INTEGRAL REPRESENTATIONS AND DIAGRAMS

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INTRODUCTION

This article is the outcome of an attempt to study integral representations by diagrammatic techniques. A *diagram* D is a finite directed graph. Given a field k, a k-*representation* of D assigns to each vertex α of D a finite dimensional k-space V_α, and to each arrow $\alpha \to \beta$ a k-linear transformation $V_\alpha \to V_\beta$. There are obvious definitions of morphisms of representations, isomorphisms, direct sum, and decomposability. It is clear that the Krull-Schmidt Theorem is valid for representations of diagrams, namely, every representation is expressible as a finite direct sum of indecomposables, unique up to isomorphism and order of occurrence. If there are only a finite number of non-isomorphic indecomposables, we call D of finite representation type.

These concepts were introduced in a fundamental article by Gabriel [9], who proved that a connected diagram D is of finite type if and only if its underlying graph is one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8. Some time thereafter, a less computational proof of this amazing result was given by Bernstein-Gelfand-Ponomarev [1], using the machinery of Coxeter functors from Lie algebras. Their approach was generalized by Dlab-Ringel [4], [5], who considered representations of a modulated graph $\mathscr{M}$. By definition, $\mathscr{M}$ consists of a finite directed graph with a skewfield k_α placed at each vertex α, and a (k_β, k_α)-bimodule ${}_\beta M_\alpha$ attached to each arrow $\alpha \to \beta$. A representation of $\mathscr{M}$ assigns to each vertex α a left k_α-space V_α, and to each arrow from α to β a k_β-homomorphism ${}_\beta M_\alpha \otimes_{k_\alpha} V_\alpha \to V_\beta$. After imposing a few reasonable hypotheses, Dlab-Ringel determined all modulated graphs of finite type.

A somewhat different approach was followed by Russian mathematicians such as Drozd, Kleiner, Nazarova, and Roiter (see the fundamental Leningrad Proceedings of 1972 [16]). They studied finite partially ordered sets (posets); a representation of a poset S is given by choosing a vector space V over some field k, and assigning to each $\alpha \in S$ a subspace V_α of V, so that $V_\alpha \subseteq V_\beta$ whenever $\alpha \leq \beta$ in S. The work of Nazarova-Roiter and Kleiner settled the question as to which posets have finite type.

It has become increasingly clear from the above-mentioned articles, as well as from related work of Donovan-Freislich [6], Gordon-Green [11], Green [12], [13], and Ringel [18], [19], that these diagrammatic methods provide a new and powerful tool for investigating representations of rings and algebras. Such

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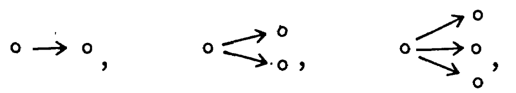
techniques were recently applied in an interesting paper of Butler [2], who gave a new proof of a classical result that there are precisely $4p + 1$ isomorphism classes of indecomposable p-adic integral representations of a cyclic group of order p^2, where p is prime. His work suggested to us that diagrammatic methods could be used to solve other problems in integral representation theory, and our goal was to use such methods to find a new proof of the fundamental results of Jacobinski [14] and Drozd-Roiter [8], which settled the question as to which commutative orders Λ are of finite representation type. It seemed likely, in view of Butler's approach, that this question could be reduced to a corresponding question about diagrams, which could then be solved by using the results of Gabriel and Dlab-Ringel.

Unfortunately, we were unable to obtain such a straightforward reduction, and we were instead led to deal with a more general type of diagram problem. Let us start with a modulated graph $\mathscr{M}$, placing at each vertex α a local ring k_α rather than a skewfield, and assigning a bimodule ${}_\beta M_\alpha$ to each arrow. We are then faced with the fundamental problem of deciding which modulated graphs are of finite representation type. In this article, we shall show the significance of this approach to integral representation theory. As we shall see, the solution of this fundamental problem for even the simplest cases, corresponding to the graphs



is quite complicated. However, even such a partial solution already yields the finiteness criteria of Jacobinski and Drozd-Roiter. In fact, many of our calculations in sections 4 and 5 are modified versions of those given by Jacobinski. In the course of this work, we were led to correct some minor errors in Jacobinski's results. It is hoped that the present article will encourage further study of "integral" representations of modulated graphs.

