THE HOCHSTER-ROBERTS THEOREM OF INVARIANT THEORY

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Let G be an algebraic k-group acting morphically on an affine k-scheme, where k is a field. The quotient $G\backslash X$ is the affine k-scheme, whose regular functions are the regular functions on X, which are invariant under the action of G.

We will assume that G is linearly reductive; *i.e.,* any finite dimensional representation of G is completely reducible. The classical finiteness theorem of Hilbert asserts that, if X is a k-scheme of finite type, then $G\backslash X$ is also [5] and [10]. A remarkable modern discovery is the

THEOREM 0.1. (*Hochster-Roberts* [7]). *If* X *is a regular* k-*scheme of finite type, then* $G\backslash X$ *is a Cohen-Macaulay* k-*scheme of finite type.*

In this paper, I will give another proof of this theorem. My proof is a modification of their proof. In fact, it relies on the proof of

THEOREM 0.2. *Let* $A \subset B$ *be two integral domains which are finitely generated algebras over a field* k. *If* B *is regular and* B *is a pure* A-*module, then* A *must be a Cohen-Macaulay ring.*

This result was conjectured by Hochster and Roberts in their paper. Furthermore, both results were established by them in finite characteristics with weaker noetherian assumptions. As B is a pure A-module if A is a direct summand of B as an A-module, my proof shows that the only fact from invariant theory, used in the proof of Theorem 0.1, is that the G-invariant projection

$$\Gamma(X, \mathcal{O}_X) \to \Gamma(G\backslash X, \mathcal{O}_{G\backslash X})$$

is a $\Gamma(G\backslash X, \mathcal{O}_{G\backslash X})$-module homomorphism.

In Hochster-Roberts' proof, Theorem 0.1 was proven by a reduction to the graded case, which is apparently not possible for Theorem 0.2. Their reduction to the graded case uses the G-action and can provide valuable information about the normal behavior of $G\backslash X$ along any stratum in terms of the invariants of linear representations of reductive subgroups of G.

In this paper, I will first prove Theorem 0.2, next explain how it implies the Theorem 0.1 and lastly show how the theorem may be used for actually computing invariants.

1. SOME BACKGROUND

Let S be a regular noetherian scheme and $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. Let F denote the support of $\mathcal{F}$. Let Z be a closed subset of F. Recall that $\mathcal{F}$ is called

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Cohen-Macaulay if and only if $\operatorname{depth}_Z \mathscr{F} \geq$ codimension of Z in F $\equiv \operatorname{cod}_F Z$ for all such Z. In Grothendieck's language, one may interpret the $\operatorname{depth}_Z \mathscr{F}$ as

$$\inf \{i: \mathscr{H}^i_Z(\mathscr{F}) \neq 0\},$$

where $\mathscr{H}^i_Z(\mathscr{F})$ is the i-th local cohomology sheaf of $\mathscr{F}$ along Z.

In this paper, we will make essential use of Grothendieck's finiteness theorem [3]. It is

THEOREM 1.1. *Assume that* $\mathscr{F}|_{S-Z}$ *is Cohen-Macaulay. Then,* $\mathscr{H}^i_Z(\mathscr{F})$ *are coherent* $\mathscr{O}_S$*-modules (which are supported by* Z) *for* $0 \leq i < \operatorname{cod}_F Z$.

We will also need to know this method of computing these local cohomology sheaves. Let T be a closed subscheme of S with ideal $\mathscr{I}$. Let T_i be the closed subscheme of S with ideal $\mathscr{I}^i$. Then, we have

THEOREM 1.2. *Let* $Z = F \cap T$. *Then,*

(a) $\mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{G}, \mathscr{F})$ *is zero if* $i < \operatorname{depth}_Z \mathscr{F}$ *for any coherent* $\mathscr{O}_S$*-module* $\mathscr{G}$ *with support in* T.

(b) *There is a natural isomorphism,*

$$\lim_{m \to \infty} \mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{O}_{T_m}, \mathscr{F}) \approx \mathscr{H}^i_Z(\mathscr{F})$$

for any i.

We will need the following

COROLLARY 1.3. *If* $\operatorname{depth}_Z \mathscr{F} \geq i$, *then we have a sequence of inclusions,*

$$\mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{O}_T, \mathscr{F}) \subset \ldots \subset \mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{O}_{T_m}, \mathscr{F}) \subset \ldots \subset \mathscr{H}^i_Z(\mathscr{F})$$

$$\text{where} \quad \mathscr{H}^i_Z(\mathscr{F}) = \bigcup \mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{O}_{T_m}, \mathscr{F}).$$

Proof. We have an exact sequence, $0 \to \mathscr{G} \to \mathscr{O}_{T_m} \to \mathscr{O}_{T_{m-1}} \to 0$, where $\mathscr{G}$ is a coherent $\mathscr{O}_S$-module supported in T. Thus, we have an exact sequence,

$$\mathscr{E}xt^{i-1}_{\mathscr{O}_S}(\mathscr{G}, \mathscr{F}) \to \mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{O}_{T_{m-1}}, \mathscr{F}) \xrightarrow{\alpha} \mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{O}_{T_m}, \mathscr{F}),$$

but $\mathscr{E}xt^{i-1}_{\mathscr{O}_S}(\mathscr{G}, \mathscr{F})$ is zero by part (a) of the theorem. Hence, α is injective and the rest follows from part (b).

We may now combine these results to get

COROLLARY 1.4. *If* $\mathscr{F}|_{S-Z}$ *is Cohen-Macaulay and* $\operatorname{depth}_Z \mathscr{F} \geq i$ *where* $i < \operatorname{cod}_F Z$, *then we have natural isomorphisms,*

$$\mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{O}_{T_j}, \mathscr{F}) \xrightarrow{\approx} \mathscr{H}^i_Z(\mathscr{F}), \qquad \text{for } j >> 0.$$

Proof. By the ascending chain condition for sub-modules of a coherent sheaf, this result follows from Theorem 1.1 and Corollary 1.3.

