the definition.

If we view $\mathscr{D}$ as an inhomogeneous dielectric medium with dielectric coefficient $K(z)$, $\mathscr{B}_j$ as a conductor with charge x_j for $j = 1, 2, \ldots, n$, and z_j as the location of a point charge x_j for $j = n+1, n+2, \ldots, m-1$, then $U(z)$ is the electrostatic potential in $\mathscr{D}$. In the special case where the only point charge in D is a unit charge at z_{n+1}, and $x_1, x_2, \ldots, x_n$ are chosen so that all constants $c_j = 0$, $U(z)$ is the dielectric Green's function of $\mathscr{D}$ with respect to $K(z)$ with pole at z_{n+1}. The theorem states that this dielectric Green's function is given by a quasiconformal mapping function composed with the real part of a linear combination of logarithmic terms plus a constant. In the proof of Theorem 4 we shall see that this quasiconformal mapping function is an extremal function for the functional χ.

First we prove

LEMMA 4. *Let Δ be a domain in the extended complex plane with boundary components $\Gamma_1, \Gamma_2, \ldots \Gamma_k$. Let U and V be real, single-valued and continuous functions in Δ which have L^2 derivatives and which satisfy the generalized Cauchy-Riemann equations*

$$KU_x = V_y, \qquad KU_y = -V_x \text{ a.e. in } \Delta$$

Further let $\lim_{z \to \Gamma_j} U(z) = c_j$ for $j = 1, 2, \ldots, k$. Then U is constant in Δ.

Proof. There exist sequences of C^∞ functions $\{S_\nu(z)\}$ and $\{T_\nu(z)\}$ such that $S_\nu(z) \to U(z)$ uniformly on $\bar{\Delta}$, and such that given any compact set E in Δ, $T_\nu(z) \to V(z)$ uniformly on E, and the partial derivatives of S_ν and T_ν converge respectively to the corresponding partial derivatives of U and V in $L^2(E)$.

For any $\delta > 0$ let $\Delta_\delta = \{z \in \Delta \colon |U(z) - c_j| > \delta \text{ for } j = 1, 2, \ldots, k\}$. Assume U is not constant; then for some $\delta > 0$, Δ_δ is nonempty. We will show that U is constant on Δ_δ, which contradicts the definition of Δ_δ. Consequently U is constant on Δ.

By Sard's theorem [13] the measure of the set

$$\bigcup_{\nu} S_\nu(\{z\colon \operatorname{grad} S_\nu(z) = 0 \text{ or } z = \infty\})$$

is zero. Therefore we can choose a number β, $0 < \beta < \delta$, such that this set is disjoint from the set $\{r\colon |r - c_j| = \beta \text{ for some } j = 1, 2, \ldots, k\}$. Let

$$\Delta_{\beta\nu} = \{z \in \Delta\colon |S_\nu(z) - c_j| > \beta \text{ for all } j = 1, 2, \ldots, k\}.$$

For large enough ν, $\Delta_\delta \subset \Delta_{\beta\nu}$ and $\bar{\Delta}_{\beta\nu} \subset \Delta$, and then

(2.9)

$$\begin{aligned}\int_{\Delta_\delta} K|\operatorname{grad} U|^2 d\mu \le \int_{\Delta_{\beta\nu}} K|\operatorname{grad} U|^2 d\mu &= \int_{\Delta_{\beta\nu}} (U_x V_y - U_y V_x)\, d\mu \\ &= \int_{\Delta_{\beta\nu}} \left(\frac{\partial S_\nu}{\partial x}\frac{\partial T_\nu}{\partial y} - \frac{\partial S_\nu}{\partial y}\frac{\partial T_\nu}{\partial x}\right) d\mu + \varepsilon(\nu),\end{aligned}$$

where $\varepsilon(\nu) \to 0$ as $\nu \to \infty$. For our choice of β, $\partial\Delta_{\beta\nu}$ is rectifiable for large enough ν, and we can use Green's identity on the last integral. Thus

$$\int_{\Delta_\delta} K|\operatorname{grad} U|^2 d\mu \le -\int_{\partial\Delta_{\beta\nu}} T_\nu\, dS_\nu + \varepsilon(\nu) = 0 + \varepsilon(\nu).$$

Consequently $U(z)$ is constant in Δ_δ, which is all we needed to show.