The organization of the present article is as follows: Section 1 fixes some notation and gives a review of some elementary facts about lattices over an R-order Λ. In particular, we show how the problem, as to whether Λ is of finite representation type, can be reduced to the "local" case where R is a complete discrete valuation ring. Furthermore, for Λ commutative, it suffices to treat the situation in which Λ itself is a local ring. In Section 2, we prove that the local problem can be reduced to a corresponding problem involving artinian rings. We are led to study the category $\mathscr{A}(\Delta, \Gamma)$, where Δ and Γ are artinian rings with $\Delta \subset \Gamma$; the objects of $\mathscr{A}(\Delta, \Gamma)$ are triples (X, Y, f) consisting of a projective Δ-module X, a projective Γ-module Y, and a Δ-homomorphism $f\colon X \to Y$ satisfying certain conditions.

Section 3 is a rather technical one, in which we adapt Jacobinski's method of changing ground rings, so as to reduce the problem to one in which various residue class fields coincide. Such a reduction helps simplify the calculations in the later sections.

In Section 4, we show that certain types of categories $\mathscr{A}(\Delta, \Gamma)$ are of infinite representation type. The proofs here are modelled upon a well known calculation due to Dade. At the same time, we establish some easy results on representation equivalences between various $\mathscr{A}$-categories.

Section 5 contains a few of the more complicated "finiteness" proofs for $\mathscr{A}$-categories satisfying certain hypotheses. The problem is first formulated in terms of matrices, and matrix manipulations are then used to prove the desired results. This approach is roughly parallel to that given by Jacobinski [14], and seems to us somewhat simpler to follow in each particular case. It is also more self-contained than the Drozd-Roiter approach to the question. The concluding Section 6 contains miscellaneous results on the categories $\mathscr{A}(\Delta, \Gamma)$, and suggests directions for further research.

1. PRELIMINARIES.

Let R be a Dedekind domain with quotient field K, and let A be a finite dimensional K-algebra. By an R-*lattice* we mean a finitely generated projective R-module (or equivalently, a finitely generated torsion-free R-module). Let Λ be an R-*order* in A, that is, Λ is a subring of A containing R, such that Λ is an R-lattice and $K\Lambda = A$. (Here, $K\Lambda$ denotes the collection of all finite sums $\sum \alpha_i x_i$, $\alpha_i \in K$, $x_i \in \Lambda$.) By virtue of the embedding $R \subset \Lambda$, we may view each Λ-module as an R-module as well.

A Λ-*lattice* is a left Λ-module M which is an R-lattice. Since M is R-torsion-free, the map $M \to K \otimes_R M$ is monic, and we may identify M with $1 \otimes M$. In that case, $K \otimes_R M = K(1 \otimes M) = K \cdot M$, a left A-module finitely generated over K.

Denote by $\mathscr{L}(\Lambda)$ the category of left Λ-lattices. For $M, N \in \mathscr{L}(\Lambda)$, we may form the R-module $\operatorname{Hom}_\Lambda(M, N)$. It is a submodule of the R-lattice $\operatorname{Hom}_R(M, N)$, hence is R-torsion-free. Thus the map $\operatorname{Hom}_\Lambda(M, N) \to K \otimes_R \operatorname{Hom}_\Lambda(M, N)$ is monic, and may be viewed as an embedding. By the Change of Rings Theorem ([17, (2.43)]), we have

$$K \otimes_R \operatorname{Hom}_\Lambda(M, N) \cong \operatorname{Hom}_A(KM, KN). \tag{1.1}$$

Thus, each $f \in \operatorname{Hom}_\Lambda(M,N)$ induces a uniquely determined $f' \in \operatorname{Hom}_A(KM, KN)$. Further, we have

$$\operatorname{Hom}_\Lambda(M, N) = \{g \in \operatorname{Hom}_A(KM, KN): g(M) \subset N\}. \tag{1.2}$$

In the case where $M = N$, the map in (1.1) is a ring isomorphism, and the map $\operatorname{End}_\Lambda(M) \to \operatorname{End}_A(KM)$ is a ring monomorphism.