Let $X = \mathrm{Spec}(A)$, where $= A_0 \oplus A_1 \oplus \ldots$ is a graded noetherian ring. Let Z equal the set of zeroes of the ideal $A_1 \oplus A_2 \oplus \ldots$ of A and let T_i be the closed subscheme of X with ideal $A_i \oplus A_{i+1} \oplus \ldots$. Let $\mathscr{F} = \tilde{M}$ for some graded A-module M of finite type. Then, we have the following theorem ([1], [2], [3]), which is a form of Serre's finiteness and vanishing theorem for projective space. It is also related to Hochster and Roberts' notion of "uniform convergence of Koszul complexes."

THEOREM 1.6. (i) *The local cohomology sheaves* $\mathscr{H}^i_Z(\mathscr{F}) = \bigoplus_{j \in \mathbb{Z}} \mathscr{H}^i_Z(\mathscr{F})_j$ *are naturally graded sheaves of* $\mathscr{O}_X$*-modules. For any integer* N, $\bigoplus_{j \geq N} \mathscr{H}^i_Z(\mathscr{F})_j$ *is a coherent* $\mathscr{O}_X$*-module supported by* Z.

(ii) *The sheaves* $\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{O}_{T_m}, \mathscr{F}) = \bigoplus_{j \in \mathbb{Z}} \mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{O}_{T_m}, \mathscr{F})_j$ *are naturally graded coherent* $\mathscr{O}_X$*-modules.*

(iii) *For any* N *and* i, *there exists an integer* $m = m(N, i)$ *such that*

$$\bigoplus_{j \geq N} \mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{O}_{T_\ell}, \mathscr{F})_j \xrightarrow{\approx} \bigoplus_{j \geq N} \mathscr{H}^i_Z(\mathscr{F})_j$$

for all $\ell \geq m$.

2. THE DISAPPEARANCE OF LOCAL COHOMOLOGY

Let U be an ℓ-scheme of finite type. Let X be a k-scheme of finite type over a subfield k of ℓ. Let $f : U \to X$ be a k-morphism. Let Z be a closed subset X.

We have a natural f-homomorphism, $\psi: \mathscr{H}^i_Z(\mathscr{O}_X) \to \mathscr{H}^i_{f^{-1}Z}(\mathscr{O}_U)$, for any integer i. Our main technical result is

THEOREM 2.1. *Assume that* U *is regular,* $X - Z$ *is Cohen-Macaulay and* i *is an integer less than* $\mathrm{cod}_X Z$ *such that* $\mathrm{depth}_Z\, \mathscr{O}_X \geq i$. *Then, the homomorphism* $\psi: \mathscr{H}^i_Z(\mathscr{O}_X) \to \mathscr{H}^i_{f^{-1}Z}(\mathscr{O}_U)$ *is zero.*

Proof. First, we will reduce to the case where $f^{-1}Z$ has codimension at least one in U. Otherwise, $f^{-1}Z$ would contain a whole connected component of U as U is regular. Thus, $\mathscr{H}^i_{f^{-1}Z}(\mathscr{O}_U)$ would be zero along this component unless $i = 0$. If $i > 0$, the theorem is trivial. If $i = 0$, we have a factorization,

$$\Gamma_Z(\mathscr{O}_X) \to \Gamma_{Z'}(\mathscr{O}_{X'}) \to \Gamma_{f^{-1}Z}(\mathscr{O}_U),$$

where Z' is the corresponding subset of $X' \equiv X_{red}$ (we can do this because U is reduced). Now, $\Gamma_{Z'}(\mathscr{O}_{X'})$ is zero unless Z' contains a component of X'; *i.e.* $\mathrm{cod}_X Z = 0$. This settles the theorem in this case as $i < \mathrm{cod}_X Z = 0$.

As the theorem is also local on X, we may assume that X is affine. Thus, we may find a k-morphism $g: X \to \mathbb{A}^n_k$ such that $Z = g^{-1}\{0\}$. Let $h = g \circ f$. Then, $f^{-1}Z = h^{-1}\{0\}$; and we have a commutative diagram,

$$\begin{array}{ccc} U & \xrightarrow{\Gamma_h} & U \times_k \mathbb{A}^n_k \equiv V \\ f \downarrow & & \downarrow \\ X & \xrightarrow{\Gamma_g} & X \times_k \mathbb{A}^n_k \equiv S, \end{array}$$

where Γ denotes the graph of a morphism. Let R be a closed subscheme of $\mathbb{A}^n_k$ with support $\{0\}$. Define $T_i = X \times_k R_i$ and $W_i = U \times_k R_i$, where R_i is the closed subscheme of $\mathbb{A}^n_k$ with ideal $\mathscr{I}^i_R$ for $i > 0$. Then, T_i is a closed subscheme of S, which meets X in Z. Similarly, W_i is a closed subscheme of V, which meets U in $f^{-1}Z$.

The virtue of this construction is that we have a commutative diagram of sheaves,

$$\begin{array}{ccc} \mathscr{E}xt^i_{\mathscr{O}_V}(\mathscr{O}_{W_j}, \mathscr{O}_U) & \longrightarrow & \mathscr{H}^i_{f^{-1}Z}(\mathscr{O}_U) \\ \uparrow \psi_i & & \uparrow \psi \\ \mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{O}_{T_j}, \mathscr{O}_X) & \xrightarrow{\gamma_j} & \mathscr{H}^i_Z(\mathscr{O}_X), \end{array}$$

which covers the morphisms of the underlying schemes.

