A NOTE ON $L^2(\Gamma\backslash SL(2,\mathbb{R}))$

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1. INTRODUCTION

Let G be the group $SL(2,\mathbb{R})$ (that is, the group of two-by-two real matrices of determinant 1). Let Γ and Γ' be discrete subgroups of G such that $\Gamma\backslash G$ and $\Gamma'\backslash G$ are compact. We shall also assume that both Γ and Γ' do not contain elements of finite order. Let σ and σ' be the right regular representations of G on $L^2(\Gamma\backslash G)$ and $L^2(\Gamma'\backslash G)$ respectively. The purpose of this note is to give a sufficient condition for σ and σ' to be unitarily equivalent, in terms of the eigenvalues of the elements of Γ and Γ' (see Theorem 3.2 in section 3). We do this by introducing a Selberg-type zeta function $\check{Z}_\Gamma(s)$ (see section 3). $\check{Z}_\Gamma(s)$ has the same kind of relationship to the nonspherical principal series of $SL(2,\mathbb{R})$ as the original Selberg zeta function ([7]) has to the spherical principal series. (It can be seen by looking at the original Selberg function that if the eigenvalues (with multiplicities) of Γ and Γ' are the same then the spherical principal series representations of G occuring in $L^2(\Gamma\backslash G)$ and $L^2(\Gamma'\backslash G)$ are the same and they occur with the same multiplicities.

2. NOTATION AND PRELIMINARIES

Let G stand for the group $SL(2,\mathbb{R})$ (that is, the group of two-by-two real matrices of determinant 1). Let Γ denote a discrete subgroup of G such that:

(a) $\Gamma\backslash G$ is compact.

(b) Γ does not contain elements of finite order (except for the identity).

It is known (see [2, p. 11]) that under these assumptions Γ contains only hyperbolic elements. (An element $\gamma \in \Gamma$ is said to be *hyperbolic* if it has distinct, real eigenvalues). An element $\gamma \in \Gamma$ is said to be primitive if it is not a positive power of any other element of Γ. Clearly any conjugate of γ will also be primitive. Let P_α $(\alpha = 1,2,\ldots)$ be a complete set of representatives of the primitive hyperbolic conjugacy classes of Γ. Let μ_α denote the eigenvalue of P_α with the larger absolute value. Let $N\{P_\alpha\} = \mu_\alpha^2$ and let $\lambda_\alpha = \text{sign}$ (eigenvalue of P_α). (Note that both eigenvalues of P_α must have the same sign).

Let

$$K = SO(2) = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};\quad 0 \le \theta \le 2\pi\right\}$$

As is well known, K is a maximal compact subgroup of G and as a symmetric space G/K can be identified with the upper half plane: $H = \{z; z \in \mathbb{C} \text{ and } \operatorname{Im} z > 0\}$.

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Then since Γ is a subgroup of $SL(2, \mathbb{R})$, Γ acts on the upper half plane H and under the assumptions on Γ, $\Gamma\backslash H$ is a compact Riemann surface and Γ is its fundamental group. Let p denote the genus of $\Gamma\backslash H$. Then again under the assumption on Γ, $p > 1$.

Fix a Haar measure dx on G and the counting measure on the discrete subgroup Γ. Then there exists a unique G invariant measure $d\dot{x}$ on $\Gamma\backslash G$ such that

$$\int_G f\,dx = \int_{\Gamma\backslash G}\left(\sum_{\gamma\in\Gamma} f(\gamma x)\right) d\dot{x}$$

Then by $\mathrm{vol}(\Gamma\backslash G)$ we mean the total measure of $\Gamma\backslash G$ with respect to the measure $d\dot{x}$. (Note that the volume of $\Gamma\backslash G$ can be computed in terms of the genus p of $\Gamma\backslash H$). By $L^2(\Gamma\backslash G)$ we mean the set of complex valued measurable functions on $\Gamma\backslash G$ which are square integrable with respect to $d\dot{x}$. Let σ be the unitary representation of G on $L^2(\Gamma\backslash G)$ given by right regular action:

$$(\sigma(x)f)(y) = f(yx) \qquad \text{for all } x \in G, y \in \Gamma\backslash G \text{ and } f \in L^2(\Gamma\backslash G).$$

Let $\hat{G}$ be the set of equivalence classes of unitary irreducible representations of G. Then it is known ([2]) that σ is a direct sum of irreducible unitary representations of G and for any $\delta \in \hat{G}$, the multiplicity of δ in σ is finite. We shall denote this number by $m(\delta)$.