Now let $\Lambda \subset \Lambda_1 \subset A$, where Λ_1 is an R-order in A. For each Λ-lattice M, we form the A-module KM generated by M, and define

$$\Lambda_1 \cdot M = \left\{\text{all finite sums} \sum \lambda_i m_i : \lambda_i \in \Lambda_1, m_i \in M\right\},$$

where these sums are computed inside the A-module KM. Then $\Lambda_1 M$ is a Λ_1-submodule of KM, and is finitely generated R-torsion-free, hence is a Λ_1-lattice. Of course, $K \cdot \Lambda_1 M = (K\Lambda_1)M = KM$.

For any Λ-lattices M,N, we have

$$\mathrm{Hom}_{\Lambda_1}(\Lambda_1 M, \Lambda_1 N) = \{f \in \mathrm{Hom}_A(KM, KN): f(\Lambda_1 M) \subset \Lambda_1 N\}.$$

But $f(\Lambda_1 M) = \Lambda_1 \cdot f(M)$ for each A-homomorphism f, so

$$\mathrm{Hom}_{\Lambda_1}(\Lambda_1 M, \Lambda_1 N) = \{f \in \mathrm{Hom}_A(KM, KN): \ \Lambda_1 f(M) \subset \Lambda_1 N\}.$$

This shows that there are embeddings

$$\mathrm{Hom}_\Lambda(M, N) \rightarrow \mathrm{Hom}_{\Lambda_1}(\Lambda_1 M, \Lambda_1 N) \subset \mathrm{Hom}_A(KM, KN).$$

When M = N, we obtain ring monomorphisms

$$\mathrm{End}_\Lambda(M) \rightarrow \mathrm{End}_{\Lambda_1}(\Lambda_1 M) \rightarrow \mathrm{End}_A(KM).$$

Remark. There is a Λ_1-surjection $\Lambda_1 \otimes_\Lambda M \rightarrow \Lambda_1 M$, but this map need not be monic. Thus, we must be careful not to identify $\Lambda_1 \otimes_\Lambda M$ with $\Lambda_1 M$ in general.

Suppose next that M,N are Λ_1-lattices; then $\Lambda_1 M = M$ in KM, and likewise $\Lambda_1 N = N$. Thus, we obtain $\mathrm{Hom}_\Lambda(M, N) = \mathrm{Hom}_{\Lambda_1}(M, N)$. Consequently, $M \cong N$ as Λ_1-lattices if and only if $M \cong N$ as Λ-lattices. Furthermore,

$$\mathrm{End}_{\Lambda_1}(M) = \mathrm{End}_\Lambda(M),$$

so M is Λ_1-indecomposable if and only if M is Λ-indecomposable.

We call $\mathscr{L}(\Lambda)$ of *finite type* if there are only a finite number of isomorphism classes of indecomposable Λ-lattices. The above discussion yields

(1.3) PROPOSITION. *Let $\Lambda \subset \Lambda_1 \subset A$. If $\mathscr{L}(\Lambda_1)$ is of infinite type, then so is $\mathscr{L}(\Lambda)$. If $\mathscr{L}(\Lambda)$ is of finite type, then so is $\mathscr{L}(\Lambda_1)$.*

Assume now that A is a separable K-algebra, and that K is a global field. For each prime ideal P of R, let Λ_P denote the P-adic completion of Λ; it is an R_P-order in the K_P-algebra A_P. We quote without proof

(1.4) THEOREM. (Jones [15]). *$\mathscr{L}(\Lambda)$ is of finite type if and only if $\mathscr{L}(\Lambda_P)$ is of finite type for each* P.