Proof of Theorem 4. If $\vec{x} = \vec{0}$, conditions i)–iv) can be trivially satisfied. We now consider the case $\vec{x} \ne \vec{0}$. Let $f \in \Sigma_K$ such that $\chi[f] = \sup_{\Sigma_K} \chi$. Let $\{w_j\}_{j=1}^n$ be a set of optimal foci for f.

Following the development in [18], [19, Chapter 13], we consider a variation of the form

$$T(w) = w - \frac{\varepsilon}{\pi}\int_C \frac{a(\zeta)}{\zeta - w}\, d\mu + O(\varepsilon^2) \tag{2.10}$$

where a has compact support in $\mathscr{D} - \{z_{n+1}, z_{n+2}, \ldots, z_m\}$, and the error term is such that T is analytic outside of the support of a. Let $f^* = T \circ f$. The varied function also maps ∞ to ∞, and for appropriate $a(\zeta)$, [19], $f^* \in \Sigma_K$.

As in the proof of Theorem 2, we want to calculate an asymptotic expansion for $\chi[f^*] - \chi[f]$. T maps D_j conformally onto $T(D_j)$ so equation (1.9) for the invariance of the Green's function applies. We also need to calculate how Φ changes:

$$\begin{aligned}\Phi(z_j; K, f^*) &= \Phi(z_j; K, f) \cdot |T'(w_j)| \\ &= \Phi(z_j; K, f)\left(1 - \frac{\varepsilon}{\pi}\operatorname{Re}\int \frac{a(\zeta)}{(\zeta - w_j)^2}\, d\mu_\zeta\right) + O(\varepsilon^2).\end{aligned} \tag{2.11}$$

In a calculation like the one for (1.11) we find

$$(2.12)\qquad 0 \geq \chi[f^*] - \chi[f] \geq \frac{\varepsilon}{\pi}\operatorname{Re}\left\{\int_{\mathrm{C}} -\left(\sum_{j=1}^{m-1}\frac{x_j}{\zeta - w_j}\right)^2 a(\zeta)\,d\mu_\zeta\right\} + O(\varepsilon^2).$$

The principal theorem for the variational method [19, Theorem 13.2] states that if f satisfies (2.12) for all variations of the form (2.10), then f satisfies the differential equation

$$(2.13)\qquad f_{\bar z}(z) = \frac{K(z)-1}{K(z)+1}\,\frac{\left|\left(\sum_{j=1}^{m-1}\frac{x_j}{f(z)-w_j}\right)^2\right|}{-\left(\sum_{j=1}^{m-1}\frac{x_j}{f(z)-w_j}\right)^2}\,\overline{f_z(z)} \quad \text{a.e. in } \mathscr{D},$$

i.e.,

$$\left(\sum_{j=1}^{m-1}\frac{x_j}{f(z)-w_j}\right) f_{\bar z} = -\frac{K-1}{K+1}\,\overline{\left(\sum_{j=1}^{m-1}\frac{x_j}{f(z)-w_j}\right) f_z} \quad \text{a.e. in} \mathscr{D}.$$

Thus if we let $U + iV = -\sum_{j=1}^{m-1} x_j \log(f(z) - w_j)$, then

$$(2.14)\qquad (U+iV)_{\bar z} = -\frac{K-1}{K+1}\,\overline{(U+iV)_z}, \quad \text{a.e. in } \mathscr{D}.$$

Separating real and imaginary parts, we find that U and V satisfy the generalized Cauchy-Riemann equations (2.7).

Next we characterize the singularities of U. Schiffer and Schober have shown in [18, Theorem 1] that if K satisfies (2.2) at a finite point z_j, and if f is a $K(z)$-q.c. function defined on $\Delta = \{|z - z_j| < \delta\}$ such that the real and imaginary parts of $-\log(f(z) - f(z_j))$ satisfy the generalized Cauchy-Riemann equations (2.7) a.e. in Δ, then $-\log|f(z) - f(z_j)| + \frac{1}{K(z_j)}\log|z - z_j|$ has a limit as $z \to z_j$. Similar calculations show that (2.8) is satisfied.

The fourth condition in the theorem can be restated to say that the boundary component B_j lies on the generalized lemniscate given by

$$-\sum_{k=1}^{m-1} x_k \log|w - w_k| = c_j - C.$$

The proof of this fact is almost exactly as it was in Theorem 2, so we will not repeat it.

