THE FUNDAMENTAL GROUP OF THE MODULUS SPACE.

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1. THE MAIN RESULT

A surface of finite type (g,n) is a compact Riemann surface of genus g with n punctures. The space of all conformal equivalence classes of surfaces of type (g,n) is the modulus space X_n. C. Maclachlan [10] proved that the modulus space of a compact surface (with no punctures) is simply-connected. In this paper we extend his result and determine the fundamental group of the modulus space for every surface of finite type.

THEOREM. *For a surface of finite type* (g,n), *the fundamental group of the modulus space* X_n *is the cyclic group of order* 5 *if* $g = 2$ *and* $n \equiv 4 (\mathrm{mod} 5)$ *and is the trivial group for all other surfaces of finite type.*

2. REDUCTION OF THE PROBLEM TO A STUDY OF THE MAPPING CLASS GROUP

We can describe the topological structure of the modulus space of a surface of finite type (g,n) by using the Teichmüller space T_n. (See the survey article of L. Bers [2] for the definition and properties of these spaces.) The points of T_n are equivalence classes of orientation-preserving homeomorphisms of a fixed Riemann surface S_n of type (g,n) onto another surface of the same type. (Two such homeomorphisms f and f′ are equivalent if there exists a conformal map h such that $f^{-1}hf'$ is homotopic to the identity map of S_n.) The Teichmüller space T_n has the structure of a finite-dimensional complex manifold. The group of all homotopy classes of orientation-preserving homeomorphisms of the reference surface S_n onto itself is the mapping class group or Teichmüller modular group M_n. The mapping class group M_n acts in a natural way as a properly discontinuous group of homeomorphisms of T_n, and the modulus space X_n is the quotient space of T_n by the action of M_n. The elements of finite order in the mapping class group M_n generate a normal subgroup F_n that plays a crucial role in the theory.

PROPOSITION 1. *The fundamental group of the modulus space* X_n *is isomorphic to the quotient group* M_n/F_n.

Proof. Following Maclachlan's method, we use the representation of X_n as the quotient of T_n by the action of M_n. Since the Teichmüller space of a surface of finite type is simply-connected, it follows from a theorem of M. A. Armstrong [1] that the fundamental group of X_n is isomorphic to the quotient of M_n by the subgroup generated by elements that fix a point of T_n. It is well known that every element of finite order in M_n has a fixed point. (See J. Nielsen [13] or

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S. Kravetz [8].) Since the group M_n acts as a properly discontinuous group, the stabilizer of a point in T_n is a finite subgroup. It follows that every element that fixes a point of T_n has finite order.

Remark on the Notation. In order to simplify the notation for the various spaces and groups associated with surfaces of type (g,n), we have suppressed the dependence on the genus g.

3. THE STRUCTURE OF THE MAPPING CLASS GROUP M_n

Let S_0 be a fixed compact surface of genus g, and let $P_n = \{p_1, \ldots, p_n\}$ be a set of n distinguished points or "punctures" on S_0. Let $S_n = S_0 - P_n$ be the corresponding punctured surface. A *self-map of* S_n is an orientation-preserving homeomorphism of S_0 that leaves the set P_n invariant. Note that we do not require that each point of P_n be fixed, but only that the set P_n be mapped onto itself. If f and g are two self-maps of S_n, then f *is isotopic to* g *on* S_n if there is an isotopy between f and g that keeps each of the points of P_n fixed during the deformation. Using this terminology, we can describe the mapping class group M_n as the group of all self-maps of S_n modulo those self-maps isotopic to the identity of S_n. We will denote the isotopy class in M_n of a self-map f of S_n by $[f]_n$. Note that each self-map of S_n also determines an isotopy class $[f]_0$ in M_0.

We shall now review some basic facts about generators and subgroups of M_n. (See [3], [5], [7] and [9]) An important type of generator is a twist τ_A about a simple closed curve A on the surface. Informally, we can describe a twist as follows: cut the surface open along the curve A, twist one end a full turn, then reattach the surface along A. We must, however, specify the direction in which the twist is to be made. When the surface is cut open, the orientation of the surface determines an orientation on each end or boundary curve. We require that the twist be in the direction indicated by the induced orientation on the end. Note that the twist τ_A depends on the orientation of the surface, but not on the orientation of the curve. Moreover, two curves that are isotopic determine the same twist (isotopy class).

