AN L_p ANALYTIC FOURIER-FEYNMAN TRANSFORM

G. W. Johnson and D. L. Skoug

0. INTRODUCTION

In [1] Brue introduced an L_1 analytic Fourier-Feynman transform. In [3] Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform. In this paper we study an L_p analytic Fourier-Feynman transform for $1 \le p \le 2$. The resulting theorems extend the theory substantially (even in the cases $p = 1$ and $p = 2$) and indicate relationships between the L_1 and L_2 theories.

Before giving the basic definitions we fix some notation. $\mathbb{R}^n$ will denote n-dimensional Euclidean space, $\mathbb{C}$ the complex numbers and $\mathbb{C}^+$ the complex numbers with positive real part. $C_0(\mathbb{R}^n)$ will denote the $\mathbb{C}$-valued continuous functions on $\mathbb{R}^n$ which vanish at ∞. Wiener space, $C[a,b]$, will denote the $\mathbb{R}$-valued continuous functions on $[a,b]$ that vanish at a. Integration over $C[a,b]$ will always be with respect to Wiener measure. If Y and Z are Banach spaces, $L(Y,Z)$ will denote the space of continuous linear operators from Y to Z.

In this paper, as in [3], the term *Wiener measurable* will always mean measurable with respect to the uncompleted Wiener measure; that is measurable with respect to the σ-algebra of Borel sets in $C[a,b]$.

Definition. Let F be a functional such that the Wiener integral

$$J(\lambda) = \int_{C[a,b]} F(\lambda^{-1/2} x)\, dx \tag{0.1}$$

exists for almost all real $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in the half-plane $\mathbb{C}^+$ such that $J(\lambda) = J^*(\lambda)$ for almost all real $\lambda > 0$, then we define this *essential analytic extension* of J to be the *analytic Wiener integral of F over* $C[a,b]$ *with parameter* λ and we write

$$\int_{C[a,b]}^{\text{anw}_\lambda} F(x)\, dx = J^*(\lambda) \qquad \text{for } \lambda \in \mathbb{C}^+. \tag{0.2}$$

Notation. For $\lambda \in \mathbb{C}^+$ and $y \in C[a,b]$ let

$$(T_\lambda F)(y) \equiv \int_{C[a,b]}^{\text{anw}_\lambda} F(x + y)\, dx. \tag{0.3}$$

Terminology. We shall say that two functionals F and G are equal s-*almost*

Received November 18, 1976. Revision received February 1, 1977.

Michigan Math. J. 26 (1979).

everywhere if for each $\rho > 0$, $F(\rho x) = G(\rho x)$ for almost all $x \in C[a,b]$; in other words, $F(x) = G(x)$ except for a scale-invariant null set. We denote this equivalence relation by $F \approx G$. The need for considering this equivalence relation is discussed in section 1 of [3]; also see the comments at the end of this introduction.

Notation. Given a number p such that $1 \le p \le \infty$, p and p′ will always be related by $1/p + 1/p' = 1$.

Definition. Let $1 < p \le 2$. Let $\{H_n\}$ and H be measurable functions such that for each $\rho > 0$,

$$\lim_{n\to\infty} \int_{C[a,b]} |H_n(\rho y) - H(\rho y)|^{p'} dy = 0. \tag{0.4}$$

Then we write

$$\underset{n\to\infty}{\text{l.i.m.}}\, (w_s^{p'})\, H_n \approx H \tag{0.5}$$

and we call H the scale invariant limit in the mean of order p′ of H_n over $C[a,b]$. A similar definition is understood when n is replaced by a continuously varying parameter.

Definition. Let q be a nonzero real number. For $1 < p \le 2$ we define the L_p analytic Fourier-Feynman transform of F, which we denote by $T_q^{(p)} F$, by the formula

$$(T_q^{(p)} F)(y) \equiv \underset{\substack{\lambda\to -iq \\ \lambda\in \mathbb{C}^+}}{\text{l.i.m.}}\, (w_s^{p'})(T_\lambda F)(y) \tag{0.6}$$

whenever this limit exists (recall that $T_\lambda F$ is given by (0.3)). Let F be a functional on Wiener space such that $(T_\lambda F)(y)$ exists in $\mathbb{C}^+$ for s-almost every y. We define the L_1 analytic Fourier-Feynman transform of F, which we denote by $T_q^{(1)} F$, as that functional (if it exists) on Wiener space such that

$$(T_q^{(1)} F)(y) = \lim_{\substack{\lambda\to -iq \\ \lambda\in \mathbb{C}^+}} (T_\lambda F)(y) \tag{0.7}$$

for s-almost every y. We note that for $1 \le p \le 2$, $T_q^{(p)} F$ is defined only s-almost everywhere.

Remarks. (i) In view of (0.6) it would seem natural and desirable to define $T_q^{(1)} F$ by requiring that for each $\rho > 0$

$$\lim_{\substack{\lambda\to -iq \\ \lambda\in \mathbb{C}^+}} [\operatorname*{ess\,sup}_{y\in C[a,b]} |(T_\lambda F)(\rho y) - (T_q^{(1)} F)(\rho y)|] = 0. \tag{0.8}$$

Unfortunately (0.8) doesn't even hold for any $\rho > 0$ for a functional as simple as $F(x) = \chi_{[-1,1]}(x(b))$.

(ii) $T_q^{(2)} F$ agrees with the L_2 analytic Fourier-Feynman transform as defined by Cameron and Storvick by equation (0.3) on page 3 of [3].

(iii) Our definition of $T_q^{(1)}F$ is more restrictive than that given by Brue in 1972 in that we require (0.7) to hold s-almost everywhere rather than just a.e.. However, for all the functionals F which Brue considered, we are able to show that $T_q^{(1)}F$ exists in this stronger sense.

Next we briefly describe the type of functionals F for which we will establish the existence of the L_p analytic Fourier-Feynman transform $T_q^{(p)}F$. Let n be a positive integer and let $a = t_0 < t_1 < \dots < t_n \le b$. For $1 \le p \le 2$ let $\mathscr{A}_n^{(p)}$ be the space of functionals F of the form $F(x) = f(x(t_1), \dots, x(t_n))$ s-almost everywhere where $f \in L_p(\mathbb{R}^n)$ and f is Borel measurable. In section 1 we obtain the existence of $T_q^{(p)}F$ for F in $\mathscr{A}_n^{(p)}$.

Next let $\Delta_n = \{(t_1, \dots, t_n) | a = t_0 < t_1 < \dots < t_n \le b\}$. For $1 \le p \le 2$ and $r \in (2p/(2p-1), \infty]$ let $L_{pr}(\Delta_n \times \mathbb{R}^n)$ be the space of all $\mathbb{C}$-valued functions f defined and Borel measurable on $\Delta_n \times \mathbb{R}^n$ such that $f(t_1, \dots, t_n; \cdot, \dots, \cdot)$ is in $L_p(\mathbb{R}^n)$ for almost all $(t_1, \dots, t_n) \in \Delta_n$ and $\|f(t_1, \dots, t_n; \cdot, \dots, \cdot)\|_p$ is in $L_r(\Delta_n)$. Let $\mathscr{S}_{n,r}^{(p)}$ be the space of functionals F of the form

$$F(x) = \int_{\Delta_n}^{(n)} \int f(t_1, \dots, t_n; x(t_1), \dots, x(t_n))\, dt_1 \dots dt_n$$

s-almost everywhere where $f \in L_{pr}(\Delta_n \times \mathbb{R}_n)$. In section 2 we show that $T_q^{(p)}F$ exists for F in $\mathscr{S}_{n,r}^{(p)}$. In section 3 we build a larger space $\mathscr{S}_r^{(p)}$ by using sums of functionals chosen from each of the spaces $\mathscr{S}_{n,r}^{(p)}$ and show that $T_q^{(p)}F$ exists for F in $\mathscr{S}_r^{(p)}$. Finally in section 4 we show that if Φ is an entire function of order less than 2p and if $\theta(t,u)$ is in $L_{pr}([a,b] \times \mathbb{R})$ then $T_q^{(p)}F$ exists for the functional

$$F(x) = \Phi\left(\int_a^b \theta(t, x(t))\, dt\right).$$

At this time we indicate briefly how our results relate to previous theorems. In the case $p = 2$, the situation studied by Cameron and Storvick, our results are slightly stronger; whereas they require $f(t_1, \dots, t_n; u_1, \dots, u_n) \in L_{2\infty}(\Delta \times \mathbb{R}^n)$ we require $f(t_1, \dots, t_n; u_1, \dots, u_n) \in \bigcup_{r>4/3} L_{2r}(\Delta_n \times \mathbb{R}^n)$. In particular we may have $f(t_1, \dots, t_n; u_1, \dots, u_n) \in L_{22}(\Delta_n \times \mathbb{R}^n) = L_2(\Delta_n \times \mathbb{R}^n)$. In the case $1 < p < 2$ our results are of course new. In the case $p = 1$, the situation studied by Brue, we obtain the existence of $T_q^{(1)}F$ for a much larger class of functionals F.

Finally we want to comment briefly on the need for considering the equivalence relation $\approx$. In [3; pp. 6 and 7], Cameron and Storvick exhibit two functionals F and G such that $F(x) = G(x)$ almost everywhere but $(T_qF)(y) \neq (T_qG)(y)$ on a set of positive Wiener measure. It is easy to see that for all $p \in [1,2]$,

$$(T_q^{(p)}F)(y) \neq (T_q^{(p)}G)(y)$$

on a set of positive Wiener measure. However the transformation $T_q^{(p)}$ preserves equivalence classes based on the relation $\approx$ as we see in the following theorem which is a restatement of Theorem 1 of [3; p. 5] in our setting.

THEOREM 0.1. *Let* q *be a non-zero real number and let* $1 \le p \le 2$ *be given. Let* F_1 *and* F_2 *be Wiener measurable functionals such that* $F_1 \approx F_2$. *Then the following statements hold:*

(i) *If* $T_q^{(p)} F_1$ *exists then* $T_q^{(p)} F_2$ *exists and* $T_q^{(p)} F_1 \approx T_q^{(p)} F_2$.