As $\mathscr{H}^i_Z(\mathscr{O}_X)$ is the direct limit of the $\mathscr{E}xt$'s by Theorem 1.2.b, we only need to prove that ψ_j is zero for all large j. On the other hand, we know that γ_j is an isomorphism for $j >> 0$ by Corollary 1.4 and our assumptions. Better yet, if γ_m is an isomorphism, then γ_j is an isomorphism for $j \geq m$ by Corollary 1.3. With these facts in mind, we see that our theorem is a consequence of the following more detailed result.

Claim. Assume that $\gamma_m: \mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{O}_{T_m}, \mathscr{O}_X) \to \mathscr{H}^i_Z(\mathscr{O}_X)$ is an isomorphism. Then, $\psi_m: \mathscr{E}xt^i_{\mathscr{O}_S}(\mathscr{O}_{T_m}, \mathscr{O}_X) \to \mathscr{E}xt^i_{\mathscr{O}_V}(\mathscr{O}_{W_m}, \mathscr{O}_U)$ is zero.

We will first prove this claim when char(k) is finite. Then we will deduce the characteristic zero case by continuity.

Assume that char(k) equals a prime p. Let g be a point of $f^{-1}Z$ such that its closure $\bar{g}$ contains the support of the coherent sheaf $\operatorname{Im} \psi_m \cdot \mathscr{O}_U$ near g. We will establish the inequalities.

$$\begin{aligned} \text{length}_{\mathscr{O}_{U,g}}(\operatorname{Im} \psi_m \cdot \mathscr{O}_{U,g}) &\geq \text{length}_{\mathscr{O}_{U,g}}(\operatorname{Im} \psi_{pm} \cdot \mathscr{O}_{U,g}) \\ &\geq p^{\dim \mathscr{O}_{U,g}} \cdot \text{length}_{\mathscr{O}_{U,g}}(\operatorname{Im} \psi_m \cdot \mathscr{O}_{U,g}). \end{aligned}$$

As $\operatorname{cod}_U f^{-1}(z) \geq 1$, $\dim \mathscr{O}_{U,g} \geq 1$. Thus, the inequalities imply that

$$\text{length}_{\mathscr{O}_{U,g}}(\operatorname{Im} \psi_m \cdot \mathscr{O}_{U,g}) = 0 \qquad \text{for all such g.}$$

Hence, $\operatorname{Im} \psi_m \cdot \mathscr{O}_{U,g}$ must be zero as its support is empty; *i.e.*, ψ_m is zero. Thus, it remains to prove the inequalities.

The first inequality is easy. We have a commutative diagram,

$$\begin{array}{ccc} \mathcal{E}xt^i_{\mathcal{O}_V}(\mathcal{O}_{W_m}, \mathcal{O}_U) & \to & \mathcal{E}xt^i_{\mathcal{O}_V}(\mathcal{O}_{W_{pm}}, \mathcal{O}_U) \\ \uparrow \psi_m & & \uparrow \psi_{pm} \\ \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_{T_m}, \mathcal{O}_X) & \overset{\approx}{\to} & \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_{T_{pm}}, \mathcal{O}_X), \end{array}$$

where the bottom arrow is an isomorphism by the remarks preceding the claim. Thus, we have an surjection of $\mathcal{O}_U$-modules, $\operatorname{Im}\psi_m \cdot \mathcal{O}_U \to \operatorname{Im}\psi_{pm} \cdot \mathcal{O}_U$. This gives the first inequality and shows that $\operatorname{length}_{\mathcal{O}_{U,g}}(\operatorname{Im}\psi_{pm} \cdot \mathcal{O}_{U,g})$ is a finite number.

The second inequality uses an analog of the Frobenius trick employed by Hochster and Roberts in their work. Let F denote the Frobenius endomorphism of any schemes involved. Recall that F fixes the points of the scheme but raises a regular function α to its p-th power, $F^*\alpha = \alpha^p$. We get the following commutative diagram,

$$\begin{array}{ccccc} \mathcal{E}xt^i_{\mathcal{O}_V}(\mathcal{O}_{W_m}, \mathcal{O}_U) & \overset{\beta}{\to} & \mathcal{E}xt^i_{\mathcal{O}_V}(\mathcal{O}_{W_m}, F_*\mathcal{O}_U) & \approx & F_*(\mathcal{E}xt^i_{\mathcal{O}_V}\mathcal{O}_{W_{pm}}, \mathcal{O}_U)) \\ \uparrow \psi_m & & \uparrow & & \uparrow F_*\psi_{pm} \\ \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_{T_m}, \mathcal{O}_X) & \to & \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_{T_m}, F_*\mathcal{O}_X) & \approx & F_*(\mathcal{E}xt_{\mathcal{O}_S}(\mathcal{O}_{T_{pm}}, \mathcal{O}_X)), \end{array}$$

where the vertical arrows are f-homomorphisms. The main point is that $F_*\mathcal{O}_U$ is a faithfully flat coherent $\mathcal{O}_U$-module as U is regular. Thus, β induces an isomorphism of $F_*\mathcal{O}_U$-modules,

$$\mathcal{E}xt^i_{\mathcal{O}_V}(\mathcal{O}_{W_m}, \mathcal{O}_U) \otimes_{\mathcal{O}_U} F_*\mathcal{O}_U \to \mathcal{E}xt^i_{\mathcal{O}_V}(\mathcal{O}_{W_m}, F_*\mathcal{O}_U).$$

Better yet, for any coherent $\mathcal{O}_U$-submodule $\mathcal{M}$ of $\mathcal{E}xt^i_{\mathcal{O}_V}(\mathcal{O}_{W_m}, \mathcal{O}_U)$, $\mathcal{M} \otimes_{\mathcal{O}_U} F_*\mathcal{O}_U$ is isomorphic to the $F_*\mathcal{O}_U$-submodule of $\mathcal{E}xt^i_{\mathcal{O}_V}(\mathcal{O}_{W_m}, F_*\mathcal{O}_U)$, which $\gamma(\mathcal{M})$ generates.

