FREE ACTIONS ON PRODUCTS OF SPHERES

Elliott Stein

1. INTRODUCTION

It has been known for some time that if the finite group G acts freely on S^n, then G satisfies the following conditions: (1) Every abelian subgroup is cyclic [2, Chap. XII]; (2) Every element of order two is central [7]. Recent work of Madsen, Thomas, and Wall has shown that these two conditions are sufficient to imply the existence of a free action on some sphere. In this paper, we consider similar questions for the product of spheres, $S^n \times S^n \times \ldots \times S^n$. The problem seems to be substantially more difficult than the spherical case, and a complete solution is not in sight. Our results, obtained by surgery theory, give an interesting class of examples.

In [3], Conner proved that condition (1) above generalizes to $S^n \times S^n \times \ldots \times S^n$ in the following way. Let $(S^n)^k$ denote the product of k copies of S^n. Conner's result is that if G acts freely on $(S^n)^k$, then any abelian subgroup of G can be generated by at most k elements. The question then arises as to whether there is an analogue of condition (2) for $(S^n)^k$ with $k > 1$. In this vein, Gene Lewis made the following conjecture in [5].

Conjecture. If G acts freely on $S^n \times S^n$ and H is a subgroup isomorphic to $Z_2 \times Z_2$, then $H \cap Z(G) \neq \{e\}$, where $Z(G)$ is the center of G.

We will show that this conjecture is false. Specifically, let D_q denote the dihedral group of order 2q, with q an odd prime. Let $(D_q)^k$ denote the product of k copies of D_q.

PROPOSITION 1. *For any* $n = 4j + 3$ *and any* $k \geq 2$, *there exist free, orientation preserving, piecewise-linear actions of* $(D_q)^k$ *on* $(S^n)^k$.

When $k = 2$, this gives the desired counterexamples, since $Z_2 \times Z_2 \subset D_q \times D_q$ but $D_q \times D_q$ has trivial center.

In Sections 2 and 3, we prove Proposition 1 for the case $k = 2$. The construction proceeds along lines suggested by [6], and we use some of the tools developed there. In Section 4, we deal with $k > 2$ and observe that some of our examples can be smoothed. The interesting arguments are in the proofs of Lemmas 5 and 7, where the surgery obstructions of certain products are analyzed. Throughout, it will be assumed that the reader is familiar with surgery theory, as given in [1] or [10].

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2. CONSTRUCTING A NORMAL MAP

Since D_q has cohomology of period four, the results of Swan [8] show that D_q acts freely and cellularly on a finite CW complex X^n with the homotopy type of S^n. In fact, Swan shows that n may be any integer of the form $8j + 7$. The following lemma improves this result. (For $q = 3$, this lemma was proven by Swan [8, p. 289].)

LEMMA 2. *There exists a free cellular action of* D_q *on a finite three-dimensional* CW *complex* X, *such that* X *is homotopy equivalent to* S^3. *The induced action on* $H_*(X)$ *is trivial.*

Proof. Let D_q be given by the following presentation:

$$\{x,y: y^2 = 1, \quad yx^{s-1}y = x^s\}$$

where $s = (q + 1)/2$. Let $\langle z_1,z_2, \ldots\rangle$ be the free $Z[D_q]$ module with $\{z_1,z_2, \ldots\}$ as basis. Define a chain complex

$$\langle e\rangle \xrightarrow{\partial_3} \langle c,c'\rangle \xrightarrow{\partial_2} \langle b,b'\rangle \xrightarrow{\partial_1} \langle a\rangle$$

by the following formulas:

$$\partial_1 b = (x - 1)a \qquad \partial_1 b' = (y - 1)a$$

$$\partial_2 c = [y(1 + x + \ldots + x^{s-2}) - (1 + x + \ldots + x^{s-1})]b + (1 + x^s y)b'$$

$$\partial_2 c' = (1 + y)b'$$

$$\partial_3 e = (x + x^2 + \ldots + x^{s-1})(x^{s-1} - 1)(1 - x^s y)c + (y - 1)c'.$$