Thus, the problem of deciding whether $\mathscr{L}(\Lambda)$ is of finite type reduces to the case where Λ is an R-order, with R a complete discrete valuation ring in the completion of a global field. If P = rad R, then we know that the residue class field $\bar{R} = R/P$ is finite. In the special case where Λ is commutative, we may write $\Lambda = \amalg \Lambda_i$, a direct sum of indecomposable ideals Λ_i. Then each Λ_i is a commutative local ring, and clearly $\mathscr{L}(\Lambda)$ is of finite type if and only if each $\mathscr{L}(\Lambda_i)$ is of finite type. Thus, the problem reduces to the study of $\mathscr{L}(\Lambda)$ for the case where Λ is a commutative local ring. In this case, $\Lambda/\mathrm{rad}\,\Lambda$ is a finite extension field of $\bar{R}$.

For Λ a commutative local R-order, where R is a complete discrete valuation ring as above, the question as to when $\mathscr{L}(\Lambda)$ has finite type has been settled by Jacobinski [14], and also by Drozd-Roiter [8] by somewhat different techniques.

Our aim here is to reprove these results by using some ideas connected with representations of diagrams, and to indicate directions for further work in this area.

As preparation for stating the Drozd-Roiter version of these results, we need some easy results about radicals of rings and modules. We shall state them in somewhat greater generality than needed below, since no extra effort is needed to prove them in the more general form.

For the moment, let Λ be an arbitrary ring (not necessarily commutative), and let X be a finitely generated left Λ-module. Define rad X as the intersection of all maximal submodules of X. We set $J = \operatorname{rad} \Lambda$, viewing Λ as left Λ-module, so J is the *Jacobson radical* of Λ. Put $\bar{\Lambda} = \Lambda / J$, $\bar{X} = X/JX$, and let $\mu_\Lambda(X)$ denote the minimal number of Λ-generators of X. Each surjection $\psi : \Lambda^{(n)} \to X$ yields a surjection $\bar{\psi} : \bar{\Lambda}^{(n)} \to \bar{X}$, and every $\bar{\psi}$ comes from some ψ. This gives

$$\mu_\Lambda(X) = \mu_{\bar{\Lambda}}(\bar{X}) \tag{1.5}$$

for each finitely generated left Λ-module X. If $\bar{\Lambda}$ is a semisimple artinian ring, then rad $\bar{X} = 0$ for every finitely generated $\bar{\Lambda}$-module $\bar{X}$, which implies that

$$\operatorname{rad} X = JX \tag{1.6}$$

in this case. Furthermore, if Λ is local (that is, $\bar{\Lambda}$ is a skewfield), then obviously

$$\mu_{\bar{\Lambda}}(\bar{X}) = \dim_{\bar{\Lambda}}(\bar{X}). \tag{1.7}$$

Now assume that Λ is an R-order in a separable K-algebra A, and let X be a finitely generated R-torsion Λ-module. We may choose a nonzero ideal $\mathfrak{a}$ of R such that $\mathfrak{a}X = 0$, and view X as a $\Lambda/\mathfrak{a}\Lambda$-module. Since

$$\Lambda/\mathfrak{a}\Lambda \cong \coprod_{P|\mathfrak{a}} \Lambda_P/\mathfrak{a}\Lambda_P, \qquad X \cong \coprod_{P|\mathfrak{a}} X_P,$$

it follows at once that

$$\mu_\Lambda(X) = \operatorname*{Max}_{P|\mathfrak{a}} \mu_{\Lambda_P}(X_P) = \operatorname*{Max}_{\text{all } P} \mu_{\Lambda_P}(X_P). \tag{1.8}$$