By the commutativity of the diagram and the last remark, we must have an injective $F_*\mathcal{O}_U$-homomorphism,

$$(\operatorname{Im}\psi_m \cdot \mathcal{O}_U) \otimes_{\mathcal{O}_U} F_*\mathcal{O}_U \hookrightarrow F_*[\operatorname{Im}\psi_{pm} \cdot \mathcal{O}_U].$$

We may rewrite this as an injection of $\mathcal{O}_U$-modules, $F^*(\operatorname{Im}\psi_m \cdot \mathcal{O}_U) \hookrightarrow \operatorname{Im}\psi_{pm} \cdot \mathcal{O}_U$. Thus,

$$\operatorname{length} \mathcal{O}_{U,g}(F^*\mathcal{O}_{U,g}/m_g) \cdot \operatorname{length}_{\mathcal{O}_{U,g}}(\operatorname{Im}\psi_m \cdot \mathcal{O}_{U,g})$$
$$\leq \operatorname{length}_{\mathcal{O}_{U,g}}(\operatorname{Im}\psi_{pm} \cdot \mathcal{O}_{U,g}).$$

The second equality follows because $\operatorname{length}_{\mathcal{O}_{U,g}}(F^*(\mathcal{O}_{U,g}/m_g)) = p^{\dim \mathcal{O}_{U,g}}$.

It remains to prove the claim when $\operatorname{char}(k) = 0$. As we are working with a property invariant under field extension we may assume that $k = \mathbb{C}$. We will find a subring A of k, which is finitely generated over $\mathbb{Z}$ so that $f: U \to X$ and all the other above structures are defined over A. If we do this carefully, our problem will be faithfully reproduced by base extension of the problem to a finite quotient field κ of A. These arguments are almost routine but I will state the necessary requirements on A and give the details at certain turning points in the arguments.

Firstly, we need to make A big enough so that we have A-morphisms $f' \colon U' \to X'$ and $g' \colon X' \to \mathbb{A}^k_A$ between A-schemes of finite type, which give the corresponding notions over k. Choose a closed subscheme R' of $\mathbb{A}^n_A$, which gives R. Let

$$Z' = (g')^{-1}(\text{support}\,(R')).$$

One may take A so that U' is smooth over A as U is smooth over ℓ because $\operatorname{char}(k) = 0$. Furthermore, we may require that X', T'_j and W'_j are flat over A; and $X' - Z'$ is Cohen-Macaulay and $\operatorname{depth}_{Z'} \mathcal{O}_{X'} \geq i$. Also, we may assume that $(f')^{-1}Z'$ has codimension at least one in any fiber of $U' \to \operatorname{Spec} A$. Lastly, we assume that

$$\mathcal{E}xt^i_{\mathcal{O}_{S'}}(\mathcal{O}_{T'_m}, \mathcal{O}_{X'}) \to \mathcal{H}^i_{Z'}(\mathcal{O}_{X'})$$

remains an isomorphism (here, it is important to use the coherence of $\mathcal{H}^i_{Z'}(\mathcal{O}_{X'})$), and $\mathcal{E}xt^i_{\mathcal{O}_{V'}}(\mathcal{O}_{W'_m}, \mathcal{O}_{U'})/\operatorname{Im}\psi'_m \cdot \mathcal{O}_{U'}$ is A-flat.

Now, let $A \to \kappa$ be a surjection of A onto a finite field κ. Such surjections exist as the points $\operatorname{Spec}\kappa$ are dense in Spec A. Let U'' be the base-extension of the above concepts over κ. Then, U'' is regular (in fact, smooth over κ). Also, $X'' - Z''$ is Cohen-Macaulay, $\operatorname{depth}_{Z''} \mathcal{O}_{X''} \geq i$ and $\operatorname{cod}_{U''}(f'')^{-1}Z''$ is at least one. As the various $\mathcal{E}xt$'s may be computed before or after base extension by the A-flatness,

$$\mathcal{E}xt^i_{\mathcal{O}_{S''}}(\mathcal{O}_{T''_m}, \mathcal{O}_{X''}) \approx \mathcal{E}xt^i_{\mathcal{O}_{S''}}(\mathcal{O}_{T''_{m+\cdot}}, \mathcal{O}_{X''})$$

and hence, to $\mathcal{H}^i_{Z''}(\mathcal{O}_{X''})$. Thus, γ''_m is an isomorphism. Furthermore, by the last flatness assumption,

$$(\operatorname{Im}\varphi'_m \cdot \mathcal{O}_{U'}) \otimes_A \kappa \approx \operatorname{Im}\varphi''_m \cdot \mathcal{O}_{U''}.$$

As the last term is zero by the finite characteristic version of the claim, the image of the support $(\operatorname{Im}\varphi'_m \cdot \mathcal{O}_{U'})$ in Spec A can not contain any point $\operatorname{Spec}(\kappa)$. On the other hand, if φ_m were not zero, then this image would contain an open dense subset of Spec A. As these possibilities are mutually exclusive, $\varphi_m = 0$.

Remark 1. Hochster and Roberts have proven a much stronger result in characteristic p.

Remark 2. It seems reasonable to ask if the analogous result is true for complex analytic varieties.

Remark 3. One may hope that the above argument may be simplified by using the proof of Theorem 1.1 rather than its statement.