Each self-map of S_n induces a permutation of the n punctures that depends only on the isotopy class of the self-map. Let M_n^* denote the subgroup of elements that fix each of the punctures. For surfaces S_n and S_0 of the same genus there is a natural homomorphism Ψ_n of M_n onto M_0 that maps $[f]_n$ onto $[f]_0$. Let K_n denote the kernel of Ψ_n, and let $K_n^* = K_n \cap M_n^*$. The groups K_n and K_n^* are the subgroups of M_n and M_n^*, respectively, represented by self-maps of the punctured surface S_n that are isotopic to the identity on the closed surface S_0.

The subgroup K_n^* is generated by elements called ξ-twists [3] or spins [5]. Each spin has the following form. Let A and B be two nonseparating, simple closed curves that bound a cylinder containing a single puncture. The isotopy class in M_n of $\tau_A \tau_B^{-1}$ is a spin. (See the footnote on p. 158 of [5].) The spin generators of K_n^* together with a finite number of twists about nonseparating curves generate M_n; therefore twists about nonseparating curves generate the subgroup M_n^*. The full group M_n is generated by these twists together with any set of elements that induce all possible permutations of the punctures.

4. ELEMENTS OF FINITE ORDER ON A PUNCTURED SURFACE

To proceed, we need to construct finite self-maps of a punctured surface with certain geometric properties. Here we discuss how to deform a finite self-map of the closed surface into a finite self-map of the punctured surface with similar properties. Let h be a self-map of the closed surface S_0. A h-invariant subset of S_0 is a set invariant under the action of the cyclic subgroup $\langle h\rangle$ generated by h. A set is h-invariant if and only if it is a union of $\langle h\rangle$-orbits.

Let Ω be a subregion of S_0 and r a nonnegative integer. We will say that h *is compatible with* Ω *and* r if Ω contains a h-invariant subset of r points. If h is compatible with Ω and r for all r, then we will simply say that h is compatible with Ω. Now consider a permutation μ of the set of integers $\{1,2,\dots,n\}$. We will say that h *is compatible with* Ω *and* μ if the region Ω contains a h-invariant set of points $\{q_1,\dots,q_n\}$ such that $h(q_i) = q_{\mu(i)} (1 \le i \le n)$.

We can express these compatibility conditions in terms of the number $N_k = N_k(h,\Omega)$ of $\langle h\rangle$-orbits of length k completely contained in the region Ω. The map h is compatible with Ω and n if and only if $n = \sum_{k\ge 0} k m_k$ for integers m_k satisfying $0 \le m_k \le N_k$. Similarly, the map h is compatible with Ω and μ if and only if the number of k-cycles in the permutation μ is less than or equal to N_k $(1 \le k \le n)$.

The next two lemmas deal with the deformation of finite self-maps of the closed surface into finite self-maps of the punctured surface. If Ω is a subregion of S_0, let $r(\Omega)$ denote the number of punctures lying in Ω.

LEMMA 1. *Let* h *be a finite self-map of* S_0 *and* Λ *a closed subset of* S_0 *disjoint from the set of punctures* P_n. *If* h *is compatible with* Ω *and* $r = r(\Omega)$ *for each component* Ω *of* $S_0 - \Lambda$ *containing* r *punctures, then there exists a finite self-map* $\hat{h}$ *of* S_n *isotopic to* h *on* S_0 *and equal to* h *on* $\Lambda \cap h^{-1}(\Lambda)$.

Proof. Let $\Omega_1,\Omega_2,\dots,\Omega_k$ be a list of the components of $S_0 - \Lambda$ that contain at least one puncture. Let $r_i = r(\Omega_i)$ be the number of punctures lying in $\Omega_i (1 \le i \le k)$. Since h is compatible with Ω_i and r_i, the region Ω_i contains a h-invariant set Q_i of r_i points. Take a self-map f_i of S_0, isotopic to the identity on S_0 and equal to the identity on the complement of Ω_i, such that

$$f_i(P_n \cap \Omega_i) = Q_i.$$

If we set $f = f_1 f_2 \dots f_k$, then the map $\hat{h} = f^{-1} h f$ has the desired properties.

LEMMA 2. *Let* h *be a finite self-map of* S_0 *and* μ *a permutation of the set of integers* $\{1,2,\dots,n\}$. *If* h *is compatible with* S_0 *and* μ, *then there exists a finite self-map* h_μ *of* S_n, *isotopic to* h *on* S_0, *such that* $h_\mu(p_i) = p_{\mu(i)}$ $(1 \le i \le n)$.

Proof. Since h is compatible with S_0 and μ, there is a set of points $\{q_1,\dots,q_n\}$ on S_0 such that $h(q_i) = q_{\mu(i)}$ $(1 \le i \le n)$. If we take a self-map f, isotopic to the identity on S_0, such that $f(p_i) = q_i$ $(1 \le i \le n)$, then the map $h_\mu = f^{-1} h f$ has the desired properties.