Finally, if $\Lambda = \coprod \Lambda_i$ is a direct sum of rings, then we may write $X = \coprod X_i$, with X_i a Λ_i-module. Clearly,

$$\mu_\Lambda(X) = \operatorname*{Max}_{i} \mu_{\Lambda_i}(X_i). \tag{1.9}$$

The Drozd-Roiter criterion for the finiteness of $\mathscr{L}(\Lambda)$ is stated in terms of the Λ-module Λ'/Λ, where Λ' is a maximal R-order in A containing Λ. Specifically, we have

(1.10) THEOREM. *Let* A *be a commutative separable* K*-algebra, where* K *is a global field. Let* Λ *be an* R*-order in* A, *and* Λ' *a maximal* R*-order containing* Λ. *Then* $\mathscr{L}(\Lambda)$ *is of finite type if and only if*

$$\mu_\Lambda(\Lambda'/\Lambda) \le 2 \quad \text{and} \quad \mu_\Lambda(\text{rad}_\Lambda(\Lambda'/\Lambda)) \le 1.$$

Now Λ'/Λ is a finitely generated R-torsion Λ-module, so by (1.8) we have $\mu_\Lambda(\Lambda'/\Lambda) = \text{Max}_P\, \mu_{\Lambda_P}(\Lambda'_P/\Lambda_P)$. Note that Λ'_P is a maximal R_P-order in A_P containing Λ_P. Likewise, $\mu_\Lambda(\text{rad}_\Lambda(\Lambda'/\Lambda))$ can be computed locally. Further, by Jones' Theorem, the question as to whether $\mathscr{L}(\Lambda)$ is of finite type can be decided locally. Finally, any commutative order over a complete discrete valuation ring can be written as a direct sum of local orders. Hence, (1.10) is equivalent to the following assertion:

(1.11) THEOREM. *Let* A *be a commutative separable* K*-algebra, where* K *is the completion of a global field with respect to a discrete valuation. Let* R *be the valuation ring of* K, Λ *a local* R*-order in* A, *and* Λ' *a maximal* R*-order in* A *containing* Λ. *Then* $\mathscr{L}(\Lambda)$ *is of finite type if and only if*

$$\mu_\Lambda(\Lambda'/\Lambda) \le 2 \quad \text{and} \quad \mu_\Lambda(\text{rad}(\Lambda'/\Lambda)) \le 1. \tag{1.12}$$

Keeping the hypotheses of the above theorem, let $\bar{R}$ be the residue class field of R, and let $J = \text{rad}\,\Lambda$. Then $\bar{\Lambda} = \Lambda/J$ is a field, finite dimensional over $\bar{R}$, and thus $\bar{\Lambda}$ is a finite field k. For any finitely generated Λ-module X, we have $\text{rad}_\Lambda X = JX$ and $\mu_\Lambda(X) = \mu_{\bar{\Lambda}}(X/JX) = \dim_k(X/JX)$. Let us now show that

$$\dim_k \Lambda'/J\Lambda' = 1 + \mu_\Lambda(\Lambda'/\Lambda). \tag{1.13}$$

For X a finitely generated Λ-module, put $\bar{X} = X/JX \cong k \otimes_\Lambda X$. Then we have seen that $\mu_\Lambda(X) = \dim_k \bar{X}$. Now apply $k \otimes_\Lambda *$ to the exact sequence of left Λ-modules: $0 \to \Lambda \to \Lambda' \to \Lambda'/\Lambda \to 0$. We obtain a k-exact sequence $k \xrightarrow{\psi} \overline{\Lambda'} \to \overline{\Lambda'/\Lambda} \to 0$, in which $\psi(1) = 1$. If $\psi(1) = 0$, then $1 \in J\Lambda'$, so $\Lambda' = J\Lambda'$; this is impossible by Nakayama's Lemma, since Λ' is a finitely generated Λ-module. Hence ψ is monic, and so $\dim_k \overline{\Lambda'} = 1 + \dim_k \overline{(\Lambda'/\Lambda)} = 1 + \mu_\Lambda(\Lambda'/\Lambda)$, as desired.