We will end this section with a complement of the previous theorem when X comes from a graded situation. Assume that X = Spec A, where

$$A = A_0 + A_1 + \dots$$

is a graded k-algebra of finite type. Assume that Z equals the set of zeroes of the ideal $A_1 \oplus A_2 \oplus \dots$. In this situation, we have the

THEOREM 2.2. *Assume that* U *is regular. For any integer* i, ψ *induces the zero* f*-homomorphism from the positive graded part* $\bigoplus_{j\geq 0} \mathcal{H}^i_Z(\mathcal{O}_X)_j$ *of* $\mathcal{H}^i_Z(\mathcal{O}_X)$ *to* $\mathcal{H}^i_U(\mathcal{O}_U)$.

Proof. The proof is almost the same as that of Theorem 2.2. One must use the Theorem 1.6 instead of the previous results from section 1. The point is that the positive graded part is invariant under the Frobenius in characteristic p but is still small enough to be a coherent $\mathcal{O}_X$-module. The reduction to the finite characteristic case is more standard and can be read in the Hochster-Roberts paper, or done by mimicking flat and projective base-extensions as in E.G.A.

3. PURE ALGEBRAS

Let $\varphi\colon M \to N$ be an A-homomorphism between two modules over a ring A. Then, φ is called pure if, for all A-modules P, $\varphi \otimes_A P\colon M \otimes_A P \to N \otimes_A P$ is injective.

Clearly, if $\varphi\colon M \to N$ is pure, then φ itself must be injective. If $\varphi\colon M \to N$ is injective and its image $\varphi(M)$ is an A-module direct summand of N, then φ is a pure A-homomorphism.

An important property of pure homomorphisms, which is evidently satisfied by the last example, is contained in

LEMMA 3.1. *Let* P^* *be a complex of* A*-modules. Then the induced* A*-homomorphism* $\varphi^*\colon H^*(M \otimes_A P^*) \to H^*(N \otimes_A P^*)$ *between the homology groups of these complexes is injective if* φ *is pure.*

Proof. We have a commutative diagram

$$\begin{array}{ccccccc}
\dots M \otimes P^{i-1} & \to & M \otimes P^{i} & \to & M \otimes P^{i+1} & \to & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
\dots N \otimes P^{i-1} & \to & N \otimes P^{i} & \to & N \otimes P^{i+1} & \to & \dots
\end{array}$$

We need to show that $\mathrm{Im}(N \otimes P^{i-1}) \cap \mathrm{Im}(M \otimes P^{i}) = \mathrm{Im}(M \otimes P^{i-1})$ as sub-modules of $N \otimes P^i$. On the other hand, we have a commutative exact diagram,

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
M \otimes P^{i-1} & \to & M \otimes P^{i} & \to & M \otimes (P^i/\mathrm{Im}\, P^{i-1}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
N \otimes P^{i-1} & \to & N \otimes P^{i} & \to & N \otimes (P^i/\mathrm{Im}\, P^{i-1}) & \to & 0,
\end{array}$$

where the vertical exactness is a consequence of the purity assumption. The desired fact follows formally from the diagram.

We can use this lemma to prove

COROLLARY 3.2. *Let* X = Spec A *where* A *is a noetherian ring. Let* Z *be a closed subset of* X.

(a) *Let* $\varphi: M \to N$ *be a pure* A*-homomorphism. Then* φ *induces an injection* $\mathscr{H}^i_Z(X, \tilde{M}) \to \mathscr{H}^i_Z(X, \tilde{N})$ *on local cohomology.*

(b) *Let* $\psi: A \to B$ *be a homomorphism of* C*-algebras, which is a pure* C*-module homomorphism. Let* f: V = Spec B → X *be the morphism given by* ψ. *Then the* ψ*-homomorphism,* $\mathscr{H}^i_Z(X, \mathscr{O}_X) \to \mathscr{H}^i_{f^{-1}Z}(V, \mathscr{O}_V)$, *is injective.*

Proof. Let $I \subset A$ be an ideal defining the closed subset Z. If one computes the induced homomorphism $\mathrm{Ext}^i_A(A/I^j, M) \to \mathrm{Ext}^i_A(A/I^j, M)$ by using a resolution of A/I^j by free A-modules of finite type, then one finds that it is injective by Lemma 2.1. As Grothendieck has proven that $H^i_Z(X, \tilde{M}) \to H^i_Z(X, \tilde{N})$ is the direct limit for these injections, we may conclude the truth of (a).

For (b), we may factor the given homomorphism as

$$H^i_Z(X, \mathscr{O}_X) \to H^i_Z(X, \tilde{B}) \to H^i_{f^{-1}Z}(V, \mathscr{O}_V).$$

The first arrow is injective by part (a) and Grothendieck has shown that the last arrow is always an isomorphism. Thus, (b) is true.

We can bring together the two divergent themes in

THEOREM 3.3. *Let* $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ *be* D*-algebra homomorphisms. Assume that* φ *is* k*-homomorphism between finitely generated algebras over the fields* $k \subset \ell$ *respectively,* $\psi \circ \varphi$ *is a pure* D*-homomorphism and* B *is a regular noetherian ring. Then,* A *is Cohen-Macaulay.*

Proof. Let $V = \mathrm{Spec}\, C \xrightarrow{g} U = \mathrm{Spec}\, B \xrightarrow{f} X = \mathrm{Spec}\, A$ be the corresponding morphisms. For any closed subset Z of X, we need to see that $\mathrm{depth}_Z \mathscr{O}_X \geq \mathrm{cod}_X Z$, when $X - Z$ is known to be Cohen-Macaulay. By induction, it will be enough to check that if $\mathrm{depth}_Z \mathscr{O}_X \geq i$ and $i < \mathrm{cod}_X Z$, then $\mathscr{H}^i_Z(\mathscr{O}_X) = 0$; *i.e.* depth $\mathscr{O}_X > i$.