Following Swan [8], this chain complex gives rise to a simply-connected finite complex X on which D_q operates freely. The cellular chain complex of X is the one given above. (Note that the two-skeleton of the quotient space X/D_q is the complex with one 0-cell, two 1-cells, and two 2-cells associated to the presentation $\{x,y: y^2, yx^{s-1}yx^{-s}\}$.) All that remains is to verify that the chain complex has the desired homology. This is a routine computation; the only nontrivial step is showing that image ∂_3 = kernel ∂_2. Since this can be done exactly as in Swan's example for $q = 3$, we omit the details. That the action on $H_*(X)$ is trivial can be seen directly from the construction, or from the Lefschetz Fixed Point Theorem.

By taking the iterated join of X with itself, we get a free action of D_q on $Y^{4j+3} \simeq S^{4j+3}$ ($\simeq$ denotes homotopy equivalence). Denote Y^{4j+3} simply by Y, and let $Z_2 \subset D_q$ be some subgroup of order two.

LEMMA 3. *Let* ξ *be a stable spherical fibration over* Y/D_q. *Let* $\pi: Y/Z_2 \to Y/D_q$ *be the* q*-fold covering. Then any* PL *structure on* $\pi^*\xi$ *is the pullback of a* PL *structure on* ξ.

Proof. The proof is given in [9, Prop.3.5]. We have to check that

$$H^i(Y/D_q; \pi_{i-1}G/PL) \xrightarrow{\pi^*} H^i(Y/Z_2; \pi_{i-1}G/PL)$$

is injective and that

$$H^i(Y/D_q; \pi_i G/PL) \xrightarrow{\pi^*} H^i(Y/Z_2; \pi_i G/PL)$$

is surjective, for all i. Recall that

$$\pi_{2i+1}(G/PL) = 0, \quad \pi_{4i}(G/PL) = Z, \quad \text{and} \quad \pi_{4i+2}(G/PL) = Z_2.$$

Also, $H^{2i+1}(Y/D_q) = H^{2i+1}(Y/Z_2) = 0$, except when $2i + 1 = 4j + 3$, in which case both groups are infinite cyclic. For $i \leq j$, $H^{4i+2}(Y/D_q) = Z_2$ and $H^{4i}(Y/D_q) = Z_{2q}$, so $H^m(Y/D_q; Z_2) = H^m(Y/Z_2; Z_2) = Z_2$ for $m \leq 4j + 3$. The conditions above now follow immediately by composing π^* with the transfer map.

LEMMA 4 ([6]). *There is a* PL *normal map* $f\colon M^{4j+3} \to Y^{4j+3}/D_q$ *such that the* q*-fold covering* $\tilde{f}\colon \tilde{M}^{4j+3} \to Y^{4j+3}/Z_2$ *is normally cobordant to a homotopy equivalence* $RP^{4j+3} \to Y^{4j+3}/Z_2$.

Proof. Since Y^{4j+3}/Z_2 is obviously homotopy equivalent to RP^{4j+3}, we give the Spivak fibration for Y^{4j+3}/Z_2 the PL structure which comes from some homotopy equivalence $RP^{4j+3} \to Y^{4j+3}/Z_2$. Because the Spivak fibration is natural for coverings, we can apply Lemma 3 and the result follows.

Note that if we pass to universal covers, $\tilde{f}\colon \tilde{M} \to Y$ is normally cobordant to a homotopy equivalence $S^{4j+3} \to Y$.

Lemma 3 will be of further use in the next section.

3. VANISHING OF A SURGERY OBSTRUCTION

The normal map $f\colon M \to Y/D_q$ must have nonzero surgery obstruction, regardless of how the normal invariant was chosen. (In fact, Milnor's condition that involutions must be central can be derived by surgery; see [4].) The map we wish to analyze is $f \times f$.