(ii) *If for s-almost every* y, $\int_{C[a,b]}^{\text{anw}_\lambda} F_1(x+y)\,dx$ *exists for* $\lambda \in \mathbb{C}^+$, *then the corresponding analytic Wiener integral of* F_2 *exists for s-almost every* y *and we have*

$$\int_{C[a,b]}^{\text{anw}_\lambda} F_1(x+y)\,dx \approx \int_{C[a,b]}^{\text{anw}_\lambda} F_2(x+y)\,dx \qquad \text{for all } \lambda \in \mathbb{C}^+. \tag{0.9}$$

(iii) *For s-almost every* y

$$F_1(\rho x + y) = F_2(\rho x + y) \tag{0.10}$$

for almost all $\rho > 0$ *and almost all* $x \in C[a,b]$.

1. THE TRANSFORM $T_q^{(p)}$ APPLIED TO FUNCTIONALS F CONTAINED IN $\mathscr{A}_n^{(p)}$

We begin this section by developing three preliminary lemmas that play key roles throughout this paper.

Lemma 1.1. *Let* $1 \le p \le 2$. *Let n be a positive integer and let*

$$a = t_0 < t_1 < \ldots < t_n \le b.$$

Given a nonzero complex number λ *with nonnegative real part and f in* $L_p(\mathbb{R}^n)$, *let*

$$(K_\lambda f)(w_1, \ldots, w_n) \equiv \lambda^{n/2}\gamma \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} f(u_1, \ldots, u_n) \exp\left(-\frac{\lambda}{2} \sum_{1}^{n} \frac{[(u_j - u_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}} \right) du_1 \ldots du_n \tag{1.1}$$

where $\gamma \equiv \gamma(t) \equiv [(2\pi)^n (t_1 - a) \ldots (t_n - t_{n-1})]^{-1/2}$. *Then* K_λ *is in* $L(L_p(\mathbb{R}^n), L_{p'}(\mathbb{R}^n))$ *and*

$$\|K_\lambda\| \le (|\lambda|^{n/2}\gamma)^{(2-p)/p}. \tag{1.2}$$

Furthermore when $p = 1$, $K_\lambda f$ *is in* $C_0(\mathbb{R}^n)$.

Remarks. (i) When n is odd we always choose $\lambda^{n/2}$ with nonnegative real part. (ii) When $1 < p \le 2$ and $\operatorname{Re}\lambda = 0$ the integral in (1.1) should be interpreted in the mean just as in the theory of the L_p Fourier transform [7]. (iii) $K_\lambda f$ is also a function of $(t_1, \ldots, t_n)$; however in this section we will be considering the t_j's as fixed and so we will suppress reference to them.

Proof. We will first treat the extreme cases $p = 1$ and $p = 2$. The intermediate cases $1 < p < 2$ will be handled by interpolation via the M. Riesz convexity theorem.

$p = 1$: The result is clear in this case since for all $(w_1, \ldots, w_n)$,

$$|(K_\lambda f)(w_1, \ldots, w_n)| \leq |\lambda|^{n/2}\gamma\, ||f||_1.$$

$p = 2$: This case was established by Cameron and Storvick in Lemma 1 of [3].

$1 < p < 2$: Fix $\lambda \neq 0$ such that $\operatorname{Re}\lambda \geq 0$. In the terminology of the M. Riesz convexity theorem as given in [7; Theorem 1.3, p. 179] we have shown that K_λ is of type $(1,\infty)$ with $(1,\infty)$ norm dominated by $|\lambda|^{n/2}\gamma$ and of type $(2,2)$ with $(2,2)$ norm dominated by 1. Applying the convexity theorem we have that K_λ is in $L(L_p, L_{p'})$ with

$$||K_\lambda|| \leq (|\lambda|^{n/2}\gamma)^{1-2/p'}(1)^{2/p'} = (|\lambda|^{n/2}\gamma)^{(2-p)/p}$$

which establishes (1.2).

Furthermore when $p = 1$, a standard argument shows that $K_\lambda f$ is in $C_0(\mathbb{R}^n)$.

LEMMA 1.2. *Let* $1 < p \leq 2$, *and* q *be a nonzero real number, and let* $f \in L_p(\mathbb{R}^n)$. *Then*

$$||K_\lambda f - K_{-iq} f||_{p'} \to 0 \quad \text{as } \lambda \to -iq \text{ through } \mathbb{C}^+. \tag{1.3}$$

Proof. First consider the substitutions

$$u_j' = \frac{u_j - u_{j-1}}{\sqrt{t_j - t_{j-1}}}, \quad \text{and} \quad w_j' = \frac{w_j - w_{j-1}}{\sqrt{t_j - t_{j-1}}}, \qquad j = 1, 2, \ldots, n.$$

Then $u_j = \sum_{k=1}^{j} \sqrt{t_k - t_{k-1}}\, u_k'$ and $w_j = \sum_{k=1}^{j} \sqrt{t_k - t_{k-1}}\, w_k'$ for $j = 1, 2, \ldots, n$. Making these substitutions on the right hand side of (1.1) we obtain

$$\begin{aligned}(K_\lambda f)(w_1, \ldots, w_n) &= (K_\lambda f)\Big(\sqrt{t_1 - a}\, w_1', \ldots, \sum_{k=1}^{n} \sqrt{t_k - t_{k-1}}\, w_k'\Big) \\ &= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} f\Big(\sqrt{t_1 - a}\, u_1', \ldots, \sum_{k=1}^{n} \sqrt{t_k - t_{k-1}}\, u_k'\Big) \\ &\qquad \exp\left(-\frac{\lambda}{2}\sum_{j=1}^{n} [u_j' - w_j']^2\right) du_1' \ldots du_n'\end{aligned}$$

which, as $\lambda \to -iq$ (using [5; Lemma 1.2, p. 100]) converges in $L_{p'}(\mathbb{R}^n)$ as a function of $(w_1', \ldots, w_n')$ to the function

$$\left(\frac{-iq}{2\pi}\right)^{n/2} \int_{-\infty}^{\infty} (n) \int_{\infty}^{\infty} f\left(\sqrt{t_1 - a}\, u_1', \ldots, \sum_{k=1}^{n} \sqrt{t_k - t_{k-1}}\, u_k'\right)$$

$$\exp\left(\frac{qi}{2} \sum_{j=1}^{n} [u_j' - w_j']^2\right) du_1' \ldots du_n'$$

$$= \lim_{A\to\infty} (-qi)^{n/2} \gamma \int_{-A}^{A} \int_{-A-u_1}^{A-u_1} \cdots \int_{-A-u_{n-1}}^{A-u_{n-1}} f(u_1, \ldots, u_n)$$

$$\exp\left(\frac{qi}{2} \sum_{1}^{n} \frac{[(u_j - u_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}\right) du_n \ldots du_1$$

$$= \lim_{A\to\infty} (-qi)^{n/2} \gamma \int_{-A}^{A} (n) \int_{-A}^{A} f(u_1, \ldots, u_n)$$

$$\exp\left(\frac{qi}{2} \sum_{1}^{n} \frac{[(u_j - u_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}\right) du_n \ldots du_1$$

$$= (K_{-iq} f)(w_1, \ldots, w_n),$$

where the limits with respect to A are taken in the $L_{p'}$-norm.

LEMMA 1.3. *Let* $f \in L_1(\mathbb{R}^n)$ *and let* $K_\lambda f$ *be given by* (1.1). *Then*

(i) *as elements of* $C_0(\mathbb{R}^n)$, $K_\lambda f$ *converges weakly to* $K_{-iq} f$ *as* $\lambda \to -iq$ *through* $\mathbb{C}^+$,

(ii) $K_\lambda f$ *converges pointwise to* $K_{-iq} f$ *in* $\mathbb{R}^n$ *as* $\lambda \to -iq$.

Proof. (ii) is a direct consequence of the Dominated Convergence Theorem. To establish (i) let $\mu \in M(\mathbb{R}^n)$, the dual of $C_0(\mathbb{R}^n)$. We need to show that

$$\lim_{\lambda\to -iq} \int_{\mathbb{R}^n} (K_\lambda f)(w_1, \ldots, w_n)\, d\mu(w_1, \ldots, w_n)$$
$$= \int_{\mathbb{R}^n} (K_{-iq} f)(w_1, \ldots, w_n)\, d\mu(w_1, \ldots, w_n). \tag{1.4}$$

But this follows quite easily using the Dominated Convergence Theorem once one substitutes for $K_\lambda f$ and $K_{-iq} f$ in (1.4) using (1.1).

Definition. Let n be a positive integer and let $a = t_0 < t_1 < \ldots < t_n \le b$. For $1 \le p < \infty$ let $\mathscr{A}_n^{(p)}$ be the space of functionals F which can be expressed in the form

$$F(x) = f(x(t_1), \ldots, x(t_n)) \tag{1.5}$$

s-almost everywhere on C [a,b] where $f \in L_p(\mathbb{R}^n)$ and f is Borel measurable. Let $\mathscr{A}_n^{(\infty)}$ be the space of functionals F which can be expressed in the form (1.5) s-almost everywhere on C [a,b] with f Borel measurable and in $C_0(\mathbb{R}^n)$.

In our first theorem we show that $T_q^{(p)} F$ exists for F in $\mathscr{A}_n^{(p)}$.

THEOREM 1.1. *Let* $1 \leq p \leq 2$, *let* q *be a nonzero real number and let* $F \in \mathscr{A}_n^{(p)}$ *be given by* (1.5). *Then the* L_p-analytic *Fourier-Feynman transform of* F, $T_q^{(p)} F$ *exists, is in* $\mathscr{A}_n^{(p')}$ *and is given by the formula*

$$(T_q^{(p)} F)(y) \approx (K_{-iq} f)(y(t_1), \ldots, y(t_n)). \tag{1.6}$$

Proof. We first note, using the Fubini Theorem, that f Borel measurable implies that $K_\lambda f$ is Borel measurable for each $\lambda \in \mathbb{C}^+$; this in turn assures us that $K_{-iq} f$ is Borel measurable. In addition, using Lemma 1.1, we see that $K_{-iq} f$ is in $L_{p'}(\mathbb{R}^n)$.