As $\mathscr{H}^i_Z(\mathscr{O}_X) \approx \widetilde{H^i_Z(X, \mathscr{O}_X)}$ for any closed subset Z of an affine scheme, we need to see that $H^i_Z(X, \mathscr{O}_X) = 0$. By Theorem 2.1, the homomorphism,

$$H^i_Z(X, \mathscr{O}_X) \to H^i_{f^{-1}Z}(U, \mathscr{O}_U),$$

is zero but, by Corollary 3.2.b, it is injective. Thus, $H^i_Z(X, \mathscr{O}_X) = 0$.

Clearly, the Theorem 0.2 mentioned in the introduction is a special case of Theorem 3.3.

Remark. Again, Hochster and Roberts have proven this type of result in characteristic p by the same argument with less restrictive finiteness assumptions.

A similar result to the last theorem is

THEOREM 3.4. *In the situation of Theorem* 3.4, *assume that* $A = A_0 \oplus A_1 \ldots$ *is a graded* k-*algebra and* Z *is the zeros of the ideal* $A_1 \oplus A_2 \oplus \ldots$ *in* Spec A = X. *Then, the positively graded part* $\bigoplus_{j \geq 0} \mathscr{H}^i_Z(\mathcal{O}_X)_j$ *of* $\mathscr{H}^i_Z(\mathcal{O}_X)$ *is zero for any integer* i.

Proof. The same as Theorem 3.3 except that one uses Theorem 2.2 in place of Theorem 2.1.

4. APPLICATION TO INVARIANT THEORY

Let G be an algebraic k-group acting morphically on an affine k-scheme X. The quotient G\X is defined to be $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X)^G)$, where $\Gamma(X, \mathcal{O}_X)^G$ is the k-algebra of G-invariants in $\Gamma(X, \mathcal{O}_X)$. The inclusion of rings induces a k-morphism $X \to G\backslash X$.

Recall that, if G is a linearly reductive group, there is a unique G-invariant projection $\mathrm{Av}: \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)^G$ of $\Gamma(X, \mathcal{O}_X)$ onto its subspace of invariant elements. A trivial calculation shows that Av is a $\Gamma(X, \mathcal{O}_X)^G$-module homomorphism. Therefore, the inclusion $\Gamma(X, \mathcal{O}_X)^G \to \Gamma(X, \mathcal{O}_X)$ is a pure homomorphism of $\Gamma(X, \mathcal{O}_X)^G$-modules.

We may now prove

THEOREM 0.1. *If* X *is a regular affine* k-*scheme of finite type, then* G\X *is a Cohen-Macaulay* k-*scheme of finite type.*

Proof. We have already noted in the introduction that G\X is a k-scheme of finite type. Thus, this theorem is a special case of Theorem 3.3. To see this, just take $A = \Gamma(X, \mathcal{O}_X)^G$ and $B = C = \Gamma(X, \mathcal{O}_X)$ where φ is the inclusion and ψ is the identity.

In classical invariant theory, X is a vector variety $\mathbb{V}$ and a reductive group G acts on $\mathbb{V}$ by linear transformations. As the G action on $\mathbb{V}$ commutes with the action of $\mathbb{G}_m$ on $\mathbb{V}$ by scalar multiplication, $\mathbb{G}_m$ acts morphically on the quotient $G\backslash\mathbb{V}$ in a natural way so that the quotient morphism $\mathbb{V} \to G\backslash\mathbb{V}$ is $\mathbb{G}_m$-equivariant. In the algebraic language, this means that $\Gamma(\mathbb{V}, \mathcal{O}_{\mathbb{V}})^G$ is a sub-graded ring of $\Gamma(\mathbb{V}, \mathcal{O}_{\mathbb{V}}) = \mathrm{Sym}^*_k V$, where V is the k-vector space of linear functions on $\mathbb{V}$.

The main problem in classical invariant theory was to explicitly determine the graded ring $\Gamma(\mathbb{V}, \mathcal{O}_{\mathbb{V}})^G$. Needless to say, this very difficult problem was never really solved. A more reasonable problem is getting an effective bound on the amount of calculation necessary to determined $\Gamma(\mathbb{V}\ \mathcal{O}_{\mathbb{V}})^G$ for given representations $\mathbb{V}$ of G.

The Hochster-Roberts Theorem provides very strong information about a part of the above process. A useful form of their theorem in this situation is

THEOREM 4.1. *There exist fundamental homogeneous invariants* $I_1, \ldots, I_d$ *and auxilary homogeneous invariants* $1 = F_1, \ldots, F_t$ *such that any invariant in* $\Gamma(\mathbb{V}, \mathcal{O}_{\mathbb{V}})^G$ *may be written uniquely in the form* $\sum G_i(I_1, \ldots, I_d) \cdot F_i$, *where the* G_i *are polynomials in* d-*variables.*

In other words, $\Gamma(\mathbb{V}, \mathcal{O}_{\mathbb{V}})^G$ is a free graded module of rank t over a graded polynomial sub-ring $k[I_1, \ldots, I_d]$ where d equals the dimension of $\Gamma(\mathbb{V}, \mathcal{O}_{\mathbb{V}})^G$.

Proof. By Max Noether's theorem (see [6]), we may choose algebraically independent homogeneous elements $I_1, \ldots, I_d$ in any finitely generated graded integral domain A over k, where d is the dimension of the domain so that the domain is a finite $k[I_1, \ldots, I_d]$-module. By a theorem of Macaulay-Serre, A is a Cohen-Macaulay ring if and only if A is free graded module. The present theorem is equivalent to Theorem 0.1 in this case.