5. THE STRUCTURE OF THE QUOTIENT M_n/F_n

We now consider the quotient of M_n by the subgroup F_n generated by elements of finite order. If two elements ω and ω' of M_n define the same element of the quotient group M_n/F_n, we will say that *ω and ω' are congruent modulo F_n*. Our primary goal is to show that the group M_n/F_n is cyclic, that is, the group is generated by a single congruence class.

Consider now two twists τ_A and τ_B about curves A and B. If ω is the isotopy class of a self-map f of S_n such that $f(A) = B$, then $\omega\tau_A\omega^{-1} = \tau_B$. (See [5] or [9].) If, in particular, the map f has finite order, then τ_A will be congruent to τ_B modulo F_n (since ω is congruent to the identity). Thus we observe that two twists τ_A and τ_B are congruent if there is a finite self-map of S_n that maps one curve onto the other. Using this observation, we can show that any two twists are congruent.

LEMMA 3. *If* A *and* B *are two nonseparating, simple closed curves on the surface* S_n, *then the twist about* A *and the twist about* B *are congruent modulo* F_n.

Proof. Since a twist depends only on the isotopy class of the curve, we may assume that A and B intersect transversely in a finite number of points. The proof is by induction on the number $r = |A \cap B|$ of points of intersection.

Case 1. $r = 0$: Assume that the curves are disjoint. Since neither curve separates the surface, the surface $S_0 - A \cup B$ obtained by removing both curves is either a (connected) surface of genus $g - 2$ with four boundary curves or a disjoint union of two surfaces of genera g_1 and g_2 ($g_1 + g_2 + 1 = g$), each with two boundary curves. By pasting the surface together along the boundary curves, we see that the original surface with the two distinguished curves is homeomorphic to one of the two topological models in Figure 1. In either case, the surface has an involution h that interchanges the two curves. Now apply Lemma 1 to the map h and the set $\Lambda = A \cup B$. Since, in either case, each component of $S_0 - \Lambda$ contains at least one fixed point of h, we see that h is compatible with each component of $S_0 - \Lambda$. It follows from Lemma 1 that there exists a finite self-map $\hat{h}$ of S_n equal to h on $\Lambda \cap h^{-1}(\Lambda) = A \cup B$. Since $\hat{h}$ is a finite self-map of S_n that maps A onto B, the twists about A and B are congruent.

Case 2. $r = 1$: Now assume that the curves intersect at only one point. If the surface has genus at least 2, then we can find a third nonseparating curve C disjoint from both A and B. (To see this, note that the set $A \cup B$ has a regular neighborhood N homeomorphic to a torus with one boundary curve. The complement of N is a surface of genus at least 1 that contains nonseparating curves.) By the previous result, the twists about A and B are both congruent to the twist about C; therefore the two twists are congruent.

Since there are no nonseparating curves on a sphere, the only remaining case is a surface of genus 1, a torus. Represent the torus as the unit square