We will use Morera's Theorem to show that for each $y \in C[a,b]$,

$$(K_\lambda f)(y(t_1), \ldots, y(t_n))$$

is an analytic function of λ in $\mathbb{C}^+$. First an application of the Dominated Convergence Theorem shows that $(K_\lambda f)(y(t_1), \ldots, y(t_n))$ is continuous in $\mathbb{C}^+$; an appropriate dominating function is obtained almost exactly as in the following argument and so the argument will be omitted here. Now let $\triangle$ be a triangular path in $\mathbb{C}^+$. We need only show that $\int_\triangle (K_\lambda f)(y(t_1), \ldots, y(t_n))\, d\lambda = 0$. But this will clearly follow from the Cauchy Integral Theorem if we can justify moving the integral with respect to λ inside the other integrals defining $K_\lambda f$ (see equation (1.1)). Let $D \equiv \sup\{|\lambda| : \lambda \in \triangle\}$ and $E \equiv \inf\{\operatorname{Re}\lambda : \lambda \in \triangle\}$. Then the function

$$D^{n/2} \gamma |f(u_1, \ldots, u_n)| \exp\left\{(-E/2) \sum_{j=1}^{n} \frac{[(u_j - u_{j-1}) - (y(t_j) - y(t_{j-1}))]^2}{t_j - t_{j-1}}\right\}$$

dominates $(K_\lambda f)(y(t_1), \ldots, y(t_n))$ and is integrable with respect to $u_1, \ldots, u_n$ and λ. Thus the use of the Fubini Theorem is justified and we have established that

$$\int_{C[a,b]}^{\text{anw}_\lambda} F(x+y)\, dx \equiv (T_\lambda F)(y)$$

exists throughout $\mathbb{C}^+$ and equals $(K_\lambda f)(y(t_1), \ldots, y(t_n))$ for s-almost every y in $C[a,b]$.

To establish the existence of $T_q^{(p)} F$ we will need to consider two cases; (a) $1 < p \leq 2$ and (b) $p = 1$.

(a) Fix $1 < p \leq 2$. To show that $T_q^{(p)} F$ exists and is given by (1.6) it suffices to show that for each $\rho > 0$

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} \int_{C[a,b]} |(K_\lambda f)(\rho y(t_1), \ldots, \rho y(t_n)) - (K_{-iq} f)(y(t_1), \ldots, y(t_n))|^{p'}\, dy = 0.$$

But

$$\int_{C[a,b]} |(K_\lambda f)(\rho y(t_1), \ldots, \rho y(t_n)) - (K_{-iq} f)(\rho y(t_1), \ldots, \rho y(t_n))|^{p'} dy$$

$$= (\gamma/\rho^n) \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} |(K_\lambda f)(u_1, \ldots, u_n) - (K_{-iq} f)(u_1, \ldots, u_n)|^{p'}$$

$$\exp\left\{(-1/2\rho^2) \sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\} du_1 \ldots du_n$$

$$\leq (\gamma/\rho^n)(\|(K_\lambda f)(\cdot, \ldots, \cdot) - (K_{-iq} f)(\cdot, \ldots, \cdot)\|_{p'})^{p'}$$

which goes to zero as $\lambda \to -iq$ through $\mathbb{C}^+$ by Lemma 1.2. Hence for $1 < p \leq 2$, $T_q^{(p)} F$ exists, belongs to $\mathscr{A}_n^{(p')}$, and is given by (1.6).

(b) Let $p = 1$. In this case the fact that $T_\lambda^{(1)} F$ exists and is given by (1.6) follows easily from Lemma 1.3.

Next we obtain an inverse transform theorem for F in $\mathscr{A}_n^{(p)}$.

THEOREM 1.2. *Let* $1 \leq p \leq 2$, *let* q *be a nonzero real number and let* $F \in \mathscr{A}_n^{(p)}$ *be given by* (1.5). *Then for each* $\rho > 0$ (*recall that* T_λ *is defined by* (0.3))

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} \int_{C[a,b]} |T_{\bar{\lambda}} T_\lambda F(\rho y) - F(\rho y)|^p dy = 0. \tag{1.7}$$

Furthermore,

$$T_{\bar{\lambda}} T_\lambda F \to F \text{ s-almost everywhere as } \lambda \to -iq. \tag{1.8}$$

Proof. We first note that for all λ in $\mathbb{C}^+$ (see pages 524 and 525 of [2] where integrals similar to those below are evaluated),

$$(K_{\bar{\lambda}} K_\lambda f)(v_1, \ldots, v_n) = |\lambda|^n \gamma^2 \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty}$$

$$\exp\left\{(-\bar{\lambda}/2) \sum_{1}^{n} \frac{[(w_j - w_{j-1}) - (v_j - v_{j-1})]^2}{t_j - t_{j-1}}\right\}$$

$$\left[\int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} f(u_1, \ldots, u_n) \exp\left\{(-\lambda/2) \sum_{1}^{n} \frac{[(u_j - u_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}\right\}\right.$$

$$\left. du_n \ldots du_1 \right] dw_n \ldots dw_1$$

$$= |\lambda|^n \gamma^2 \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} f(u_1, \ldots, u_n) \left[\int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty}\right.$$

$$\exp\left\{(-\bar{\lambda}/2) \sum_{1}^{n} \frac{[(w_j - w_{j-1}) - (v_j - v_{j-1})]^2}{t_j - t_{j-1}}\right\}$$

$$\exp\left\{(-\lambda/2)\sum_1^n \frac{[(w_j - w_{j-1}) - (u_j - u_{j-1})]^2}{t_j - t_{j-1}}\right\} dw_n \ldots dw_1 \Bigg] du_n \ldots du_1$$

$$= \frac{|\lambda|^n}{(2\pi)^n (t_1 - a) \ldots (t_n - t_{n-1})} \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} f(u_1, \ldots, u_n) \prod_{j=1}^n \left[\frac{\pi(t_j - t_{j-1})}{\operatorname{Re}\lambda}\right]^{1/2}$$

$$\exp\left\{(-|\lambda|^2/4\operatorname{Re}\lambda)\sum_1^n \frac{[(u_j - u_{j-1}) - (v_j - v_{j-1})]^2}{t_j - t_{j-1}}\right\} du_n \ldots du_1$$

$$= (f * \phi_\varepsilon)(v_1, \ldots, v_n)$$

where

$$\phi(v_1, \ldots, v_n) \equiv \prod_{j=1}^n [2\pi(t_j - t_{j-1})]^{-1/2} \exp\left\{\frac{-(v_j - v_{j-1})^2}{2(t_j - t_{j-1})}\right\},$$

$$\varepsilon \equiv \sqrt{2\operatorname{Re}\lambda}\,/|\lambda|, \quad \text{and} \quad \phi_\varepsilon(v_1, \ldots, v_n) \equiv \frac{1}{\varepsilon^n}\phi\left(\frac{v_1}{\varepsilon}, \ldots, \frac{v_n}{\varepsilon}\right).$$

Now

$$\int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \phi(v_1, \ldots, v_n)\, dv_1 \ldots dv_n = 1 \quad \text{and}$$

$$\phi(v_1, \ldots, v_n) > 0 \qquad \text{for all } (v_1, \ldots, v_n)$$

and so using [7; Theorem 1.18, page 10] we obtain that

(1.9) $$\|f * \phi_\varepsilon - f\|_p \to 0 \qquad \text{as } \varepsilon \to 0 \ (\textit{i.e.}, \text{ as } \lambda \to -iq).$$

But (1.7) now follows easily from (1.9) and the observation that for each $\rho > 0$

$$\int_{C[a,b]} |T_{\bar\lambda} T_\lambda F(\rho y) - F(\rho y)|^p\, dy$$

$$= (\gamma/\rho^n) \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} |(K_{\bar\lambda} K_\lambda f)(v_1, \ldots, v_n) - f(v_1, \ldots, v_n)|^p$$

$$\exp\left\{\frac{-1}{2\rho^2}\sum_1^n \frac{(v_j - v_{j-1})^2}{t_j - t_{j-1}}\right\} dv_1 \ldots dv_n$$

$$= (\gamma/\rho^n) \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} |(f * \phi_\varepsilon)(v_1, \ldots, v_n) - f(v_1, \ldots, v_n)|^p$$

$$\exp\left\{\frac{-1}{2\rho^2}\sum_1^n \frac{(v_j - v_{j-1})^2}{t_j - t_{j-1}}\right\} dv_1 \ldots dv_n$$

$$\le (\gamma/\rho^n)\|f * \phi_\varepsilon - f\|_p.$$

Next using [7; Theorem 1.25, page 13] it follows that the function

$$(K_{\bar\lambda} K_\lambda f)(v_1, \ldots, v_n) = (f * \phi_\varepsilon)(v_1, \ldots, v_n)$$

converges pointwise to the function $f(v_1, \ldots, v_n)$ as $\varepsilon \to 0$ (*i.e.*, as $\lambda \to -iq$). (There is a little work involved in showing that the hypotheses are satisfied.) Hence for each $\rho > 0$ and for almost all y in $C[a,b]$

$$\begin{aligned} T_{\bar\lambda} T_\lambda F(\rho y) &= K_{\bar\lambda} K_\lambda f(\rho y(t_1), \ldots, \rho y(t_n)) \\ &= (f * \phi_\varepsilon)(\rho y(t_1), \ldots, \rho y(t_n)) \to f(\rho y(t_1), \ldots, y(t_n)) = F(\rho y) \end{aligned}$$

which establishes (1.8) and concludes the proof of the theorem.