Let $\varphi(s) = \sum \dim(\text{i-graded piece of } \Gamma(\mathbb{V}, \mathcal{O}_{\mathbb{V}})^G) s^i$ be the generating function of the graded ring $\Gamma(\mathbb{V}, \mathcal{O}_{\mathbb{V}})^G$. By Frobenius-Schur-Weyl methods, $\varphi(s)$ may be computed directly from G and $\mathbb{V}$ without determining the ring of invariants. The last theorem says that the generating function has a special form.

COROLLARY 4.2. *The generating function* $\varphi(s)$ *has the form*

$$\frac{\sum s^{f_i}}{\pi(1 - s^{g_j})}, \quad \textit{where } f_i = \deg F_i \textit{ for } 1 \leq i \leq t$$

$$\textit{and } g_j = \deg I_j \textit{ for } 1 \leq j \leq d.$$

Furthermore, $\varphi(s)$ *has a pole of order* d *at* $s = 1$. *In fact*

$$\lim_{s \to 1} \varphi(s)(s-1)^d = \sum f_i \Big/ (\pi g_j).$$

Proof. The first statement is an easy exercise once one knows Theorem 4.1. The other statement is apparent.

Unfortunately, the generating function $\varphi(s)$ only allows one to guess what the g_i may be. Hilbert described the problem of actually finding fundamental invariants as the hardest problem of invariant theory. This is shown quite dramatically by an interpretation of another result of Hochster and Roberts, which is

THEOREM 4.3. *With the notation of Theorem* 4.1, *for* $1 \leq i \leq t$,

$$\deg F_i \leq \sum \deg I_j,$$

where $1 \leq j \leq d$.

I will give the proof of this result later after which I will give a little discussion of the sharpening of the above theorems which is possible when G is finite. Before I begin the last part of this section, I will sketch Hilbert's method of finding fundamental invariants.

In [5], Hilbert established his famous finiteness theorem. He next founded geometric invariant theory in [6]. His Nullstellensatz was crucial to the proof of the next result.

THEOREM 4.4. (*Hilbert*) *Let* $f: \mathbb{V} \to G\backslash\mathbb{V}$ *be the quotient morphism. Let* Z *be the closed subset of* $G\backslash\mathbb{V}$ *defined by all homogeneous invariants of positive degree. Let* K *be the set of homogeneous invariants such that* $f^{-1}Z$ *is the zeroes of* f^*K. *Then, there is a finite subset* $J_1, \ldots, J_r$ *of* K *such that* $\Gamma(\mathbb{V}, \mathcal{O}_{\mathbb{V}})^G$ *is a finite graded module over the subring generated by the* J_i*'s. Furthermore, if* k *is infinite and* d *is the least common multiple of the degrees* d_i *of the* J_i, *then we may choose fundamental invariants* $I_1, \ldots, I_d$ *to be linear combinations of the* J_i^{d/d_i}.

This result was combined with his geometric description of the locus of null-forms, $f^{-1}Z$, intrinsically without knowing any invariants. Mumford has modernized and generalized his description in [10]. He proved that $f^{-1}Z$ consists of the vectors v in V such that the closure of orbit $G \cdot v$ contains 0. More precise further results may be found in [10], [11] and [9].

One possible approach to proving the Hochster-Roberts theorem in the linear case uses the

LEMMA 4.5. *Pull back via* f *gives natural isomorphism of* $\mathbb{G}_m$*-modules from* $H^i_Z(G\backslash\mathbb{V}, \mathcal{O}_{G\backslash\mathbb{V}})$ *to the* G*-invariants* $H^i_{f^{-1}Z}(\mathbb{V}, \mathcal{O}_{\mathbb{V}})^G$ *in* $H^i_{f^{-1}Z}(\mathbb{V}, \mathcal{O}_{\mathbb{V}})$.

As I will not need this trivial result, I will not prove it but it shows clearly how the difficulty of proving the Hochster-Roberts theorem is related to the codimension of $f^{-1}Z$ in $\mathbb{V}$ being smaller than the codimension of Z in $G\backslash\mathbb{V}$. The next result will be useful. It is

THEOREM 4.6. (*Hochster-Roberts*) (a) *The local cohomology groups* $H^i_Z(G\backslash\mathbb{V}, \mathcal{O}_{G\backslash\mathbb{V}})$ *are zero if* $i < \dim G\backslash\mathbb{V} = d$.

(b) *The positively graded part* $\bigoplus_{j\geq 0} H^d_Z(G\backslash\mathbb{V}, \mathcal{O}_{G\backslash\mathbb{V}})$ *is zero.*

Proof. Part (a) is a special case of Theorem 0.1 because it says that

$$\operatorname{depth}_Z \mathcal{O}_{G\backslash\mathbb{V}} = \operatorname{codimension}_{G\backslash\mathbb{V}} Z\,(= \dim G\backslash V).$$

In fact, the statement (a) itself implies that $G\backslash\mathbb{V}$ is Cohen-Macaulay because the worst singularity of the "cone" $G\backslash\mathbb{V}$ is at its vertex.

Part (b) follows from Theorem 3.4 in the same way as Theorem 0.1 follows from Theorem 3.3.

We will now see how Theorem 4.3 is equivalent to part (b) of Theorem 4.6.