$$\{(x, y)\colon 0 \le x \le 1,\ 0 \le y \le 1\}$$

with its opposite sides identified. For the curves A and B, we take the vertical

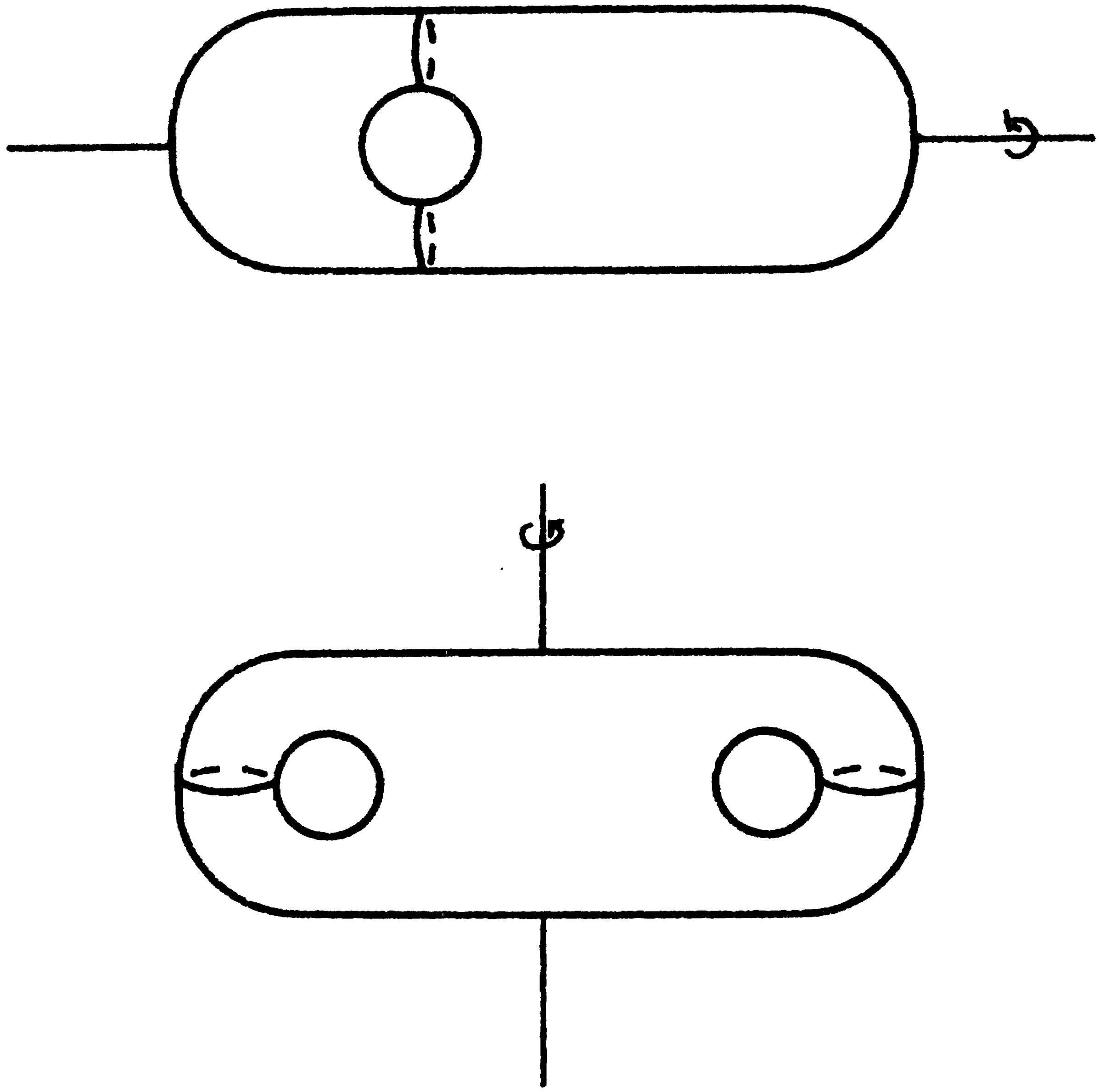

Figure 1

and horizontal line segments through the point (1/4,1/4). The positive rotation of order 4 about the center of the square represents a finite self-map h of the torus that carries A onto B. Set $\Lambda = A \cup B$ and apply Lemma 1 as before. The complement of Λ has a single component Ω that contains two orbits of length 1, one orbit of length 2, and an infinite number of orbits of length 4. The map h is compatible with Ω; therefore there exists a finite self-map $\hat{h}$ of S_n equal to h on $\Lambda \cap h^{-1}(\Lambda) = A$. Since $\hat{h}$ maps A onto B, we see that the two twists are congruent.

Case 3. $r \geq 2$: Now assume that A and B intersect at r points and that the result is true for each pair of curves that intersect in fewer than r points. It suffices to show that there exists a third nonseparating curve C that intersects both A and B in fewer than r points. The argument given here is based on one used by J. Birman to show that the mapping class group is generated by twists [4].