Finally, we remark that $\text{rad}_\Lambda(\Lambda'/\Lambda) = J\cdot(\Lambda'/\Lambda) = (J\Lambda' + \Lambda)/\Lambda$. Therefore $\mu_\Lambda(\text{rad}_\Lambda(\Lambda'/\Lambda)) = \mu_\Lambda((J\Lambda' + \Lambda)/\Lambda) = \dim_k \dfrac{(J\Lambda' + \Lambda)/\Lambda}{J\cdot(J\Lambda' + \Lambda)/\Lambda} = \dim_k \dfrac{J\Lambda' + \Lambda}{J^2\Lambda' + \Lambda}$. Conditions (1.12) may thus be rewritten as

$$\dim_k \Lambda'/J\Lambda' \le 3, \qquad \dim_k (J\Lambda' + \Lambda)/(J^2\Lambda' + \Lambda) \le 1. \tag{1.14}$$

2. REDUCTION TO THE ARTINIAN CASE

Suppose now that R is a complete discrete valuation ring with quotient field K, and that Λ is an R-order in a finite dimensional semisimple K-algebra A. Let $\Lambda \subset \Lambda'$, where Λ' is a hereditary R-order in A. Eventually we shall consider the situation occurring in (1.11), where Λ is a local ring and Λ' is a maximal order in a commutative separable algebra A, but for the time being we shall treat the more general situation.

Let I be a proper two-sided Λ'-ideal in Λ, such that $K \cdot I = A$, and set

$$\Delta = \Lambda/I, \qquad \Gamma = \Lambda'/I, \tag{2.1}$$

so there is an inclusion of rings $\phi : \Delta \to \Gamma$, making each Γ-module, and Γ itself, into a Δ-module. Since $1 \notin I$, we have $\Delta \neq 0$. Both Δ and Γ are finitely generated R-torsion R-algebras, and of course $\Gamma/\Delta \cong \Lambda'/\Lambda$.

Given a left Λ-lattice M, define $M' = \Lambda' M$ computed inside KM. Thus M' is a Λ'-lattice, and since Λ' is hereditary, it follows (see [17, (10.7)]) that $M' \in \mathbf{P}(\Lambda')$, the set of finitely generated projective Λ'-modules. Clearly $IM' = I\Lambda' M = IM$, so the inclusion $M \subset M'$ induces an inclusion $M/IM \subset M'/IM'$. Note that M/IM is a finitely generated Δ-module, and that $Y = M'/IM' \in \mathbf{P}(\Gamma)$. Next, since R is complete and Δ is an R-algebra, the Δ-module M/IM has a Δ-projective cover X, unique up to isomorphism. Thus there is a Δ-surjection $X \to M/IM$, whose kernel lies in $(\operatorname{rad} \Delta)X$. Now let $f \in \operatorname{Hom}_\Delta(X, Y)$ be defined by composition of maps:

$$f: X \to M/IN \to M'/IM' = Y.$$

Then we have

$$Y = \Gamma \cdot f(X), \qquad \ker f \subset (\operatorname{rad} \Delta)\, X. \tag{2.2}$$