Remark. In [1], Brue, for a more restricted class of functionals F, showed that $T_{-q}^{(1)} T_q^{(1)} F = F$. Actually he only considered the case $q = 1$ but clearly his results are valid for all real $q \neq 0$. In order to obtain this result he put additional assumptions of f guaranteeing that $(K_\lambda f)(w_1, \ldots, w_n)$ would be in $L_1(\mathbb{R}^n)$. That is to say, he restricted F so that $T_q^{(1)} F$ would be in $\mathscr{A}_n^{(1)}$ rather than only in $\mathscr{A}_n^{(\infty)}$; and thus it made sense to apply $T_{-q}^{(1)}$ to the functional $T_q^{(1)} F$. If for $1 \le p < 2$ we would put additional assumptions on f guaranteeing that $(K_\lambda f)(w_1, \ldots, w_n)$ would be in $L_p(\mathbb{R}^n)$, then $T_q^{(p)} F$ would be in $\mathscr{A}_n^{(p)}$ from which it follows that $T_{-q}^{(p)} T_q^{(p)} F \approx F$. However for $p = 2$, $p' = 2$ and so for F in $\mathscr{A}_n^{(2)}$ we see, using Theorem 1.1, that $T_q^{(2)} F$ is in $\mathscr{A}_n^{(2)}$. Thus we have the following Theorem.

THEOREM 1.3. (Theorem 3 on page 16 of [3].) *Let* q *be a non-zero real number and let* F *be in* $\mathscr{A}_n^{(2)}$. *Then* $T_{-q}^{(2)} T_q^{(2)} F \approx F$.

2. THE TRANSFORM $T_q^{(p)}$ APPLIED TO FUNCTIONALS F CONTAINED IN $\mathscr{S}_{n,r}^{(p)}$

For $n \ge 1$ let $\Delta_n = \{(t_1, \ldots, t_n): a < t_1 < \ldots < t_n \le b\}$. For $1 \le p \le \infty$ and $1 \le r \le \infty$ let $L_{pr}(\Delta_n \times \mathbb{R}^n)$ be the space of all $\mathbb{C}$-valued functions f defined and Borel measurable on $\Delta_n \times \mathbb{R}^n$ such that $f(t_1, \ldots, t_n; \cdot, \ldots, \cdot)$ is in $L_p(\mathbb{R}^n)$ (In case $p = \infty$ we require $f(t_1, \ldots, t_n; \cdot, \ldots, \cdot)$ to be in $C_0(\mathbb{R}^n)$.) for almost all $(t_1, \ldots, t_n) \in \Delta_n$ and $\|f(t_1, \ldots, t_n; \cdot, \ldots, \cdot)\|_p$ is in $L_r(\Delta_n)$. For $f \in L_{pr}(\Delta_n \times \mathbb{R}^n)$ let

$$N_n(f) \equiv \|f\|_{pr} = \left\{ \int_{\Delta_n} (n) \int \|f(t_1, \ldots, t_n; \cdot, \ldots, \cdot)\|_p^r \, dt_1 \ldots dt_n \right\}^{1/r}. \tag{2.1}$$

For $n \ge 1$, let $\mathscr{S}_{n,r}^{(p)}$ be the space of functionals F such that for some $f \in L_{pr}(\Delta_n \times \mathbb{R}^n)$

$$F(x) = \int_{\Delta_n} (n) \int f(t_1, \ldots, t_n; x(t_1), \ldots, x(t_n)) \, dt_1 \ldots dt_n \tag{2.2}$$

for s-almost all x. The function f is called a defining kernel for F. For notational purposes we will let $\mathscr{S}_{0,r}^{(p)}$ denote the constant C-valued functionals.

Remarks. (i) In what follows we will have $1 \le p \le 2$ and $r \in (2p/(2p-1), \infty]$ and p' and r' will satisfy $1/p + 1/p' = 1$ and $1/r + 1/r' = 1$ as usual.

(ii) In the case $p = 2$, our basic assumption on f will always be that $f(t_1, \ldots, t_n; u_1, \ldots, u_n) \in \bigcup_{r>4/3} L_{2r}(\Delta_n \times \mathbb{R}^n)$. In particular we may have

$$f(t_1, \ldots, t_n; u_1, \ldots, u_n) \in L_{22}(\Delta_n \times \mathbb{R}^n) = L_2(\Delta_n \times \mathbb{R}^n).$$

In order to show that $T_q^{(p)} F$ exists for F in $\mathscr{S}_{n,r}^{(p)}$ it will be useful to first establish two lemmas.

LEMMA 2.1. *Let* $1 \le p \le 2$ *and* $r \in (2p/(2p-1), \infty]$. *For* $f \in L_{pr}(\Delta_n \times \mathbb{R}^n)$ *and* $\operatorname{Re}\lambda \ge 0$, $\lambda \ne 0$, *let*

$$(2.3)\qquad \begin{aligned}&(J_\lambda f)(t_1, \ldots, t_n; w_1, \ldots, w_n)\\ &\quad \equiv \lambda^{n/2}\gamma \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} f(t_1, \ldots, t_n; u_1, \ldots, u_n)\\ &\quad \exp\left\{(-\lambda/2)\sum_{j=1}^{n} \frac{[(u_j - u_{j-1}) - (w_j - w_{j-1})]^2}{t_j - t_{j-1}}\right\} du_1 \ldots du_n.\end{aligned}$$

Then for all $\operatorname{Re}\lambda \ge 0$, $\lambda \ne 0$ *and almost all* $(t_1, \ldots, t_n) \in \Delta_n$,

$$(J_\lambda f)(t_1, \ldots, t_n; \cdot, \ldots, \cdot) \in L_{p'}(\mathbb{R}^n)$$

(as before in the case $p = 1$, $J_\lambda f \in C_0(\mathbb{R}^n)$) *with*

$$(2.4)\quad \|(J_\lambda f)(t_1, \ldots, t_n; \cdot, \ldots, \cdot)\|_{p'} \le |\lambda|^{n(2-p)/2p} \gamma^{(2-p)/p} \|f(t_1, \ldots, t_n; \cdot, \ldots, \cdot)\|_p.$$

In addition for all $\operatorname{Re}\lambda \ge 0$, $\lambda \ne 0$ *and* $1 \le p < 2$

$$(2.5a)\qquad J_\lambda f \in L_{p'\alpha}(\Delta_n \times \mathbb{R}^n) \qquad \textit{for all } \alpha < \frac{2pr}{2p + r(2-p)}$$

while for $p = 2$

$$(2.5b)\qquad J_\lambda f \in L_{2r}(\Delta_n \times \mathbb{R}^n).$$

Remarks. (i) In case $\operatorname{Re}\lambda = 0$ and $1 < p \le 2$ the integral in (2.3) is of course interpreted in the mean.

(ii) For fixed $(t_1, \ldots, t_n) \in \Delta_n$, $(J_\lambda f)(t_1, \ldots, t_n; w_1, \ldots, w_n) \equiv (K_\lambda g)(w_1, \ldots, w_n)$ where K_λ is given by (1.1) and $g(w_1, \ldots, w_n) \equiv f(t_1, \ldots, t_n; w_1, \ldots, w_n)$.

Proof. In view of (ii) above, Lemma 1.1 immediately implies that $J_\lambda f$ is in $L_{p'}(\mathbb{R}^n)$ and satisfies (2.4). In the case $p = 1$, it is again easy to see that $J_q f$ is in $C_0(\mathbb{R}^n)$. Also (2.5b) follows immediately from (2.4). Thus all that remains to be established is (2.5a). So we fix $p \in [1,2)$ and note that the function

$$\gamma^{(2-p)/p} = \left[\frac{1}{(2\pi)^n (t_1 - a) \ldots (t_n - t_{n-1})} \right]^{(2-p)/2p}$$

belongs to $L_{2p/(2-p)-\varepsilon}(\Delta_n)$ for all $\varepsilon \in (0, 2p/(2-p) - 1)$. In addition $\|f(t_1; \ldots, t_n; \cdot, \ldots, \cdot)\|_p \in L_r(\Delta_n)$ by assumption. Next we recall that if $f_1 \in L_{s_1}(\mathbb{R}^n)$ and $f_2 \in L_{s_2}(\mathbb{R}^n)$ then $f_1 f_2 \in L_k(\mathbb{R}^n)$ where $1/k = 1/s_1 + 1/s_2$. Thus for each $\varepsilon \in (0, 2p/(2-p) - 1)$,

$$\gamma^{(2-p)/p} \|f(t_1, \ldots, t_n; \cdot, \ldots, \cdot)\|_p \in L_k(\Delta^n),$$

where

$$1/k = 1/r + \frac{1}{2p/(2-p) - \varepsilon} = \frac{2p - \varepsilon(2-p) + r(2-p)}{r[2p - \varepsilon(2-p)]}.$$

Now using (2.4) and letting $\varepsilon \to 0$ yields (2.5a).

LEMMA 2.2. *Let* $1 \le p \le 2$ *and* $r \in (2p/(2p-1), \infty]$. *Let* $F \in \mathscr{S}^{(p)}_{n,r}$ *be given by* (2.2) *with defining kernel* f. *Let* $J_\lambda f$ *be given by* (2.3). *Then for s-almost every* y *the analytic Wiener integral* $(T_\lambda F)(y) \equiv \int_{C[a,b]}^{\mathrm{anw}_\lambda} F(x+y)\,dx$ *exists and is given by*

$$(2.6) \qquad (T_\lambda F)(y) \approx \int_{\Delta_n} (n) \int (J_\lambda f)(t_1, \ldots, t_n; y(t_1), \ldots, y(t_n))\, dt_1 \ldots dt_n.$$

In addition for $1 \le p \le 2$ *and each* $\rho > 0$,

$$(2.7) \quad \begin{aligned} \|(T_\lambda F)(\rho(\cdot))\|_{w,p'} &\equiv \left\{ \int_{C[a,b]} |(T_\lambda F)(\rho y)|^{p'} dy \right\}^{1/p'} \\ &\le \frac{|\lambda|^{n(2-p)/2p} N_n(f)}{\rho^{n/p'}} \left\{ \int_{\Delta_n} (n) \int \gamma^{r'/p} dt_1 \ldots dt_n \right\}^{1/r'} \\ &\le \frac{|\lambda|^{n(2-p)/2p} N_n(f)(b-a)^{n(1-r'/2p)/r'} \left[\Gamma\left(1 - \frac{r'}{2p}\right) \right]^{(n+1)/r'}}{\rho^{n/p'} (2\pi)^{n/2p} \left\{ \Gamma\left[(n+1)\left(1 - \frac{r'}{2p}\right) \right] \right\}^{1/r'}} \end{aligned}$$

where Γ *denotes the Gamma function.*

Remark. In case $p = 1$, $p' = \infty$ and in (2.7) we mean

$$\|(T_\lambda F)(\rho(\cdot))\|_{w,\infty} \equiv \sup_{y \in C[a,b]} |(T_\lambda F)(\rho y)|.$$

Proof. For $\lambda > 0$, one obtains equation (2.6) by a fundamental Wiener integration formula and use of the Fubini Theorem. Then, much as in the proof of Theorem

1.1, one shows that the right hand side of equation (2.6) has an analytic extension throughout $\mathbb{C}^+$.