Representations of $SL(2, \mathbb{R})$. Since the notation used by different authors differ widely, we shall collect here the necessary facts concerning the unitary representations of $SL(2, \mathbb{R})$. (See [5] or [11] for details). The Iwasawa decomposition $G = KAN$ is given by:

$$K = SO(2) = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};\ 0 \le \theta \le 2\pi\right\}$$

$$A = \left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix};\ \lambda > 0\right\}$$

$$N = \left\{\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix};\ \xi \in \mathbb{R}\right\}$$

Let

$$M = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \subseteq K.$$

For each $\lambda \in \mathbb{C}$ define the following homomorphism ϕ_λ of A into $\mathbb{C}\backslash\{0\}$:

$$\phi_\lambda(a) = \xi^{2i\lambda+1} \qquad \text{where } a = \begin{pmatrix} \xi & 0 \\ 0 & 1/\xi \end{pmatrix} \in A$$

Now let ϕ_λ^+, ϕ_λ^- be representations of MAN defined by:

$\phi_\lambda^+|_M$ is trivial, $\phi_\lambda^+|_A = \phi_\lambda$, $\phi_\lambda^+|_N$ is trivial,

$\phi_\lambda^-|_M$ is the unique non trivial character of M,

$\phi_\lambda^-|_A = \phi_\lambda$, $\phi_\lambda^-|_N$ is trivial.

Let π_λ^+ (respectively π_λ^-) be the representations of G induced by the representations ϕ_λ^+ (respectively ϕ_λ^-) of MAN. The representations $\{\pi_\lambda^+\}$ are called the spherical principal series of G and the representations $\{\pi_\lambda^-\}$ are called the *non*-spherical principal series of G. If λ is real or if λ is purely imaginary and $|\lambda| < 1/2$ then π_λ^+ is irreducible and infinitesimally equivalent to a unitary representation. π_λ^- is equivalent to a unitary representation if and only if λ is real. $\{\pi_\lambda^-\}_{\lambda\in\mathbb{R}}$ are all irreducible except when $\lambda = 0$. π_0^- is the direct sum of two non-equivalent irreducible unitary representations which we will denote by $D_{1/2}$ and $D_{-1/2}$ (see [5]). Recall that $\pi \in \hat{G}$ is said to be a discrete series representation if it occurs as a subrepresentation of the right (or left) regular representation of G on $L^2(G)$. Then we have the following fact: Any $\delta \in \hat{G}$ is equivalent to one of the following representations:

(a) the trivial representation

(b) π_λ^+ for some λ real or λ purely imaginary and $|\lambda| < 1/2$

(c) π_λ^- for some λ real and nonzero

(d) $D_{1/2}$

(e) $D_{-1/2}$

(f) a discrete series representation.

The Selberg trace formula. Let $\chi_{1/2}$ be the character of $K(=SO(2))$ defined as follows: $\chi_{1/2}(U_\theta) = e^{i\theta/2}$ where

$$U_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in K.$$

We say a function f on G 'transforms according to the character $\chi_{1/2}$' if

$$f(kxk') = \overline{\chi_{1/2}(k)}\, f(x)\, \overline{\chi_{1/2}(k')}, \qquad \text{for all } x \in G,\ k, k' \in K.$$

We will now define the concept of a Fourier transform for functions transforming according to the character $\chi_{1/2}$. Let $f \in L^1(G)$ transform according to the character $\chi_{1/2}$. Then the Fourier transform is a function $\hat{f}$ defined on $\mathbb{C}$ by:

$$\hat{f}(\lambda) = \operatorname{trace} \pi_\lambda^-(f) \qquad \text{(if it exists).}$$

Then, retaining the notation introduced earlier, we have the following version of the Selberg trace formula for a suitable class of functions f that transform according to the character $\chi_{1/2}$ (see [2]):

$$\sum m_k \hat{f}(r_k) = (2p-2)\int_{-\infty}^{\infty} r\hat{f}(r)\coth \pi r\,dr + 2\sum_{\alpha}\sum_{k}\frac{\lambda_\alpha^k g(k\log N\{P_\alpha\})\log N\{P_\alpha\}}{N\{P_\alpha\}^{k/2}-N\{P_\alpha\}^{-k/2}}$$

where $g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru}\hat{f}(r)\,dr$, r_k are the real numbers such that $\pi_{r_k}^-$, $k = 1,2,\ldots$ are precisely those non-spherical principal series representations of $SL(2,\mathbb{R})$ occuring in the decomposition of $L^2(\Gamma\backslash G)$ under the right regular representation σ of G on $L^2(\Gamma\backslash G)$, and m_k is the multiplicity of $\pi_{r_k}^-$ in σ.

(*Note.* (i) The class of functions for which the trace formula is valid are called admissible functions (see [2]).