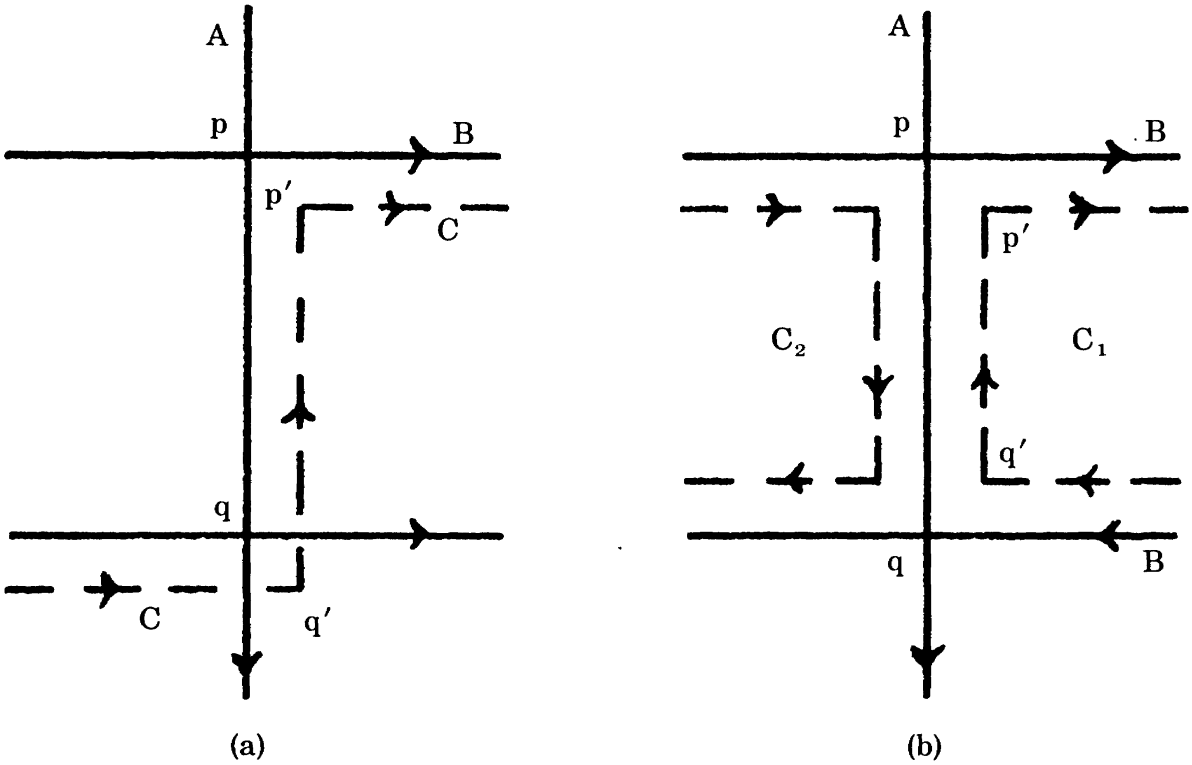

Figure 2

We can construct a curve with these properties as follows. Let p be a point of intersection. Assign an orientation to both A and B and move along A in the positive direction until you reach the next point of intersection q. The curve B crosses A at each point of intersection in a definite direction. There are two possibilities. In the first case (Figure 2a) the directions are the same and in the second case (Figure 2b) the directions are opposite.

In the first case, we construct the new curve C shown in Figure 2a. The curve C starts at a point p' near p, runs along but slightly to the right of B until it reaches a point q' near q, then returns to p' by crossing both A and B once. Since the curve C intersects B at only one point, it must be a nonseparating curve. The curve C has the desired properties, since $|B \cap C| = 1$ and

$$|A \cap C| < |A \cap B| = r.$$

In the second case, we construct the two new curves C_1 and C_2 shown in Figure 2b. The curve C_1 starts at a point p' near p, runs along but slightly to the right of B until it reaches a point q' near q, then returns to p' without crossing either A or B. The curve C_2 is constructed similarly on the other side of A. Since $|B \cap C_i| = 0$ and $|A \cap C_i| < |A \cap B|$ $(1 \leq i \leq 2)$, each of the curves C_1 and C_2 has the desired properties provided that it is a nonseparating curve. We claim that at least one of these is a nonseparating curve. The curve B is homologous to the sum of C_1 and C_2. If both C_1 and C_2 were separating curves, then both of these curves would be homologous to zero and, hence, the curve B would also be homologous to zero. This, however, is impossible, since B is a nonseparating curve.

LEMMA 4. *For a punctured surface* S_n *of genus at least* 1, *each permutation of the punctures is induced by an element of the subgroup* F_n.

Proof. We only need to consider the case when $n \geq 2$. A surface of genus at least 1 has an involution h with at least four fixed points. Set $m = n$ if n is even and $m = n - 1$ if n is odd. Let μ be the permutation $(12)(34) \ldots (m-1\ m)$ and ν the permutation $(34) \ldots (m-1\ m)$. (If $m = 2$, set ν equal to the identity.)

Since the map h is compatible with both permutations, it follows from Lemma 2 that there exist finite self-maps h_μ and h_ν of S_n, isotopic to h on S_0, such that $h_\mu(p_i) = p_{\mu(i)}$ and $h_\nu(p_i) = p_{\nu(i)}$ $(1 \leq i \leq n)$. The isotopy class of $h_\mu h_\nu$ is an element of F_n that interchanges p_1 and p_2 and fixes each of the remaining punctures. Since transpositions generate the full permutation group, it follows that each permutation of the punctures is induced by an element of F_n. We have actually proven a slightly stronger result. Since the map $h_\mu h_\nu$ is isotopic to h^2, the identity, on S_0, it follows that each permutation of the punctures is induced by an element of the subgroup $K_n \cap F_n$.