Next we will establish (2.7) for $1 < p \le 2$. The proof for the case $p = 1$ is similar, but somewhat easier. In (2.8) below, the first inequality follows from Minkowski's inequality for integrals [6, p. 271], the third from (2.4) and the fourth from Hölder and (2.1).

$$
\begin{aligned}
&\left\{\int_{C[a,b]} |(T_\lambda F)(\rho y)|^{p'}\,dy\right\}^{1/p'} \\
&= \left\{\int_{C[a,b]} \left|\int_{\triangle_n}^{(n)}\int (J_\lambda f)(t_1,\ldots,t_n;\rho y(t_1),\ldots,\rho y(t_n))\,dt_1\ldots dt_n\right|^{p'} dy\right\}^{1/p'} \\
&\le \int_{\triangle_n}^{(n)}\int \left[\int_{C[a,b]} \left|(J_\lambda f)(t_1,\ldots,t_n;\rho y(t_1),\ldots,\rho y(t_n))\right|^{p'} dy\right]^{1/p'} dt_1\ldots dt_n \\
&= \int_{\triangle_n}^{(n)}\int \left[\frac{\gamma}{\rho^n}\int_{-\infty}^{\infty}{}^{(n)}\int_{-\infty}^{\infty} |(J_\lambda f)(t_1,\ldots,t_n;u_1,\ldots,u_n)|^{p'} \right. \\
&\qquad\qquad \left. \exp\left\{-\frac{1}{2\rho^2}\sum_1^n \frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\right\} du_1\ldots du_n\right]^{1/p'} dt_1\ldots dt_n \qquad (2.8) \\
&\le \rho^{-n/p'}\int_{\triangle_n}^{(n)}\int \gamma^{1/p'}\,\|(J_\lambda f)(t_1,\ldots,t_n;\cdot,\ldots,\cdot)\|_{p'}\,dt_1\ldots dt_n \\
&\le \frac{|\lambda|^{n(2-p)/2p}}{\rho^{n/p'}}\int_{\triangle_n}^{(n)}\int \gamma^{1/p}\,\|f(t_1,\ldots,t_n;\cdot,\ldots,\cdot)\|_p\,dt_1\ldots dt_n \\
&\le \frac{|\lambda|^{n(2-p)/2p}}{\rho^{n/p'}} N_n(f)\left[\int_{\triangle_n}^{(n)}\int \gamma^{r'/p}\,dt_1\ldots dt_n\right]^{1/r'}
\end{aligned}
$$

But (see [5, pp. 106-107] for the key equality)

$$
\begin{aligned}
&\int_{\triangle_n}^{(n)}\int \gamma^{r'/p}\,dt_1\ldots dt_n \\
&= (2\pi)^{-nr'/2p}\int_a^b\int_a^{t_n}\cdots\int_a^{t_2}[(t_1-a)\ldots(t_n-t_{n-1})]^{-r'/2p}\,dt_1\ldots dt_n \qquad (2.9)\\
&\le \left[\frac{b-a}{(2\pi)^n}\right]^{r'/2p}\int_a^b\int_a^{t_n}\cdots\int_a^{t_2}[(t_1-a)\ldots(t_n-t_{n-1})(b-t_n)]^{-r'/2p}\,dt_1\ldots dt_n \\
&= \frac{\left[\dfrac{b-a}{(2\pi)^n}\right]^{r'/2p}(b-a)^{n-(n+1)r'/2p}\left[\Gamma\left(1-\dfrac{r'}{2p}\right)\right]^{n+1}}{\Gamma\left[(n+1)\left(1-\dfrac{r'}{2p}\right)\right]}
\end{aligned}
$$

from which (2.7) follows easily.

THEOREM 2.1. *Let* $1 \le p \le 2$ *and* $r \in (2p/(2p-1),\infty]$. *Let* $F \in \mathscr{S}^{(p)}_{n,r}$ *be given by* (2.2) *with defining kernel* f. *Then for all real* $q \neq 0$, *the* L_p *analytic Fourier-Feynman transform* $T^{(p)}_q$ *exists s-almost everywhere and is given by*

$$(2.10)\qquad (T^{(p)}_q F)(y) \approx \int_{\triangle_n}^{(n)}\int (J_{-iq} f)(t_1, \ldots, t_n; y(t_1), \ldots, y(t_n))\, dt_1 \ldots dt_n$$

where $J_\lambda f$ *is given by* (2.3).

Proof. *Case* 1: $1 < p \le 2$. First we note that by Minkowski's inequality for integrals [6, p. 271]

$$(2.11)\qquad \begin{aligned} &\int_{C[a,b]} \left| \int_{\triangle_n}^{(n)}\int [(J_\lambda f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n)) - \right. \\ &\qquad \left. (J_{-iq} f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n))]\, dt_1 \ldots dt_n \right|^{p'} dy \\ &\le \int_{\triangle_n}^{(n)}\int \left[\int_{C[a,b]} |(J_\lambda f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n)) \right. \\ &\qquad \left. - (J_{-iq} f)(t_1, \ldots, t_n; y(t_1), \ldots, y(t_n))|^{p'} dy \right]^{1/p'} dt_1 \ldots dt_n \end{aligned}$$

for all $\rho > 0$ and $\lambda \in \mathbb{C}^+$. To establish (2.10) it suffices to show that for each $\rho > 0$ the limit as $\lambda \to -iq$ $(\operatorname{Re}\lambda > 0)$ of the left side of (2.11) is zero; we will show that the limit of the right hand side is zero using the Dominated Convergence Theorem. The use of the Dominated Convergence Theorem is justified by (1) and (2) below.

(1) Using Lemma 1.2 it is easy to see that for almost all $(t_1, \ldots, t_n) \in \triangle_n$

$$\begin{aligned} \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} \int_{C[a,b]} &|(J_\lambda f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n)) \\ &- (J_{-iq} f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n))|^{p'} dy = 0. \end{aligned}$$

(2) To obtain a dominating function we note that for all $\lambda \in \mathbb{C}^+$ such that $|\lambda| < |q| + 1$ we have using (2.4),

$$\begin{aligned} &\left[\int_{C[a,b]} |(J_\lambda f)(t_1, \ldots, t_n; y(t_1), \ldots, y(t_n)) - \right. \\ &\qquad \left. (J_{-iq} f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n))|^{p'} dy \right]^{1/p'} \\ &= \left[\frac{\gamma}{\rho^n} \int_{-\infty}^{\infty} {}^{(n)} \int_{-\infty}^{\infty} |(J_\lambda f)(t_1, \ldots, t_n; u_1, \ldots, u_n) - \right. \\ &\qquad (J_{-iq} f)(t_1, \ldots, t_n; u_1, \ldots, u_n)|^{p'} \end{aligned}$$

$$\exp\left\{-\frac{1}{2\rho^2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\right\}du_1 \dots du_n\Bigg]^{1/p'}$$

$$\leq \frac{\gamma^{1/p'}}{\rho^{n/p'}}\,[\,\|(J_\lambda f)(t_1, \dots, t_n; \cdot, \dots, \cdot)\|_{p'}$$

$$+ \|(J_{-iq} f)(t_1, \dots, t_n; \cdot, \dots, \cdot)\|_{p'}\,]$$

$$\leq \frac{2\,[|q|+1]^{n(2-p)/2p}}{\rho^{n/p'}}\gamma^{1/p}\|f(t_1, \dots, t_n; \cdot, \dots, \cdot)\|_p$$

which is in $L_1(\triangle_n)$ since $\gamma^{1/p} \in L_{r'}(\triangle_n)$ and $\|f(t_1, \dots, t_n; \cdot, \dots, \cdot)\|_p \in L_r(\triangle_n)$.

Case 2: $p = 1$. In this case we show that for each $\rho > 0$ and almost all $y \in C\,[a,b]$

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}^+}} \int_{\triangle_n}(n)\int (J_\lambda f)(t_1, \dots, t_n; \rho y(t_1), \dots, \rho y(t_n))\,dt_1 \dots dt_n$$

$$= \int_{\triangle_n}(n)\int (J_{-iq} f)(t_1, \dots, t_n; \rho y(t_1), \dots, \rho y(t_n))\,dt_1 \dots dt_n.$$

Again this follows by use of the Dominated Convergence Theorem. To obtain a dominating function we note that for $\lambda \in \mathbb{C}^+$ satisfying $|\lambda| < |q| + 1$ we have, using (2.4),

$$|(J_\lambda f)(t_1, \dots, t_n; \rho y(t_1), \dots, \rho y(t_n))|$$
$$\leq \|(J_\lambda f)(t_1, \dots, t_n; \cdot, \dots, \cdot)\|_\infty$$
$$\leq [|q|+1]^{n/2}\gamma\,\|f(t_1, \dots, t_n; \cdot, \dots, \cdot)\|_1$$

which is in $L_1(\triangle_n)$ since $\gamma \in L_{r'}(\triangle_n)$ and $\|f(t_1, \dots, t_n; \cdot, \dots, \cdot)\|_1 \in L_r(\triangle_n)$.