To show that U is unique, assume $\hat{U}$ also satisfies conditions i)-iv). Let $\hat{V}$ correspond to V. The period of V around $\mathscr{B}_j$ or z_j is the period of $-\sum x_j \log(w - w_j)$ around B_j or w_j which is $2\pi x_j$. $\hat{V}$ has the same periods so $V - \hat{V}$ is well defined in $\Delta = \mathscr{D} - \{z_{n+1}, z_{n+2}, \ldots, z_m\}$. Corresponding singularities of U and $\hat{U}$ have the same form so $U - \hat{U}$ is continuous in $\bar{\mathscr{D}}$. We can apply Lemma 4 to $U - \hat{U}$ and $V - \hat{V}$, and thus $U - \hat{U}$ is constant. Since U and $\hat{U}$ equal 0 on $\mathscr{B}_n$, $U \equiv \hat{U}$.

In the proof of Theorem 4 we have derived some properties of f, an extremal function for the functional χ over the class Σ_K. In general these properties are not enough to calculate $\chi[f] = \sup_{\Sigma_K} \chi$ because we cannot find a useful representation of the solution to the generalized Cauchy-Riemann equations (2.7). If we make further restrictions on $K(z)$, we can derive a useful expression for $U(z)$ and calculate $\sup_{\Sigma_K} \chi$.

To describe the restrictions we will make on K, we return to our electrostatic model. $U(z)$ is the potential in $\mathscr{D}$ if $\mathscr{D}$ is a dielectric medium with dielectric coefficient $K(z)$, $\mathscr{B}_j$ is a conductor containing a charge x_j for $j = 1, 2, \ldots n$, and isolated point charges x_j are located at z_j for $j = n+1, \ldots, m-1$. Let $\mathscr{p}(z)$ be the potential with the same charges and conductors but no dielectric. To make $U(z)$ easy to calculate, we now introduce the restriction that $K(z)$ is constant on each component of each level line of $\mathscr{p}$ in $\mathscr{D}$.

THEOREM 5. *With notation as in the beginning of section* 2, *let*

$$\mathscr{p}(z) = \sum_{j=1}^{n} c_j \Omega_j(z) + \sum_{j=n+1}^{m} x_j \mathscr{g}(z, z_j) \tag{2.15}$$

with the constants c_j chosen so that $c_n = 0$ and

$$-\frac{1}{2\pi} \int_{\Gamma_j} \frac{\partial \mathscr{p}}{\partial n} ds = x_j \quad \text{for} \quad j = 1, 2, \ldots, n, \tag{2.16}$$

where Γ_j is a smooth curve in $\mathscr{D}$ homologous to $\mathscr{B}_j$. Let K: $\mathscr{D} \to [1,\infty)$ be a measurable function such that $\|K\|_\infty < \infty$; K satisfies (2.2) at $z_{n+1}, z_{n+2}, \ldots, z_m$, and K is constant on each component of each level line of $\mathscr{p}$. Let $f \in \Sigma_K$. Let $w_j \in E_j$ for each $j = 1, 2, \ldots, n$. Then

(2.17)

$$\sum_{\substack{j,k=1 \\ j \neq k}}^{m-1} x_j x_k \log|w_j - w_k| + \sum_{j=1}^{n} x_j^2 h_j(w_j)$$

$$+ \sum_{j=n+1}^{m} x_j^2 \log \Phi(z_j; K, f)$$

$$\leq -\sum_{j=1}^{n-1} x_j \int_{\mathscr{B}_n}^{\mathscr{B}_j} \frac{d\mathscr{p}(\zeta)}{K(\zeta)} + \sum_{j=n+1}^{m} x_j \left[\frac{-\sigma_j}{K(z_j)} + \int_{\mathscr{B}_n}^{z_j} \left(\frac{1}{K(z_j)} - \frac{1}{K(\zeta)} \right) d\mathscr{p}(\zeta) \right]$$

where

$$\sigma_j = \begin{cases} \lim_{z\to z_j} (\mathscr{p}(z) + x_j \log|z - z_j|)\ if\ j = n+1, \ldots m-1 \\ \\ \lim_{z\to\infty} (\mathscr{p}(z) - x_m \log|z|)\ if\ j = m. \end{cases} \tag{2.18}$$

Proof. Let $\mathscr{q}(z)$ be a harmonic conjugate of $\mathscr{p}$. We will show that the functions U and V of Theorem 3 can be expressed as

$$U(z) = \int_{\mathscr{B}_n}^{z} dp(\zeta)/K(\zeta) \quad \text{and} \quad V(z) = \mathscr{q}(z). \tag{2.19}$$