PROPOSITION 2. *For a surface of genus at least* 1,

(a) *the quotient* M_n/F_n *is a cyclic group generated by the common congruence class of twists about nonseparating curves,*

(b) *the homomorphism* Ψ_n *of* M_n *onto* M_0 *induces an isomorphism of* M_n/F_n *onto* $M_0/\Psi_n(F_n)$, *and*

(c) *the order of* M_n/F_n *divides* 12 *if* $g = 1$, *divides* 10 *if* $g = 2$, *and is* 1 *if* $g \geq 3$.

Proof. The group M_n is generated by twists about nonseparating curves together with any set of elements that induce all possible permutations of the punctures. Since, by Lemma 4, the elements of F_n induce all possible permutations, we see that M_n is generated by twists and the elements of F_n. The elements of F_n are all trivial in the quotient M_n/F_n and, by Lemma 3, all twists about nonseparating curves lie in the same congruence class; therefore the quotient M_n/F_n is the cyclic group generated by this single congruence class.

To prove part (b), we first show that $K_n \subseteq F_n$. The subgroup K_n^* is generated by spins, each of which is the product of a twist and an inverse of a twist. Since two twists about nonseparating curves are congruent modulo F_n, it follows that each spin is congruent to the identity modulo F_n and, therefore, $K_n^* \subseteq F_n$. Now take an element $\omega \in K_n$. The proof of Lemma 4 shows that there is an element $\zeta \in K_n \cap F_n$ that induces the same permutation of the punctures as ω does. It follows that $\zeta^{-1}\omega \in K_n^*$; hence $\omega \in \zeta K_n^* \subseteq F_n$.

The homomorphism Ψ_n of M_n onto M_0 followed by the natural projection of M_0 onto the quotient $M_0/\Psi_n(F_n)$ is a homomorphism of M_n onto $M_0/\Psi_n(F_n)$ with kernel equal to $K_n F_n$. Since $K_n \subseteq F_n$, the kernel is F_n; therefore M_n/F_n is isomorphic to $M_0/\Psi_n(F_n)$ as stated in part (b). Since the group $M_0/\Psi_n(F_n)$ is an abelian quotient of M_0, its order divides the order of the commutator quotient group M_0/M_0'. The group M_0/M_0' has order 12 if $g = 1$, order 10 if $g = 2$, and order 1 if $g \geq 3$. (See [12], [4], and [14].) From these two results, we see that the order of the quotient group M_n/F_n is as stated in part (c).

6. THE GROUP M_n/F_n

We can now determine the group M_n/F_n for all surfaces of finite type.

Case 1. $g \geq 3$: Proposition 2 shows that the group M_n/F_n is trivial.

Case 2. $g = 0$: For a surface of genus $g = 0$, we use the presentation of M_n given in [11]. The group M_n is generated by elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$. The element $\hat{\alpha} = \sigma_1\sigma_2 \ldots \sigma_{n-2}$ has order $n-1$ and the element $\hat{\beta} = \sigma_1\sigma_2 \ldots \sigma_{n-1}$ has order n; therefore the generator $\sigma_{n-1} = \hat{\alpha}^{-1}\hat{\beta}$ lies in F_n. Since each of the other generators is conjugate to σ_{n-1}, the group M_n is generated by elements of finite order. Hence the quotient M_n/F_n is also trivial if $g = 0$.

For a surface of genus 1 or 2, we shall use the following technique to determine the order of M_n/F_n. Using the isomorphism established in Proposition 2, we may replace the group M_n/F_n by its isomorphic image $M_0/\Psi_n(F_n)$. Let $\bar{\tau}$ denote the cyclic generator of the quotient group $M_0/\Psi_n(F_n)$ corresponding to the common congruence class modulo $\Psi_n(F_n)$ of twists about nonseparating curves. If

$$\omega = \tau_1^{\varepsilon_1}\tau_2^{\varepsilon_2} \ldots \tau_k^{\varepsilon_k}$$

is an element of M_0 expressed as a product of twists τ_i, then its image in the quotient $M_0/\Psi_n(F_n)$ is $(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_k)\bar{\tau}$. If ω lies in $\Psi_n(F_n)$, then its image is 0. Thus each element in $\Psi_n(F_n)$ determines a relation $(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_k)\bar{\tau} = 0$ in the quotient.

To determine whether an element lies in $\Psi_n(F_n)$, we shall use the following Lemma.