Proof of Theorem 4.3. By Theorem 4.1, we know that the graded ring $\Gamma(\mathbb{V}, \mathcal{O}_V)^G$ is a free graded module with basis $F_1, \ldots, F_t$ over its subring $k[I_1, \ldots, I_d]$. Let $\pi: G\backslash\mathbb{V} \to \operatorname{Spec}(k[I_1, \ldots, I_d]) = Y$ be the morphism corresponding to the inclusion of the rings. Now, $\pi^{-1}\{0\} = Z$ and π is a finite flat morphism. Therefore, we have a natural isomorphism

$$H^d_Z(G\backslash\mathbb{V}, \mathcal{O}_{G\backslash\mathbb{V}}) \approx H^d_{\{0\}}(Y, \pi_* \mathcal{O}_{G\backslash\mathbb{V}}) \approx \oplus H^d_{\{0\}}(Y, F_j \cdot \mathcal{O}_Y).$$

Now, $H^d_{\{0\}}(Y, \mathcal{O}_Y)$ is graded and the highest degree of a nonzero homogeneous component is $-\sum \deg I_m$. Thus, the highest degree of a nonzero homogeneous component of $H^d_Z(G\backslash\mathbb{V}, \mathcal{O}_{G\backslash\mathbb{V}})$ is $\max \deg F_j - \sum \deg I_m$. Thus, the part (b) of Theorem 4.6 is equivalent to the inequality, $\deg F_j \leq \sum \deg I_m$ for all j, which is Theorem 4.3.

Remark. If G is a finite group and k is infinite, Dade (unpublished) has shown that in Theorem 4.1 one may choose the fundamental invariants $I_1, \ldots, I_d$ to have degree less than or equal to order (G) where $d = \dim \mathbb{V}$. In fact, he notes that we may take $I_i = \pi_{g \in G}\, g^* X_i$, where $X_1, \ldots, X_d$ are coordinate functions on $\mathbb{V}$, which are in general enough position so that I_j does not vanish on any component (which must be linear) of $I_1 = 0, \ldots, I_{j-1} = 0$ for all j. See [12] for background material on invariants of finite groups.

Furthermore, for finite groups, the inequality of Theorem 4.3 may be strengthened to an equality. Let $H^d_{\{0\}}(\mathbb{V}, \mathcal{O}_V)_m$ be the m^{th}-graded piece of $H^d_{\{0\}}(\mathbb{V}, \mathcal{O}_V)$. By Serre duality, we have a perfect pairing

$$H^d_{\{0\}}(\mathbb{V}, \mathcal{O}_V)_m \otimes \mathrm{Sym}^{-m-d} V \to H^d_{\{0\}}(\mathbb{V}, \mathcal{O}_V)_{-d}$$

where $\mathrm{Sym}^* V$ is the $*$-th symmetric product of the linear functions V on $\mathbb{V}$ and $H^d_{\{0\}}(\mathbb{V}, \mathcal{O}_V)_{-d}$ is a line.

As G acts on the line $H^d_{\{0\}}(\mathbb{V}, \mathcal{O}_V)_{-d}$ by the determinant $(\rho)^{-1}$, where $\rho\colon G \to GL(\mathbb{V})$ is the given representation and the pairing is equivariant, we have a perfect pairing,

$$H^d_{\{0\}}(\mathbb{V}, \mathcal{O}_V)^G_m \otimes (\mathrm{Sym}^{-m-d} V \otimes H^d_{\{0\}}(\mathbb{V}, \mathcal{O}_V)^{\wedge}_{-d})^G \to k.$$

As $H^d_{\{0\}}(\mathbb{V}, \mathcal{O}_V)^G_m \approx H^d_Z(G\backslash\mathbb{V}, \mathcal{O}_{G\backslash\mathbb{V}})_m$, the above pairing shows that, if $n = \max m$ such that $H^d_Z(G\backslash\mathbb{V}, \mathcal{O}_{G\backslash\mathbb{V}})_m \neq 0$, then $-d - n = \min i$ such that

$$(\mathrm{Sym}^i V \otimes H^d_{\{0\}}(\mathbb{V}, \mathcal{O}_V)_{-d})^G \neq 0;$$

i.e., minimal degree p such that $\mathrm{Sym}^p V$ contains a nonzero semi-invariant of weight $\det(\rho)^{-1}$. Thus,

$$-d - \max \deg I_j + \sum \deg I_m = p;$$

or, rather,

$$\max \deg F_j = \sum \deg I_m - d - p \leq \sum \deg I_m - d.$$

The reader may prefer another proof of the last equation. We have a canonical isomorphism $\pi_* \Omega^d_V \approx \mathcal{H}om_{\mathcal{O}_Y}(\pi_* \mathcal{O}_V, \Omega^d_Y)$ [3] where we are using the notation of the proof of Theorem 4.3. As $\Omega^d_V \approx \Lambda^d V \otimes_k \mathcal{O}_V$ and $\Omega^d_Y \approx dI_1 \wedge \ldots \wedge dI_d \cdot \mathcal{O}_Y$, the duality isomorphism gives a direct relationship between covariants in $\Lambda^d V \otimes_k k[V]$ and covariants in $\mathrm{Hom}_{k[I_1,\ldots,I_d]}(k[V], I_1 \cdots I_d \cdot k[I_1,\ldots,I_d])$. From this one may deduce more general relationships of the above types.

Remark. The reduction to the graded case in Hochster and Roberts' proof gives valuable information about the graded approximation $A = \oplus\, n_i / n_{i+1}$ to the local ring $\mathcal{O}_{G\backslash X,x}$ at any point x of the quotient variety $G\backslash X$, where n_i is the integral closure of the i-th power m^i of the maximal ideal m of $\mathcal{O}_{G\backslash X,x}$. In characteristic zero, one may show that A is up to extension of scalars just the ring of invariants of a reductive subgroup H of G acting on a representation of H. One takes H to be the stabilizer of a point of a minimal G-orbit lying over x. The representation of H is given by the action of H on the normal bundle of X along the orbit. When one works out what this means in practice, one may use the information from the smaller subgroup H and its smaller representation. See [8] for more indication of this procedure.

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