LEMMA 5. *The surgery obstruction of* $f \times f\colon M \times M \to Y/D_q \times Y/D_q$ *is zero.*

Proof. By the construction of f, the covering of $f \times f$ corresponding to a 2-Sylow subgroup, $\tilde{f} \times \tilde{f}\colon \tilde{M} \times \tilde{M} \to Y/Z_2 \times Y/Z_2$, has zero obstruction. From the proof of Theorem 4.1 of [6], it follows that we only need to show that $(Y \times Y)/H$ has the homotopy type of a manifold for each hyperelementary subgroup $H \subset D_q \times D_q$. Recall that H is hyperelementary if it is the split extension of a cyclic group by a group of prime power order.

Let p_1 and p_2 denote the projections from $D_q \times D_q$ onto its factors. We now divide the hyperelementary subgroups into two types and treat these cases separately.

Type I: For at least one value of $i \in \{1,2\}$, $p_i(H) \neq D_q$. Assume $p_1(H) \neq D_q$; the same proof will work for $i = 2$. We want to show that $(Y \times Y)/H$ has the homotopy type of a manifold. Since $H \subseteq p_1(H) \times D_q$, it suffices to show that

$$Y/p_1(H) \times Y/D_q$$

has the homotopy type of a manifold. Note, however, that $p_1(H)$ is cyclic of order 1,2, or q, so $Y/p_1(H)$ has the homotopy type of S^{4j+3}, RP^{4j+3}, or some linear Z_q lens space, L^{4j+3}. Let N denote any of these three manifolds and form the normal map

$$g = \mathrm{id} \times f\colon N \times M \to N \times Y/D_q.$$

When $N = S^{4j+3}$, $\sigma(g)$, the surgery obstruction of g, is obviously zero. When $N = RP^{4j+3}$, $\sigma(g) = 0$ by the following argument. Form the normal map

$$g' = \mathrm{id} \times f\colon CP^{2j+1} \times M \to CP^{2j+1} \times Y/D_q.$$

Since CP^{2j+1} is an oriented boundary, $\sigma(g') = 0$. Let W be a normal cobordism from g' to a homotopy equivalence. There is a circle bundle over CP^{2j+1} with total space RP^{4j+3}. This induces a circle bundle over W whose total space is a normal cobordism from g to a homotopy equivalence.

If L^{4j+3} fibres over CP^{2j+1}, then once again we have that $\sigma(g) = 0$. On the other hand, the lens spaces which fibre over CP^n generate the Z_q bordism over the bordism of a point. Thus, L will be cobordant to some multiple of a fibred lens space. By bordism invariance of the surgery obstruction, $\sigma(g) = 0$. Therefore, in all three cases $\sigma(g) = 0$, so $N \times Y/D_q \simeq Y/p_1(H) \times Y/D_q$ has the homotopy type of a manifold.

Type II: $p_1(H) = p_2(H) = D_q$. An example of such a subgroup is the diagonal $\Delta \subset D_q \times D_q$; let us treat this example first.

The covering corresponding to Δ is the fibre product $Y \times_{D_q} Y$, the quotient of $Y \times Y$ by the diagonal action of D_q. There is a fibre bundle

$$(\xi) \qquad Y \to Y \times_{D_q} Y \to Y/D_q,$$

which we view as a stable spherical fibration over Y/D_q. If we lift ξ to Y/Z_2 by the covering π, we get

$$(\pi^*\xi) \qquad Y \to Y \times_{Z_2} Y \to Y/Z_2.$$

The action of Z_2 on Y is equivariantly homotopy equivalent to the antipodal action on S^{4j+3}, so $\pi^*\xi$ is fibre homotopy equivalent to the linear sphere bundle

$$S^{4j+3} \to S^{4j+3} \times_{Z_2} Y \to Y/Z_2.$$

In particular, $\pi^*\xi$ has a PL structure. By Lemma 3, ξ has a PL structure, which we denote by η. Thus, $Y \times_{D_q} Y$ has the homotopy type of $E(\eta)$, the total space of a PL sphere bundle. Consider the diagram

$$\begin{array}{ccc} \tilde{M} = E(f^*\eta) & \xrightarrow{\tilde{f}} & E(\eta) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & Y/D_q. \end{array}$$

We can make $\tilde{f}$ into a normal map. Since η bounds a PL disk bundle, $\tilde{f}$ is null cobordant and has zero surgery obstruction. It follows that $(Y \times Y)/\Delta$ has the homotopy type of a manifold.