LEMMA 5. *If* h *is a finite self-map compatible with* S_0 *and* n, *then the isotopy class of* h *lies in* $\Psi_n(F_n)$.

Proof. Apply Lemma 1 to the map h and the set $\Lambda = \emptyset$. There exists a finite self-map $\hat{h}$ of S_n isotopic to h on S_0. The class $[\hat{h}]_n$ is an element of F_n whose image under the homomorphism Ψ_n is $[h]_0$.

We need to know the properties of some elements of finite order in M_0. Consider the system of nonseparating curves $\{A_i\}$ $(1 \leq i \leq 2g+1)$ shown in Fig. 3. Let τ_i denote the twist about the curve A_i $(1 \leq i \leq 2g+1)$. The elements $\alpha = \tau_1\tau_2 \ldots \tau_{2g}$ and $\beta = \tau_1\tau_2 \ldots \tau_{2g+1}$ are the isotopy classes of finite self-maps h_α and h_β, respectively, with the properties listed in Table 1.

Table 1

isotopy class	self-map	order	orbit structure: N_k is the number of orbits of length k.
α	h_α	$4g+2$	$N_1 = 1, N_2 = 1, N_{2g+1} = 1, N_{4g+2} = \infty$
β	h_β	$2g+2$	$N_1 = 2, N_2 = 1, N_{2g+2} = \infty$.

For a surface of genus 1, we can construct h_α and h_β explicitly and verify their properties. (Note that $\beta = \tau_1\tau_2\tau_3$ can also be written as $\tau_1\tau_2\tau_1$.) For a surface

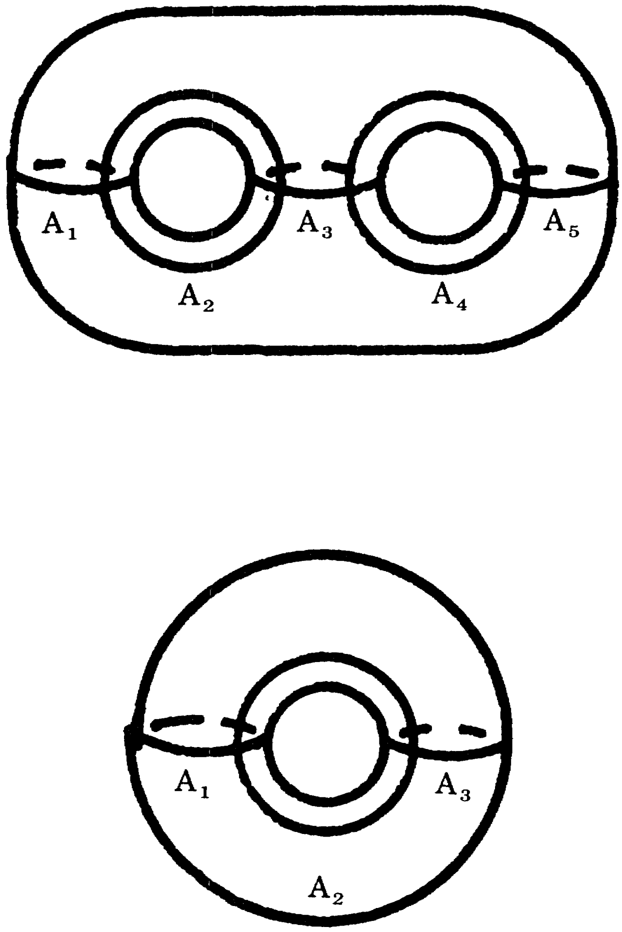

Figure 3

of genus 2, the elements α and β have the following simple interpretations. We can represent the surface as a two-sheeted covering of the sphere branched over six points. There is a natural homomorphism from the mapping class group of the surface onto the mapping class group of the sphere with six punctures. (See [6].) The map h_α is the lift to the covering surface of a rotation of order 5 on the punctured sphere, while h_β is the lift of a rotation of order 6. (The fifth power of h_α is a covering transformation; therefore h_α has order 10, not 5.)

Case 3. $g = 1$: Consider the element h_α with the properties listed in Table 1. Since an integer n can be written as $1m_1 + 2m_2 + 3m_3 + 6m_6$ $(0 \leq m_1 \leq 1, 0 \leq m_2 \leq 1, 0 \leq m_3 \leq 1, 0 \leq m_6)$, it follows from the discussion in section 3 that h_α is compatible with S_0 and n. Using Lemma 5, we see that α is in $\Psi_n(F_n)$. A similar argument shows that β is also in $\Psi_n(F_n)$. Since α is a product of two twists and β is a product of three twists, we obtain the relations $2\bar{\tau} = 0$ and $3\bar{\tau} = 0$ in the quotient $M_n/\Psi_n(F_n)$. Taken together these relations imply that $\bar{\tau} = 0$; therefore the group $M_n/\Psi_n(F_n)$ is trivial.