COROLLARY 1 TO THEOREM 2.1. *Let* $1 \leq p < 2$ *and* $r \in (2p/(2p-1), \infty]$. *Let* $F \in \mathscr{S}^{(p)}_{n,r}$ *be given by* (2.2) *with defining kernel* f. *Then for all real* $q \neq 0$, $T^{(p)}_q F$ *is in* $\mathscr{S}^{(p')}_{n,\alpha}$ *for all* $\alpha \in \left[1, \dfrac{2pr}{2p + r(2-p)}\right)$.

COROLLARY 2 TO THEOREM 2.1. *Let* $p = 2$ *and* $r \in (4/3, \infty]$. *Let* $F \in \mathscr{S}^{(2)}_{n,r}$ *be given by* (2.2) *with defining kernel* f. *Then* $T^{(2)}_q F$ *belongs to* $\mathscr{S}^{(2)}_{n,r}$ *for all real* $q \neq 0$.

If we choose $r = +\infty$ in Corollary 2 above we have the situation studied by Cameron and Storvick in [3].

COROLLARY 3 TO THEOREM 2.1. *Theorem* 4 *on page* 21 *of* [3].

COROLLARY 4 TO THEOREM 2.1. *Under the hypotheses of Theorem* 2.1, *for each* $\rho > 0$ *and* $1 \leq p \leq 2$

$$\|(T^{(p)}_q F)(\rho(\cdot))\|_{w,p'} \leq \frac{|q|^{n(2-p)/2p}N_n(f)}{\rho^{n/p'}}\left\{\int_{\triangle_n}(n)\int \gamma^{r'/p}\,dt_1 \dots dt_n\right\}^{1/r'}$$

$$\leq \frac{|q|^{n(2-p)/2p} N_n(f)(b-a)^{n(1-r'/2p)/r'} \left[\Gamma\left(1-\frac{r'}{2p}\right)\right]^{(n+1)r'}}{\rho^{n/p'}(2\pi)^{n/2p}\left\{\Gamma\left[(n+1)\left(1-\frac{r'}{2p}\right)\right]\right\}^{1/r'}}.$$

THEOREM 2.2. *Let* $1 \leq p \leq 2$ *and* $r \in (2p/(2p-1), \infty]$. *Let* $F \in \mathscr{S}^{(p)}_{n,r}$ *be given by* (2.2) *with defining kernel* f. *Then for all real* $q \neq 0$

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbf{C}^+}} \int_{C[a,b]} |T_{\bar\lambda} T_\lambda F(\rho y) - F(\rho y)|^p \, dy = 0 \qquad \textit{for each } \rho > 0 \tag{2.12}$$

where T_λ *is given by* (0.3).

Proof. For $\rho > 0$ and $\mathrm{Re}\,\lambda > 0$

$$(T_{\bar\lambda} T_\lambda F)(\rho y) = \int_{\triangle_n} (n) \int (J_{\bar\lambda} J_\lambda f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n)) \, dt_1 \ldots dt_n$$

for almost all $y \in C[a,b]$. Thus for $\rho > 0$ and $\mathrm{Re}\,\lambda > 0$

$$(T_{\bar\lambda} T_\lambda F)(\rho y) = \int_{\triangle_n} (n) \int (J_{\bar\lambda} J_\lambda f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n)) \, dt_1 \ldots dt_n$$

for almost all $y \in C[a,b]$. But by use of Theorem 1.2 and its proof we obtain that for almost all $(t_1, \ldots, t_n) \in \triangle_n$ and each $\rho > 0$

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbf{C}^+}} (J_{\bar\lambda} J_\lambda f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n)) = f(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n)) \tag{2.13}$$

and

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbf{C}^+}} \int_{C[a,b]} |J_{\bar\lambda} J_\lambda f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n)) - f(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n))|^p \, dy = 0 \tag{2.14}$$

for almost all $y \in C[a,b]$.

Again by Minkowski [6, p. 271] we see that

$$\left\{\int_{C[a,b]} |T_{\bar\lambda} T_\lambda F(\rho y) - F(\rho y)|^p \, dy\right\}^{1/p}$$
$$= \left\{\int_{C[a,b]} \left|\int_{\triangle_n} (n) \int [(J_{\bar\lambda} J_\lambda f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n))\right.\right.$$

$$- f(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n))] \, dt_1 \ldots dt_n \Big|^p dy \Big\}^{1/p}$$

$$\leq \int_{\Delta_n}^{(n)} \Big\{ \int_{C[a,b]} |(J_{\bar\lambda} J_\lambda f)(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n)) - f(t_1, \ldots, t_n; \rho y(t_1), \ldots, \rho y(t_n))|^p dy \Big\}^{1/p} dt_1 \ldots dt_n$$

which goes to zero as $\lambda \to -iq$ by use of the Dominated Convergence Theorem. This, however, establishes (2.12). A dominating $L_1(\Delta_n)$ function is the quantity

$$(2\gamma^{1/p}/\rho^{n/p}) \| f(t_1, \ldots, t_n; \cdot, \ldots, \cdot) \|_p .$$

Thus the proof of Theorem 2.2 is finally complete.

In the case $p = 2$ we can however obtain a stronger inverse transform theorem since for F in $\mathscr{S}^{(2)}_{n,r}$, $T^{(2)}_q F$ is again in $\mathscr{S}^{(2)}_{n,r}$ and so it makes sense to apply $T^{(2)}_{-q}$ to it.

THEOREM 2.3. *Let* $p = 2$ *and* $r \in (4/3, \infty]$. *Let* $F \in \mathscr{S}^{(2)}_{n,r}$ *be given by* (2.2) *with defining kernel* $f \in L_{2r}(\Delta_n \times \mathbb{R}^n)$. *Then for all real* $q \neq 0$, $T^{(2)}_{-q} T^{(2)}_q F \approx F$.

Again if we choose $r = +\infty$ in Theorem 2.3 we have the situation studied by Cameron and Storvick in [3].

COROLLARY TO THEOREM 2.3. *Theorem* 5 *on page* 23 *of* [3].

3. THE TRANSFORM $T^{(p)}_q$ APPLIED TO FUNCTIONALS F CONTAINED IN $\mathscr{S}^{(p)}_r$

For $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$ let $\mathscr{S}^{(p)}_r$ be the space of functionals F such that there exists a sequence $\{F_n\}$ with $F_n \in \mathscr{S}^{(p)}_{n,r}$ having corresponding kernel $f_n \in L_{pr}(\Delta_n \times \mathbb{R}^n)$ such that

$$F \approx \sum_0^\infty F_n \tag{3.1}$$

and

$$[N_n(f_n)]^{1/n} = O(n^{(1-r'/2p)/r'}) \qquad \text{as } n \to \infty. \tag{3.2}$$

We shall call $\{F_n\}$ a defining sequence for F and $\{f_n\}$ a corresponding kernel sequence.

The following two lemmas play an important role in establishing the existence of $T^{(p)}_q F$ for F in $\mathscr{S}^{(p)}_r$.

LEMMA 3.1. *Let* $1 \leq p \leq 2$ *and* $r \in (2p/(2p-1), \infty]$. *Let* $F \in \mathscr{S}^{(p)}_{n,r}$ *be given by* (2.2) *with defining kernel* f. *Then for each* $\rho > 0$ *we have*

$$
\begin{aligned}
\|F(\rho(\cdot))\|_{w,p} &\equiv \left\{\int_{C[a,b]} |F(\rho x)|^p\, dx\right\}^{1/p} \\
&\le (N_n(f)/\rho^{n/p})\left\{\int_{\Delta_n}^{(n)}\int \gamma^{r'/p}\, dt_1 \ldots dt_n\right\}^{1/r'} \\
&\le \frac{N_n(f)(b-a)^{n(1-r'/2p)/r'}\left[\Gamma\left(1-\dfrac{r'}{2p}\right)\right]^{(n+1)/r'}}{\rho^{n/p}(2\pi)^{n/2p}\left\{\Gamma\left[(n+1)\left(1-\dfrac{r'}{2p}\right)\right]\right\}^{1/r'}}.
\end{aligned} \tag{3.3}
$$

Proof. Proceeding as in the proof of Lemma 2.2 we obtain

$$
\begin{aligned}
&\left\{\int_{C[a,b]} |F(\rho x)|^p\, dx\right\}^{1/p} \\
&= \left\{\int_{C[a,b]} \left|\int_{\Delta_n}^{(n)}\int f(t_1, \ldots, t_n; \rho x(t_1), \ldots, \rho x(t_n))\, dt_1 \ldots dt_n\right|^p dx\right\}^{1/p} \\
&\le \int_{\Delta_n}^{(n)}\int \left\{\int_{C[a,b]} |f(t_1, \ldots, t_n; \rho x(t_1), \ldots, \rho x(t_n))|^p\, dx\right\}^{1/p} dt_1 \ldots dt_n \\
&= \int_{\Delta_n}^{(n)}\int \left\{(\gamma/\rho^n)\int_{-\infty}^{\infty}\overset{(n)}{\cdots}\int_{-\infty}^{\infty} |f(t_1, \ldots, t_n; u_1, \ldots, u_n)|^p \right. \\
&\quad \left. \exp\left[(-1/2\rho^2)\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\right] du_1 \ldots du_n\right\}^{1/p} dt_1 \ldots dt_n \\
&\le \rho^{-n/p}\int_{\Delta_n}^{(n)}\int \gamma^{1/p}\|f(t_1, \ldots, t_n; \cdot, \ldots, \cdot)\|_p\, dt_1 \ldots dt_n \\
&\le \rho^{-n/p} N_n(f)\left\{\int_{\Delta_n}^{(n)}\int \gamma^{r'/p}\, dt_1 \ldots dt_n\right\}^{1/r'}
\end{aligned}
$$

which in view of (2.9) establishes (3.3).