Except at the isolated stationary points of $\mathscr{p}$, we can use $\mathscr{p}$ and $\mathscr{q}$ as local coordinates in the z-plane and calculate

$$KU_{\mathscr{p}} = 1 = V_{\mathscr{q}} \qquad KU_{\mathscr{q}} = 0 = -V_{\mathscr{p}} \text{ a.e. in } \mathscr{D}$$

Except at isolated stationary points of $\mathscr{p}$ the mapping $\mathscr{p} + i\mathscr{q} \to U + iV$ is $\|K\|_\infty$-quasiconformal. Since $\|K\|_\infty$-quasiconformal mappings are differentiable a.e. [8, p. 161], we can transform back to the usual (x,y) coordinates and obtain the generalized Cauchy-Riemann equations in their usual form (2.7).

To analyze the singularities of U we consider the improper integral

(2.20)

$$\int_{\mathscr{B}_n}^{z_j} \left(\frac{1}{K(z_j)} - \frac{1}{K(\zeta)}\right) d\mathscr{p}(\zeta) = \lim_{z\to z_j}\left(\frac{\mathscr{p}(z)}{K(z_j)} - U(z)\right), \qquad j = n+1, \ldots m$$

We can show that the integral is convergent using the condition on $K(z)$, (2.2), and the fact that $\mathscr{p}$ has only a logarithmic singularity at z_j. The fact that the limit in (2.20) exists implies that U satisfies (2.8). U is clearly constant on each boundary component of $\mathscr{D}$.

The only other condition used to determine $U(z)$ uniquely in Theorem 3 was that the multiple-valued function V has periods x_j. With $V(z)$ defined as the harmonic conjugate of $\mathscr{p}$, (2.15) and (2.16) ensure that this condition is also satisfied.

We can use the representation of U (2.19) to evaluate $\chi[f]$. We can see from the definition of Φ that if $z_j \neq \infty$ and $\lim_{z\to z_j} x_j \log(|f(z) - f(z_j)|/|z - z_j|^{1/K(z_j)})$ exists, it equals $\Phi(z_j; K, f)$. We can show that the limit does exist by regrouping terms:

$$\begin{aligned} &x_j \log \Phi(z_j; K, f) \\ &= \lim_{z\to z_j} x_j \log(|f(z) - f(z_j)|/|z - z_j|^{1/K(z_j)}) \end{aligned} \tag{2.21}$$

$$= \lim_{z \to z_j} \left[\left(\frac{\mu(z)}{K(z_j)} - U(z) \right) + \left(C - \sum_{\substack{k=1 \\ k \neq j}}^{m-1} x_k \log |f(z) - w_k| \right) - \frac{1}{K(z_j)} \left(\mu(z) + x_j \log |z - z_j| \right) \right]$$

$$= \int_{\mathscr{B}_n}^{z_j} \left(\frac{1}{K(z_j)} - \frac{1}{K(\zeta)} \right) d\mu(\zeta) + \left(C - \sum_{\substack{k=1 \\ k \neq j}}^{m-1} x_k \log |w_j - w_k| \right) - \frac{\sigma_j}{K(z_j)}.$$

The calculation of $\Phi(\infty; K, f)$ is similar. We can compute $h_j(w_j)$ from (1.13), as we did in Section 1, and then put the calculations for the different terms of $\chi[f]$ together to obtain the right hand side of (2.17). Since this is $\sup_{\Sigma_K} \chi$, the inequality (2.17) is true for any $f \in \Sigma_K$.

Example. Let
$$L(z) = \begin{cases} K_o & \text{if } |z| \leq \rho \\ 1 & \text{otherwise} \end{cases}$$

Let $\mathscr{D}$ be the domain in the extended complex plane whose complement consists of $\mathscr{E}_1 = \{|z + a| \leq r\}$ and $\mathscr{E}_2 = \{|z - a| \leq r\}$, where $0 < r < a$. Let $f \in \Sigma_L$; $w_1 \in E_1$, and $w_2 \in E_2$. Then

$$(2.22) \qquad 2 \log |w_1 - w_2| + h_1(w_1) + h_2(w_2) - 4 \log |f'(\infty)| \leq \left(2 - \frac{2}{K_o} \right) \log \frac{\rho^2 + a^2}{2ar - r^2} + 2 \log (2ar + r^2).$$