(ii) If λ is real π_λ^- and $\pi_{-\lambda}^-$ are equivalent. Hence if $x \in \{r_k\}_{k=1}^{\infty}$ then $-x \in \{r_k\}_{k=1}^{\infty}$.

(iii) As observed already π_0^- is not irreducible. In fact $\pi_0^- = D_{1/2}\oplus D_{-1/2}$. However it can be shown (see for example [8]) that $\mathrm{mult}(D_{1/2},\sigma) = \mathrm{mult}(D_{-1/2},\sigma)$. Hence it is meaningful to talk about the multiplicity of π_0^- in σ.

(iv) Selberg's original trace formula [7] was for K-biinvariant functions. However one can derive a general trace formula for a function f transforming according to any character of K (see [2]). If we take the character to be $\chi_{1/2}$ then there is no contribution from the discrete series representations and we get the formula given above.)

Finally we recall some well known facts about the original Selberg zeta function Z_Γ introduced in [7]. (Proofs can be found in [3], [4] or [1])

$$Z_\Gamma(s) = \prod_{\alpha=1}^{\infty}\prod_{n=0}^{\infty}(1-N\{P_\alpha\}^{-s-n}).$$

$Z_\Gamma(s)$ has 'trivial' zeros at $-k$ for $k = 0,1,2,\ldots$ of multiplicity $(2k+1)(2p-2)$. Apart from this $Z_\Gamma(s)$ has zeros $1/2 + ir_k$, $k = 1,2,\ldots$ of multiplicity m_k where $\pi_{r_k}^+$ are precisely the spherical principal series representations of $SL(2,\mathbb{R})$ occuring in $L^2(\Gamma\backslash G)$ with multiplicity m_k (*i.e.,* the occurence along with multiplicities of the spherical principal series representations of $SL(2,\mathbb{R})$ is completely determined by $Z_\Gamma(s)$.

3. A SELBERG TYPE ZETA FUNCTION AND AN APPLICATION

We retain the notation of section 2. Let Γ be a discrete subgroup of G as in section 2 (*i.e.,* $\Gamma\backslash G$ is compact and Γ does not contain elements of finite order). In analogy with the Selberg zeta function of the previous section consider the 'zeta' function defined by

$$\check{Z}_\Gamma(s) = \prod_{a=1}^{\infty}\prod_{n=0}^{\infty}(1-N\{P_\alpha\}^{-s-n}\lambda_\alpha) \qquad \ldots (*)$$

Then we have the following theorem for our zeta function which tells us that $\check{Z}_\Gamma(s)$ bears the same kind of relationship to the nonspherical principal series as the original Selberg zeta function $Z_\Gamma(s)$ bears to the spherical principal series.

THEOREM 3.1. *The infinite product* (*) *converges for* Re $s > 1$ *to a holomorphic function* $\check{Z}_\Gamma(s)$. *Further* $\check{Z}_\Gamma(s)$ *can be extended to an entire function and has the following zeros:*

(a) *'trivial' zeros at* $s = -n + 1/2$ *of multiplicity* $2n(2p-2)$ (*where* p *is the genus of* $\Gamma\backslash G/K$), $n = 1,2,\ldots$

(b) *'Spectral' zeros at* $s = 1/2 + ir_k$ *of multiplicity* m_k, $k = 1,2,\ldots$ (*where* $\pi^-_{r_k}$ *are precisely the nonspherical unitary principal series representations of G occuring with multiplicity* m_k *in the decomposition of* $L^2(\Gamma\backslash G)$ *under the right regular representation of* G).

The proof of the above theorem is carried out exactly as the corresponding proof for the original Selberg zeta function. The only difference is that instead of using the original trace formula of Selberg for a K-biinvariant function one uses the trace formula given in section 2 for functions transforming according to the character $\chi_{1/2}$. Since several authors have given proofs of the properties of the original Selberg zeta function we will not repeat the proof here. (see [1], [3], [4] or [8]).

We now give an application of Theorem 3.1. Let Γ and Γ' be discrete subgroups of G such that $\Gamma\backslash G$ and $\Gamma'\backslash G$ are compact and such that Γ and Γ' do not have elements of finite order. Let P_α, $N\{P_\alpha\}$, μ_α, λ_α be defined for Γ as in section 2. Let P'_α, $N\{P'_\alpha\}$, μ'_α, λ'_α be similarly defined for Γ'. Let σ and σ' denote the right regular representations of G on $L^2(\Gamma\backslash G)$ and $L^2(\Gamma'\backslash G)$ respectively. Then:

THEOREM 3.2. *If* (*after a suitable reordering if necessary*), $\mu_\alpha = \mu'_\alpha$ $\alpha = 1,2,\ldots$, *then* σ *and* σ' *are unitarily equivalent.*

Proof. As observed in the introduction it is known ([2]) that σ (resp σ') is a direct sum of irreducible unitary representations of G and that for any $\delta \in \hat{G}$, mult(δ,σ) (resp mult (δ,σ')) is finite. Therefore to show σ and σ' are unitarily equivalent it is enough to show that for any $\delta \in \hat{G}$, mult(δ,σ) = mult(δ,σ').