LEMMA 3.2. *Let* $1 \le p \le 2$ *and* $r \in (2p/(2p-1), \infty]$. *For* $n = 0, 1, 2, \ldots$ *let* F_n *be in* $\mathscr{S}^{(p)}_{n,r}$ *with defining kernel function* $f_n \in L_{pr}(\Delta_n \times \mathbb{R}^n)$ *satisfying* (3.2). *Then for all* $\rho > 0$ *we have*

$$
\sum_{n=0}^{\infty} \|F_n(\rho(\cdot))\|_{w,p} < \infty, \tag{3.4}
$$

and the series

$$
\sum_{n=0}^{\infty} F_n(\rho x) \tag{3.5}
$$

converges absolutely for almost all $x \in C[a,b]$ *and converges in the* $L_p(C[a,b])$ *mean. Moreover for s-almost every y and almost every* $\rho > 0$

$$\sum_{n=0}^{\infty} \int_{C[a,b]} |F_n(\rho x + y)|\, dx < \infty. \tag{3.6}$$

Proof. To establish (3.4) we will use (3.3), Stirling's Theorem and the root test. We first recall that for positive z sufficiently large $1/\Gamma(z) < 2e^z \sqrt{z}/\sqrt{2\pi}\, z^z$. Thus for n sufficiently large,

$$\frac{1}{\Gamma\left[(n+1)\left(1-\frac{r'}{2p}\right)\right]} \le \frac{2e^{(n+1)(1-r'/2p)}\sqrt{(n+1)\left(1-\frac{r'}{2p}\right)}}{(2\pi)^{1/2}\left[(n+1)\left(1-\frac{r'}{2p}\right)\right]^{(n+1)(1-r'/2p)}}.$$

Hence using (3.3) we obtain that for n sufficiently large and each $\rho > 0$.

$$\begin{aligned}
&[\|F_n(\rho(\cdot))\|_{w,p}]^{1/n} \\
&\le \frac{[N_n(f_n)^{1/n}(b-a)^{(1-r'/2p)/r'}\left[\Gamma\left(1-\frac{r'}{2p}\right)\right]^{(n+1)/r'} 2^{1/nr'} e^{(n+1)(1-r'/2p)/nr'}}{\rho^{1/p}(2\pi)^{1/2p+1/2nr'}\left[(n+1)\left(1-\frac{r'}{2p}\right)\right]^{(n+1)(1-r'/2p)/nr'-1/2nr'}} \\
&\le \frac{[N_n(f_n)]^{1/n}}{n^{(1-r'/2p)/r'}}\left[\frac{(b-a)^{(1-r'/2p)/r'}\left[\Gamma\left(1-\frac{r'}{2p}\right)\right]^{(n+1)/nr'} 2^{1/2nr'}}{\rho^{1/p}(2\pi)^{(1/p+1/nr')/2}(n+1)^{(1-r'/p)/2nr'}}\right. \\
&\qquad \left.\frac{\exp\left\{\left(\frac{n+1}{nr'}\right)\left(1-\frac{r'}{2p}\right)\right\}}{\left(1-\frac{r'}{2p}\right)^{((n+1)(1-r'/2p)-1/2)/nr'}}\right]
\end{aligned}$$

which goes to zero as $n \to \infty$ since (3.2) implies that $\frac{[N_n(f_n)]^{1/n}}{n^{(1-r'/2p)/r'}} \to 0$ as $n \to \infty$. Thus, by the root test, (3.4) is established.

The absolute convergence of the series (3.5) follows easily from the observation that for $1 \le p \le 2$

$$\sum_{0}^{\infty} \int_{C[a,b]} |F_n(\rho x)|\, dx = \sum_{0}^{\infty} \|F_n(\rho(\cdot))\|_{w,1} \tag{3.7}$$

$$\leq \sum_{0}^{\infty} ||F_n(\rho(\cdot))||_{w,p} < \infty.$$

We still need to establish (3.6). Since the translation theorem for Wiener integrals does not allow us to proceed directly from (3.4) to (3.6) we need to examine the corresponding kernels of the functionals $F_n(\rho x + y)$. Let

$$F_n^*(x) = \int_{\triangle_n} (n) \int f(t_1, \ldots, t_n; x(t_1), \ldots, x(t_n))\, dt_1 \ldots dt_n$$

for all x for which the integral exists. For each $y \in C[a,b]$ and $\rho > 0$, let

$$H_n(x) \equiv F_n^*(\rho x + y)$$

so that

$$H_n(x) \equiv \int_{\triangle_n} (n) \int h_n(t_1, \ldots, t_n; x(t_1), \ldots, x(t_n))\, dt_1 \ldots dt_n$$

where

$$h_n(t_1, \ldots, t_n; u_1, \ldots, u_n) \equiv f_n(t_1, \ldots, t_n; \rho u_1 + y(t_1), \ldots, \rho u_n + y(t_n)).$$

But $f_n \in L_{pr}(\triangle_n \times \mathbb{R}^n)$ implies that $h_n \in L_{pr}(\triangle_n \times \mathbb{R}^n)$ and a direct calculation shows that

$$N_n(h_n) \equiv ||h_n||_{pr} = \rho^{-n/p} ||f_n||_{pr} \equiv \rho^{-n/p} N_n(f_n).$$

Thus $H_n \in \mathscr{S}_{n,r}^{(p)}$ and so H_n with defining kernel h_n satisfies the hypotheses of this lemma. Hence using (3.7) we obtain that for each $y \in C[a,b]$ and $\rho > 0$

$$\sum_{n=0}^{\infty} \int_{C[a,b]} |H_n(x)|\, dx = \sum_{n=0}^{\infty} \int_{C[a,b]} |F_n^*(\rho x + y)|\, dx < \infty.$$

But $F_n^* \approx F_n$ and so (3.6) follows from (0.10).

COROLLARY TO LEMMA 3.2. *Under the hypotheses of Lemma* 3.2, *the functional* F *defined by the formula* $F \equiv \sum_{0}^{\infty} F_n$ *belongs to* $\mathscr{S}_r^{(p)}$.

THEOREM 3.1. *Let* $1 \leq p \leq 2$ *and* $r \in (2p/(2p-1), \infty]$. *Let* $F \in \mathscr{S}_r^{(p)}$ *be given by* (3.1) *with corresponding kernel sequence* $\{f_n\}$ *satisfying* (3.2). *Then for all real* $q \neq 0$, *the* L_p *analytic Fourier-Feynman transform* $T_q^{(p)}F$ *exists and is given by*

$$T_q^{(p)} F \approx \sum_{n=0}^{\infty} T_q^{(p)} F_n. \tag{3.8}$$

Proof. Since $F \approx \sum_0^\infty F_n$ and since (see Lemma 2.2) for s-almost every $y \in C[a,b]$ the analytic Wiener integral $(T_\lambda F_n)(y) \equiv \int_{C[a,b]}^{anw_\lambda} F_n(x+y)\,dx$ exists, it is quite easy to see that for s-almost every $y \in C[a,b]$ the analytic Wiener integral $(T_\lambda F)(y) \equiv \int_{C[a,b]}^{anw_\lambda} F(x+y)\,dx$ exists and satisfies the equation

$$(T_\lambda F)(y) \approx \sum_0^\infty (T_\lambda F_n)(y) \tag{3.9}$$

for $\operatorname{Re}\lambda > 0$. First we will consider the case $1 < p \leq 2$. In this case in order to show that $T_q^{(p)} F$ exists and is given by (3.8) it will suffice to show that

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} \int_{C[a,b]} \left| \sum_0^\infty (T_\lambda F_n)(\rho y) - \sum_0^\infty (T_q^{(p)} F_n)(\rho y) \right|^{p'} dy = 0 \tag{3.10}$$

for each $\rho > 0$. Next we observe that

$$\begin{aligned} &\left\{ \int_{C[a,b]} \left| \sum_0^\infty (T_\lambda F_n)(\rho y) - \sum_0^\infty (T_q^{(p)} F_n)(\rho y) \right|^{p'} dy \right\}^{1/p'} \\ &= \left\| \sum_0^\infty [(T_\lambda F_n)(\rho(\cdot)) - (T_q^{(p)} F_n)(\rho(\cdot))] \right\|_{w,p'} \\ &\leq \sum_0^\infty \|(T_\lambda F_n)(\rho(\cdot)) - (T_q^{(p)} F_n)(\rho(\cdot))\|_{w,p'}. \end{aligned} \tag{3.11}$$

Next, using (2.7) and Corollary 4 to Theorem 2.1, we see that (for all $\lambda \in \mathbb{C}^+$ such that $|\lambda| < 1 + |q|$) the series on the right hand side of (3.1)) is dominated by the series

$$\sum_0^\infty [\|(T_\lambda F_n)(\rho(\cdot))\|_{w,p'} + \|(T_q^{(p)} F_n)(\rho(\cdot))\|_{w,p'}]$$

$$\leq \sum_0^\infty \frac{[1+|q|]^{n(2-p)/2p} N_n(f_n)(b-a)^{n(1-r'/2p)/r'} \left[\Gamma\left(1 - \frac{r'}{2p}\right)\right]^{(n+1)/r'}}{\rho^{n/p'}(2\pi)^{n/2p} \left\{\Gamma\left[(n+1)\left(1-\frac{r'}{2p}\right)\right]\right\}^{1/r'}}.$$

But, as shown in the proof of Lemma 3.2, this series converges. Hence, since for each n,

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} \|(T_\lambda F_n)(\rho(\cdot)) - (T_q^{(p)} F_n)(\rho(\cdot))\|_{w,p'} = 0,$$

it follows that

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} \sum_{0}^{\infty} \|(T_\lambda F_n)(\rho(\cdot)) - (T_q^{(p)} F_n)(\rho(\cdot))\|_{w,p'} = 0.$$

Thus (3.10) is established and so $T_q^{(p)} F$ exists and is given by (3.8).

The case $p = 1$ is handled in a similar way by showing that for each $\rho > 0$

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} \sum_{0}^{\infty} (T_\lambda F_n)(\rho y) = \sum_{0}^{\infty} (T_q^{(1)} F_n)(\rho y)$$

for almost all $y \in C[a,b]$.