Let us introduce the category $\mathscr{A} = \mathscr{A}(\Delta,\Gamma,\phi)$, whose objects are triples (X,Y,f), where $X \in \mathbf{P}(\Delta)$, $Y \in \mathbf{P}(\Gamma)$, and where $f : X \to Y$ is a Δ-homomorphism satisfying (2.2). A morphism $(\alpha, \beta) : (X_1, Y_1, f_1) \to (X_2, Y_2, f_2)$ in $\mathscr{A}$ is a pair (α,β) such that $\alpha \in \operatorname{Hom}_\Delta(X_1,X_2)$, $\beta \in \operatorname{Hom}_\Gamma(Y_1,Y_2)$, and $f_2\alpha = \beta f_1$. Thus each $M \in \mathscr{L}(\Lambda)$ gives rise to an object $F(M) = (X, Y, f)$ by the preceding construction. Furthermore, each $\mu \in \operatorname{Hom}_\Lambda(M_1, M_2)$ in the category $\mathscr{L}(\Lambda)$ gives rise to a commutative diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & M_1/IM & \longrightarrow & M'_1/IM'_1 \\ {\scriptstyle \mu_0}\downarrow & & {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \mu_2} \\ X_2 & \longrightarrow & M_2/IM_2 & \longrightarrow & M'_2/IM'_2 \end{array}$$

The map μ_1 lifts to a map μ_0 of projective covers, and we thus obtain a morphism $(\mu_0, \mu_2) : F(M_1) \to F(M_2)$ in $\mathscr{A}$. However, F is *not* a functor from $\mathscr{L}(\Lambda)$ to $\mathscr{A}$, since F need not preserve compositions of morphisms. Nevertheless, if μ is an isomorphism, so is $F(\mu)$. Furthermore, $F(M_1 \oplus M_2) \cong F(M_1) \oplus F(M_2)$, since projective covers are "additive."

We wish to obtain information about $\mathscr{L}(\Lambda)$ by studying the category $\mathscr{A}$, which is easier to handle because Δ and Γ are artinian rings. This "reduction to the artinian case" occurs in one form or another in many of the earlier calculations with lattices, and is of course closely related to Jacobinski's technique of working modulo the conductor of Λ' in Λ. We shall show that in fact the categories $\mathscr{L}(\Lambda)$ and $\mathscr{A}$ are *representation equivalent,* that is, there is a bijection between the sets of isomorphism classes of objects in the two categories, and this bijection preserves indecomposability. For this purpose, we must construct a map $G : \mathscr{A} \to \mathscr{L}(\Lambda)$.

Starting with $(X,Y,f) \in \mathscr{A}$, we set $\bar{M} = f(X)$, a Δ-submodule of Y such that $\Gamma \cdot \bar{M} = Y$. Since $\ker f \subset (\operatorname{rad}\Delta)X$, the surjection $X \to \bar{M}$ gives a Δ-projective cover of $\bar{M}$. On the other hand, since Λ' is an R-order and R is complete, the method of lifting idempotents shows that every $Y \in \mathbf{P}(\Gamma)$ is of the form $Y \cong M'/IM'$ for some $M' \in \mathbf{P}(\Lambda')$, and then M' is the Λ'-projective cover of Y. Let M be defined as a pullback

$$\begin{array}{ccc} M & \dashrightarrow & M' \\ \downarrow & & \downarrow \\ \bar{M} & \longrightarrow & M'/IM' \cong Y, \end{array}$$

so M is a Λ-module. The inclusion $M \to M'$ shows that M is in fact a Λ-lattice. Further, from $\Gamma \cdot \bar{M} = Y$ we obtain $M' = \Lambda'M + IM'$. But $IM' \subset (\operatorname{rad}\Lambda')M'$, since $M' \to Y$ is a Λ'-projective cover, and thus we have $M' = \Lambda'M + (\operatorname{rad}\Lambda')M'$. By Nakayama's Lemma, we may conclude that $M' = \Lambda'M$; thus $IM' = IM$, and we obtain $\bar{M} \cong M/IM' = M/IM$. We set $G(X,Y,f) = M$, and it is then obvious that $F(M) \cong (X, Y, f)$, that is, $FG \approx 1$. Conversely, starting with any $M \in \mathscr{L}(\Lambda)$, it is clear from the above construction that $GF \approx 1$, that is, $GF(M) \cong M$.