For a typical subgroup of type II, we make the following group theoretic observation, the proof of which can be safely left to the reader. If $H \subset D_q \times D_q$ is hyperelementary and $p_1(H) = p_2(H) = D_q$, then there is an automorphism $\theta: D_q \to D_q$ such that

$$\theta \times \mathrm{id}: D_q \times D_q \to D_q \times D_q$$

carries H onto a subgroup which is conjugate to Δ. By the classification of coverings, we may regard conjugate subgroups as being identical. It can now be easily checked that the proof above for Δ is left undisturbed by the automorphism $\theta \times \mathrm{id}$. For example, we will have the same spherical fibration ξ, except that the action of the structure group will be changed by θ. Clearly, this does not affect whether there is a lifting to a PL structure. This takes care of subgroups of type II and completes the proof of Lemma 5.

PROPOSITION 6. $D_q \times D_q$ *acts freely on* $S^{4j+3} \times S^{4j+3}$.

Proof. After completing surgery on $f \times f: M \times M \to Y/D_q \times Y/D_q$, we obtain a PL manifold $Z^{8j+6} \simeq Y/D_q \times Y/D_q$. Obviously, $D_q \times D_q$ acts freely on the universal cover $\tilde{Z}$. We now show that $\tilde{Z}$ is PL homeomorphic to $S^{4j+3} \times S^{4j+3}$.

Let $h: Z \to Y/D_q \times Y/D_q$ be the homotopy equivalence which is normally cobordant to $f \times f$. The universal covering map $\tilde{h}: \tilde{Z} \to Y \times Y$ is cobordant to $\tilde{f} \times \tilde{f}$. By construction, $\tilde{f} \times \tilde{f}$ is cobordant to a homotopy equivalence

$$h': S^{4j+3} \times S^{4j+3} \to Y \times Y.$$

The normal cobordism from $\tilde{h}$ to h' is simply connected and odd-dimensional, so further surgery in its interior produces a h-cobordism from $\tilde{Z}$ to $S^{4j+3} \times S^{4j+3}$.

4. FURTHER PRODUCTS AND SOME REMARKS

We will merely sketch the proof of the remainder of Proposition 1, since most of the details are the same as in the case already treated. Continue to denote by Z the manifold produced in Proposition 6.

LEMMA 7. *The surgery obstruction of* $f \times \mathrm{id}: M^{4j+3} \times Z \to Y/D_q \times Z$ *is zero.*

Proof. The argument is the same as in Lemma 5. If $H \subset D_q \times D_q \times D_q$ is hyperelementary and $p_i(H) \neq D_q$ for some $i \in \{1,2,3\}$, then the proof for subgroups of type I applies. If $p_1(H) = p_2(H) = p_3(H) = D_q$, then there are automorphisms θ_1 and θ_2 such that

$$\theta_1 \times \theta_2 \times \mathrm{id}: D_q \times D_q \times D_q \to D_q \times D_q \times D_q$$

carries H onto a conjugate of the diagonal Δ. To see that $(Y \times Y \times Y)/\Delta$ has the homotopy type of a manifold, consider the "external product" bundle:

$$(\xi \times \xi) \qquad \begin{array}{c} (Y \times_{D_q} Y) \times (Y \times_{D_q} Y) \\ \downarrow \\ Y/D_q \times Y/D_q. \end{array}$$