We actually calculate the bound for $f \in \Sigma_K \supset \Sigma_L$, where K satisfies the conditions of Theorem 5. We set $m = 3$, $x_1 = x_2 = 1$, and $x_3 = -2$. Let μ_o be the minimum value of μ on $\{|z| \leq \rho\} \cap \bar{\mathscr{D}}$ and for $z \in \mathscr{D}$ let

$$K(z) = \begin{cases} K_o \text{ if } \mu(z) \geq \mu_o \\ 1 \text{ otherwise}. \end{cases}$$

Applying (2.17) with all of our present assumptions we can calculate

$$(2.23) \qquad 2 \log |w_1 - w_2| + h_1(w_1) + h_2(w_2) - 4 \log |f'(\infty)| \leq -\mu_o(2 - (2/K_o)) + 2\sigma_3.$$

$\mathscr{D}$ is not a convenient domain for which to calculate μ_o and σ_3 exactly, but we may bound quantities such as μ_o and σ_3 by comparing $\mathscr{D}$ with domains bounded by generalized lemniscates. Let $\psi(z) = -\log |z^2 - a^2|$. It is easy to check that the maximum value of ψ on $\partial \mathscr{D}$ occurs at $z = \pm(a - r)$, and the minimum value, at $z = \pm(a + r)$. Thus

$$\mathscr{D}^+ = \{z: \psi(z) < \psi(a-r)\} \supset \mathscr{D} \supset \mathscr{D}^- = \{z: \psi(z) < \psi(a+r)\}.$$

By the maximum principle we have the following relations between the Green's functions for these domains, $\mathscr{g}^+(z,\infty) > \mathscr{g}(z,\infty) > \mathscr{g}^-(z,\infty)$. Thus

$$-2\mathscr{g}^-(z,\infty) > -2\mathscr{g}(z,\infty) > -2\mathscr{g}^+(z,\infty),$$

and equivalently $\psi(z) - \psi(a+r) > p(z) > \psi(z) - \psi(a-r)$. One can check that the minimum value of $\psi(z)$ on $\{|z| \leq \rho\}$ is $\psi(i\rho)$. Thus

$$-p_o \leq -[\psi(i\rho) - \psi(a-r)].$$

Also,

$$\sigma_3 = \lim_{z\to\infty} (p(z) + 2\log|z|) \leq \lim_{z\to\infty} [\psi(z) - \psi(a+r) + 2\log|z|] = -\psi(a+r).$$

Finally we can use the inequality (2.23) and the bounds for $-p_o$ and σ_3 to obtain the explicit bound on distortion in (2.22).

The following corollary to Theorem 5 gives a bound on the change in the domain dependent quantity $\sum_{j,k=1}^{n-1} x_j x_k \mathscr{R}_{jk}$ in terms of the bound K on the distortion of f.

COROLLARY. *Assume the hypotheses of Theorem* 5 *and let* $x_{n+1} = x_{n+2} = \ldots, x_m = 0$, *then*

$$\sum_{j,k=1}^{n-1} (\mathscr{R}_{jk} - R_{jk}) x_j x_k \leq \sum_{j=1}^{n-1} x_j \int_{\mathscr{B}_n}^{\mathscr{B}_j} (1 - (1/K(\zeta))\, dp(\zeta), \tag{2.24}$$

where the matrices $\mathscr{R}$ *and* R *are defined as in* (1.20) *and* (1.24).

Proof. In this special case where χ does not involve Φ, χ is defined exactly like the corresponding functional in Section 1, so we will not distinguish between them. Let $f \in \Sigma_K$. Let $\Sigma(D)$ be the class of functions defined like Σ except with domain D. Let χ_D correspond to χ except be defined for functions in $\Sigma(D)$. Let F maximize χ_D over $\Sigma(D)$. By (1.21) we have

$$\chi_D[F] = -\sum_{j,k=1}^{n-1} x_j x_k R_{jk}. \tag{2.25}$$

$F \circ f \in \Sigma_K$ so

$$\sup_{\Sigma_K} \chi \geq \chi[F \circ f] = \chi_D[F]. \tag{2.26}$$

In the conformal case $K \equiv 1$ the right hand sides of (1.21) and (2.17) are both $\sup_{\Sigma} \chi$. Comparing these to (2.17) with the more general $K(z)$ in light of (2.25) and (2.26), we can obtain the conclusion of the corollary (2.24).

Example. If we let K be constant, we find

$$\sum_{j,k=1}^{n-1} x_j x_k R_{jk} \geq \frac{1}{K} \sum_{j,k=1}^{n-1} x_j x_k \mathscr{R}_{jk}$$

when $f \in \Sigma_K$, and $x_1, x_2, \ldots, x_{n-1}$ are any real numbers.

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