Case 4. $g = 2$: According to Table 1, the element β^3 is the isotopy class of an involution h_β^3 that has two orbits of length 1 and an infinite number of orbits of length 2. Clearly the map h_β^3 is compatible with S_0 and n; therefore the element β^3 lies in $\Psi_n(F_n)$. Since β^3 is the product of 15 twists, we obtain the relation

$15\bar{\tau} = 0$ in the quotient group. Moreover, since the order of the quotient divides 10, the relation $10\bar{\tau} = 0$ also holds. Taken together these relations imply that $5\bar{\tau} = 0$.

Now α^2 is the isotopy class of self-map h_α^2 of order 5. From Table 1, we see that h_α^2 has three orbits of length 1 and an infinite number of orbits of length 5. Now an integer n can be written as $1m_1 + 5m_5$ $(0 \leq m_1 \leq 3,\ 0 \leq m_5)$ if $n \not\equiv 4 \pmod 5$; therefore the map h_α^2 is compatible with S_0 and n and, hence, α^2 lies in $\Psi_n(F_n)$ if $n \not\equiv 4 \pmod 5$. The element α^2 is a product of 8 twists; therefore we obtain the additional relation $8\bar{\tau} = 0$ if $n \not\equiv 4 \pmod 5$. This relation together with the previous ones show that the quotient is trivial if $n \not\equiv 4 \pmod 5$.

Assume now that $n \equiv 4 \pmod 5$ and consider the original group M_n/F_n. We shall show directly that this group is not trivial. First we show that M_n does not contain elements of order divisible by 5. Each element of finite order in M_n is the isotopy class of a conformal self-map of the surface S_n [8]; therefore, if there were an element of order 5, then we could find a conformal self-map h of the closed surface S_0 that leaves the set of n punctures invariant.

The Riemann-Hurwitz formula for Riemann surfaces states that, if the surface S is a branched cover of a surface S′, then the genus g of S, the genus g′ of S′, the number of sheets k, and the branch number b satisfying the equation:

$$2 - 2g = k(2 - 2g') - b.$$

If we apply the formula to the natural covering of S_0 over the quotient surface $S_0/\langle h \rangle$, we obtain:

$$-2 = 5(2 - 2g') - b.$$

Since the only branching occurs at the m fixed points of h (each of which has branch order 4), the branch number is 4m. Since g′ and m are nonnegative integers, the only solution is $g' = 0$ and $m = 3$. Therefore the only orbits of h are the three orbits of length 1 and the infinite number of orbits of length 5. Since the map h must permute the n punctures in these orbits of length 1 or 5, the integer n is of the form $1m_1 + 5m_5$ $(0 \leq m_1 \leq 3, 0 \leq m_5)$. This is impossible if $n \equiv 4 \pmod 5$; therefore M_n has no elements of order 5. Moreover, the group M_n has no elements of order divisible by 5.

Since the commutator quotient group of M_0 is a cyclic group of order 10, there is a homomorphism of M_0 onto Z_5, the cyclic group of order 5. If we precede this homomorphism by the homomorphism Ψ_n of M_n onto M_0, we obtain a homomorphism Φ_n of M_n onto Z_5. Let L_n denote the kernel of the homomorphism Φ_n. If ω is an element of finite order in M_n, then the order of its image $\Phi_n(\omega)$ divides both the order of ω and the order of the image group Z_5. Since the order of ω is not divisible by 5, the order of its image $\Phi_n(\omega)$ must be 1 and, therefore, ω is in the kernel L_n. It follows that the subgroup F_n is contained in L_n; therefore the index of F_n in M_n is divisible by 5 (the index of L_n in M_n). On the other hand, we know from our previous work that the order of M_n/F_n divides 5; consequently we see that M_n/F_n has order 5 if $n \equiv 4 \pmod 5$.

We summarize these results in Proposition 3. The main Theorem follows immediately from Propositions 1 and 3.

PROPOSITION 3. *The group* M_n/F_n *is a cyclic group of order* 5 *if* $g = 2$ *and* $n \equiv 4 (\mathrm{mod}\ 5)$ *and order* 1 *otherwise.*

Remark. We can modify the proof above to show that the quotient of M_n by the subgroup generated by involutions is trivial for surfaces of genus $g \geq 3$.

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