COROLLARY 1 TO THEOREM 3.1. *Let* $1 \le p < 2$ *and* $r \in (2p/(2p-1), \infty]$. *Let* $F \in \mathscr{S}_r^{(p)}$. *Then for all real* $q \neq 0$, $T_q^{(p)} F$ *belongs to* $\mathscr{S}_\alpha^{(p')}$ *for all* $\alpha \in \left[1, \dfrac{2pr}{2p + r(2-p)}\right)$.

COROLLARY 2 TO THEOREM 3.1. *Let* $p = 2$ *and* $r \in (4/3, \infty]$. *Let* $F \in \mathscr{S}_r^{(2)}$. *Then for all real* $q \neq 0$, $T_q^{(2)} F \in \mathscr{S}_r^{(2)}$.

THEOREM 3.2. *Let* $1 \le p \le 2$ *and* $r \in (2p/(2p-1), \infty]$. *Let* $F \in \mathscr{S}_r^{(p)}$ *be given by* (3.1) *with corresponding kernel sequence* $\{f_n\}$ *satisfying* (3.2). *Then for all real* $q \neq 0$

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} \int_{C[a,b]} |T_{\bar\lambda} T_\lambda F(\rho y) - F(\rho y)|^p \, dy = 0 \qquad \textit{for each } \rho > 0. \tag{3.12}$$

Proof. First we observe that

$$\begin{aligned}
&\left\{\int_{C[a,b]} |T_{\bar\lambda} T_\lambda F(\rho y) - F(\rho y)|^p \, dy\right\}^{1/p} \\
&= \left\{\int_{C[a,b]} \left|\sum_{0}^{\infty} [T_{\bar\lambda} T_\lambda F_n(\rho y) - F_n(\rho y)]\right|^p dy\right\}^{1/p} \\
&= \left\|\sum_{0}^{\infty} [T_{\bar\lambda} T_\lambda F_n(\rho(\cdot)) - F_n(\rho(\cdot))]\right\|_{w,p} \\
&\le \sum_{0}^{\infty} \|T_{\bar\lambda} T_\lambda F_n(\rho(\cdot)) - F_n(\rho(\cdot))\|_{w,p}.
\end{aligned} \tag{3.13}$$

But the series on the right hand side of (3.13) is dominated, uniformly in λ, by the convergent series

$$\sum_{0}^{\infty} \frac{2N_n(f_n)(b-a)^{n(1-r'/2p)/r'}\left[\Gamma\left(1-\frac{r'}{2p}\right)\right]^{(n+1)/r'}}{\rho^{n/p}(2\pi)^{n/2p}\left\{\Gamma\left[(n+1)\left(1-\frac{r'}{2p}\right)\right]\right\}^{1/r'}}.$$

Hence, since for each n,

$$\lim_{\substack{\lambda\to -iq \\ \lambda\in \mathbf{C}^+}} \|T_{\bar\lambda}T_\lambda F_n(\rho(\cdot)) - F_n(\rho(\cdot))\|_{w,p} = 0,$$

it follows that

$$\lim_{\substack{\lambda\to -iq \\ \lambda\in \mathbf{C}^+}} \sum_{0}^{\infty} \|T_{\bar\lambda}T_\lambda F_n(\rho(\cdot)) - F_n(\rho(\cdot))\|_{w,p} = 0,$$

which establishes (3.12).

THEOREM 3.3. *Let* $p = 2$ *and* $r \in (4/3, \infty]$. *Let* $F \in \mathscr{S}_r^{(2)}$. *Then for all real* $q \neq 0$, $T_q^{(2)}F \in \mathscr{S}_r^{(2)}$. *In addition,* $T_{-q}^{(2)}T_q^{(2)}F \approx F$.

Again if we choose $r = +\infty$ in Theorem 3.3 we have the situation studied by Cameron and Storvick in [3].

COROLLARY TO THEOREM 3.3. *Theorem* 6 *on page* 26 *of* [3].

4. THE TRANSFORM $T_q^{(p)}$ APPLIED TO ENTIRE FUNCTIONS OF INTEGRALS

THEOREM 4.1. *Let* $1 \le p \le 2$ *and* $r \in (2p/(2p-1), \infty]$. *Let* $\Phi(z) = \sum_0^\infty a^n z^n$ *be an entire function of order less than* 2p *and let* $\theta(t,u) \in L_{pr}([a,b] \times \mathbb{R})$. *Let*

$$F(x) = \Phi\left[\int_a^b \theta(t, x(t))\,dt\right]. \tag{4.1}$$

Then $F \in \mathscr{S}_r^{(p)}$ *and so the* L_p *analytic Fourier-Feynman transform* $T_q^{(p)}F$ *exists for all real* $q \neq 0$. *In addition*

$$\lim_{\substack{\lambda\to -iq \\ \lambda\in \mathbf{C}^+}} \int_{C[a,b]} |T_{\bar\lambda}T_\lambda F(\rho y) - F(\rho y)|^p\,dy = 0 \qquad \textit{for each } \rho > 0. \tag{4.2}$$

Proof. In view of Theorems 3.1 and 3.2 we need only show that $F \in \mathscr{S}_r^{(p)}$. First note that we can write $F(x)$ in the form $F(x) = \sum_0^\infty F_n(x)$ where for each n,

$$F_n(x) \approx \int_{\triangle_n}(n)\int f_n(t_1, \ldots, t_n; x(t_1), \ldots, x(t_n))\, dt_1 \ldots dt_n$$

and $f_n(t_1, \ldots, t_n; u_1, \ldots, u_n) = a_n\, n! \prod_1^n \theta(t_j, u_j)$. We need to show that the sequence $\{f_n\}$ satisfies (3.2). That is to say that $[N_n(f_n)]^{1/n} = O(n^{(1-r'/2p)/r'})$ as $n \to \infty$. But

$$\begin{aligned} N_n(f_n) &= ||f_n||_{pr} \\ &= \left[\int_{\triangle_n}(n)\int |a_n|^r (n!)^r \prod_1^n ||\theta(t_j, \cdot)||_p^r\, dt_1 \ldots dt_n\right]^{1/r} \\ &= |a_n|\, n! \left[\int_{\triangle_n}(n)\int \prod_1^n ||\theta(t_j, \cdot)||_p^r\, dt_1 \ldots dt_n\right]^{1/r} \\ &= |a_n|\, n! \left[\frac{1}{n!}\int_a^b (n)\int_a^b ||\theta(t_j, \cdot)||_p^r\, dt_1 \ldots dt_n\right]^{1/r} \\ &= |a_n|(n!)^{1/r'}\, ||\theta||_{pr}^n \\ &= |a_n|(n!)^{1/r'}\, [N_1(\theta)]^n \end{aligned}$$

so that

$$[N_n(f_n)]^{1/n} = |a_n|^{1/n} (n!)^{1/r'n} N_1(\theta). \tag{4.3}$$

Next since Φ is an entire function of order less than 2p, there exists an $\varepsilon > 0$ such that for sufficiently large n, $\dfrac{n \log n}{\log\left(\dfrac{1}{|a_n|}\right)} < 2p - \varepsilon$, and so $|a_n| < n^{-n/(2p-\varepsilon)}$. But by Stirling's formula, for sufficiently large n, $n! \le (n/e)^n \sqrt{2\pi n}\, e^{1/12n}$. Thus, using (4.3), we obtain

$$[N_n(f_n)]^{1/n} < \frac{n^{1/r'}}{n^{1/(2p-\varepsilon)}} \{e^{-1/r'} (2\pi n)^{1/nr'} e^{1/12n^2r'} N_1(\theta)\},$$

so that

$$\frac{[N_n(f_n)]^{1/n}}{n^{(1-r'/2p)/r'}} < \frac{\{e^{-/r'} (2\pi n)^{1/nr'} e^{1/12n^2r'} N_1(\theta)\}}{n^{\varepsilon/2p(2p-\varepsilon)}}$$

which goes to zero as $n \to \infty$ since $0 < \varepsilon < 2p$. Hence

$$[N_n(f_n)]^{1/n} = O(n^{(1-r'/2p)/r'}) \qquad \text{as } n \to \infty$$

and the proof is complete.

Again in the case p = 2 we obtain a stronger inverse transform theorem.

THEOREM 4.2. *Let* $p = 2$ *and* $r \in (4/3,\infty]$. *Let* $\Phi(z)$ *be an entire function of order less than* 4 *and let* $\theta(t,u) \in L_{2r}([a,b] \times \mathbb{R})$. *Let*

$$F(x) = \Phi\left(\int_a^b \theta(t, x(t))\, dt\right).$$

Then $T_q^{(2)} F$ *exists for all real* $q \neq 0$, $T_q^{(2)} F \in \mathscr{S}_r^{(2)}$, *and* $T_{-q}^{(2)} T_q^{(2)} F \approx F$.

Once again if we choose $r = +\infty$ we have the situation studied by Cameron and Storvick [3].

COROLLARY. *Theorem* 7 *on page* 29 *of* [3].

Remark. Note that for all $p \in [1,2]$, Theorem 4.1 applies to functionals of the form $\exp\left(\int_a^b \theta(t, x(t))\, dt\right)$; there is considerable interest in function space integrals of functionals of this type.

BIBLIOGRAPHY

1. M. D. Brue, *A functional transform for Feynman integrals similar to the Fourier transform,* thesis, University of Minnesota, 1972.
2. R. H. Cameron and D. A. Storvick, *An operator valued function space integral and a related integral equation.* J. Math. Mech. 18 (1968), 517–552.
3. ———, *An* L_2 *analytic Fourier-Feynman transform.* Michigan Math. J. 23 (1976), 1–30.
4. G. W. Johnson and D. L. Skoug, *The Cameron-Storvick function space integral: the* L_1 *theory.* J. Math. Anal. Appl. 50 (1975), 647–667.
5. ———, *The Cameron-Storvick function space integral: an* $L(L_p, L_{p'})$ *theory.* Nagoya Math. J. 60 (1976), 93–137.
6. E. M. Stein, *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
7. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces.* Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971.

Department of Mathematics and Statistics
University of Nebraska
Lincoln, Nebraska 68588