This bundle has the fibre homotopy type of a bundle η' over $Y/D_q \times Y/D_q$, whose fibre is $S^{4j+3} \times S^{4j+3}$ and whose structure group is $PL(S^{4j+3}) \times PL(S^{4j+3})$, the product of PL homeomorphism groups. If $\delta: Y/D_q \to Y/D_q \times Y/D_q$ is the diagonal, then $\delta^*(\xi \times \xi)$ has $(Y \times Y \times Y)/\Delta$ as total space. As in Lemma 5, we form the diagram

$$\begin{array}{ccc} E(f^*\delta^*n') & \longrightarrow & E(\delta^*\eta') \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad f \quad} & Y/D_q \end{array}$$

Since $PL(S^{4j+3}) \times PL(S^{4j+3}) \subset PL(D^{4j+4} \times S^{4j+3})$, the map between total spaces is null cobordant (as a normal map), so $E(\delta^*\eta') \simeq (Y \times Y \times Y)/\Delta$ has the homotopy type of a manifold. The remaining details are left to the reader.

PROPOSITION 8. *$(D_q)^3$ acts freely on $(S^{4j+3})^3$.*

Proof. Completing surgery on $f \times \mathrm{id}: M \times Z \to Y/D_q \times Z$ gives a PL manifold $T \simeq (Y/D_q)^3$. As in Proposition 6, the homotopy equivalence of universal covers, $\tilde{T} \to Y \times \tilde{Z}$, is normally cobordant to an equivalence $(S^{4j+3})^3 \to Y \times \tilde{Z}$. The cobordism is simply-connected and $(4\ell + 2)$-dimensional. After adding a Kervaire manifold in the interior, if necessary, surgery can be completed on the cobordism to get a h-cobordism.

Evidently, Proposition 1 follows immediately from Propositions 6 and 8.

Observe now that at least some of these actions can be smoothed. In fact, since PL/O is 6-connected, the Poincaré space $\tilde{X}^3/D_q$ has a smooth normal invariant. Once a smooth normal invariant is obtained for Y, the surgery analysis proceeds just as in the PL case, except that in Proposition 8 more care must be taken with differentiable structures. Rather than pursue this in detail, we simply give the following as an example.

PROPOSITION 9. *For $k = 2$ or $k \geq 4$, there is a smooth, free action of $(D_q)^k$ on $(S^3)^k$. There is a smooth, free action of $(D_q)^3$ on $S^3 \times S^3 \times S^3$ or $S^3 \times S^3 \times S^3 \# \Sigma^9$, where Σ^9 is the Kervaire sphere.*

The difficulty with Σ^9 disappears when $k \geq 4$ because $\Sigma^9 \times S^3$ is diffeomorphic to $S^9 \times S^3$.

It is reasonable to expect that if G has periodic cohomology, then $G \times G$ will act freely on $S^n \times S^n$ for some n. However, the simple arguments given here, which depend on having a group with a small lattice of conjugacy classes of subgroups, do not immediately generalize. Of course, the larger question is whether the condition that every abelian subgroup of G has rank less than or equal to k is sufficient to imply that G acts freely on $(S^n)^k$. If true, such a theorem appears very difficult, even to the extent of proving the analogue of Swan's theorem for actions on CW complexes. The smallest test case is the alternating group A_4, so we close with the following problem: Does A_4 act freely on $S^n \times S^n$ or even on a finite CW complex with that homotopy type?

Added in proof (December, 1978). The question above has been answered negatively by Robert Oliver in "Free compact group actions on products of spheres," to appear in the proceedings of the 1978 Aarhus topology conference.

The theorem of Conner quoted in the introduction was proved in [3] only when $k = 2$. In a private communication with Oliver, Conner has indicated that his proof for $k > 2$ may have been incomplete. The general case is unsettled.

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Department of Mathematics
University of Chicago
Chicago, Illinois 60637

Current address:
Department of Mathematics
State University of N. Y. at Stony Brook
Stony Brook, NY 11794