If $\mu_\alpha = \mu'_\alpha$ for all α then $N\{P_\alpha\} = N\{P'_\alpha\}$ for all α. However this would mean that the Selberg zeta functions $Z_\Gamma = Z_{\Gamma'}$ and hence the spherical principal series representations occuring in $L^2(\Gamma\backslash G)$ and $L^2(\Gamma'\backslash G)$ are the same and they occur with the same multiplicities (see the last part of section 2). Since the 'trivial zeros' of the Selberg zeta function determines the genus we have

$$\text{genus}(\Gamma\backslash G/K) = \text{genus}(\Gamma'\backslash G/K).$$

However this implies vol$(\Gamma\backslash G)$ = vol$(\Gamma'\backslash G)$. Again it follows from the work of Gelfand and Langlands (see [10, p. 174]) that if π is a discrete series representation of G then the number of times π occurs in $L^2(\Gamma\backslash G)$ is completely determined by vol$(\Gamma\backslash G)$. Thus the proof of our theorem is complete once we show that if δ is a non-spherical principal series representation of G then

$$\text{mult}(\delta,\sigma) = \text{mult}(\delta,\sigma').$$

Now under the assumption $\mu_\alpha = \mu'_\alpha$ for all α, we have $N\{P_\alpha\} = N\{P'_\alpha\}$ and $\lambda_\alpha = \lambda'_\alpha$ for all α. Hence the Selberg type zeta functions introduced in Theorem 3.1 are the same for Γ and Γ'; *i.e.,* $\check{Z}_\Gamma = \check{Z}_{\Gamma'}$ and hence by (b) of Theorem 3.1 if δ is a nonspherical principal series representation of G,

$$\operatorname{mult}(\delta, \sigma) = \operatorname{mult}(\delta, \sigma')$$

and the proof of our theorem is complete.

Remarks. (i) The fact that the numbers $N\{P_\alpha\}$ determine the occurence of the spherical principal series representations has also been observed by Mckean in [6]. The numbers $N\{P_\alpha\}$ can also be given a differential geometric interpretation (see [6]) and in fact if $N\{P_\alpha\} = N\{P'_\alpha\}$ for all α, Γ and Γ' are isomorphic (see [9]).

(ii) If σ and σ' are unitarily equivalent it is a conjecture of Gelfand [2, p. 87] that Γ and Γ' should be conjugate.

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REFERENCES

1. R. Gangolli, *Zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one.* Illinois J. Math. 21 (1977), 1–41.

2. I. M. Gelfand, M. I. Graev, and I. I. Pjateckiĭ-Šapiro, *Representation theory and automorphic functions.* Translated from the Russian by K. A. Hirsch. W. B. Saunders Co., Philadelphia, Pa.-London-Toronto, Ont., 1969.

3. D. A. Hejhal, *The Selberg trace formula for PSL*$(2,\mathbb{R})$. *Vol. 1.* Lecture Notes in Mathematics, Vol.548, Springer-Verlag, Berlin-New York, 1976.

4. M. Kuga, *Topological analysis and its applications in weakly symmetric Riemannian Spaces.* (Introduction to the work of A. Selberg). (Japanese) Sûgaku 9 (1957/58), 166–185.

5. S. Lang, $SL_2(R)$. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1975.

6. H. P. McKean, *Selberg's trace formula as applied to a compact Riemann surface.* Comm. Pure Appl. Math. 25 (1972), 225–246.

7. A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series.* J. Indian Math. Soc. (N.S)20 (1956), 47–87.

8. A. Sitaram, *A Selberg type zeta function and the representations of* $SL(2,\mathbb{R})$. Dissertation, University of Washington, 1975.

9. A. Sitaram, *On the isomorphism of discrete subgroups of* $SL(2,\mathbb{R})$. Michigan Math. J. 23 (1976), 31–32.

10. N. R. Wallach, *On the Selberg trace formula in the case of compact quotient.* Bull. Amer. Math. Soc. 82 (1976), 171–195.

11. G. Warner, *Harmonic analysis on semi-simple Lie groups* I. Springer-Verlag, Berlin-New York, 1972.

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