We observe also that a morphism $(\alpha, \beta) : (X_1, Y_1, f_1) \to (X_2, Y_2, f_2)$ in $\mathscr{A}$ gives rise to a commutative diagram

$$\begin{array}{ccc} \bar{M}_1 & \longrightarrow & Y_1 \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \beta} \\ \bar{M}_2 & \longrightarrow & Y_2 \end{array},$$

where α induces α_*. Then β lifts to a map $M_1' \to M_2'$ of projective covers, and there is an induced map on pullbacks. Hence G carries (α,β) onto a morphism in $\mathscr{L}$. Clearly G preserves isomorphisms and direct sums. Just as before, however, G is not a functor since it need not preserve compositions of morphisms. In any case, the maps F,G give a bijection between the isomorphism classes of objects in $\mathscr{L}(\Lambda)$ and those in $\mathscr{A}$, preserving indecomposability. Therefore $\mathscr{L}(\Lambda)$ and $\mathscr{A}$ are representation equivalent, as claimed.

Conditions (2.2) often play a minor role in various calculations, so it is convenient to introduce the category $\mathscr{B}(\Delta,\Gamma,\phi)$ of triples (X,Y,f) with $X \in \mathbf{P}(\Delta)$, $Y \in \mathbf{P}(\Gamma)$, and $f \in \operatorname{Hom}_\Delta(X, Y)$. No further restrictions are imposed on f, and indeed ϕ may be any ring homomorphism from Δ to Γ, not necessarily monic. Thus $\mathscr{A}$ is a full subcategory of $\mathscr{B}$. Furthermore, there is a canonical isomorphism

$$\operatorname{Hom}_\Delta(X, Y) \cong \operatorname{Hom}_\Delta({}_\Gamma\Gamma_\Delta \otimes_\Delta X, Y),$$

where Δ acts on Γ via ϕ. This suggests a new point of view: let ${}_\Gamma M_\Delta$ be a finitely generated (Γ,Δ)-bimodule, and let $\mathscr{C}(\Delta,\Gamma;M)$ be the category whose objects are triples (X,Y,g), where $X \in \mathbf{P}(\Delta)$, $Y \in \mathbf{P}(\Gamma)$, and g is a left Γ-homomorphism

(2.3) $$g: M \otimes X \to Y .$$

We may thus identify $\mathscr{B}(\Delta,\Gamma,\phi)$ with the category $\mathscr{C}(\Delta,\Gamma;{}_{\Gamma}\Gamma_{\Delta})$, if desired.

This new category $\mathscr{C}(\Delta,\Gamma;M)$ is a familiar one from several standpoints. On the one hand, an object $(X,Y,g) \in \mathscr{C}$ may be considered as a representation of the modulated graph $\overset{\Delta}{\circ} \xrightarrow[{}_{\Gamma}M_{\Delta}]{} \overset{\Gamma}{\circ}$ The representation assigns to the vertex Δ the module $X \in \mathbf{P}(\Delta)$, to the vertex Γ the module $Y \in \mathbf{P}(\Gamma)$, and to the arrow a Γ-homomorphism g as in (2.3). Such representations have been studied extensively by Gabriel and Dlab-Ringel, among others, for the special case where Δ and Γ are skewfields. However, their results do not apply directly to our category $\mathscr{C}(\Delta,\Gamma;M)$, except in a few isolated instances. Nevertheless, the analogy is extremely suggestive, and leads to a more natural interpretation of the Drozd-Roiter conditions. It also suggests many questions worthy of further investigation.

A second approach to the study of $\mathscr{C}(\Delta,\Gamma;M)$ is to view $\mathscr{C}$ as a full subcategory of the category mod T of all finitely generated left T-modules, where $T = \begin{pmatrix} \Delta & 0 \\ M & \Gamma \end{pmatrix}$ is a ring of triangular matrices (see [9]). To be explicit, given any

$$(X, Y, g) \in \mathscr{C}(\Delta, \Gamma; M) ,$$

Δ define a Γ-module $\begin{pmatrix} X \\ Y \end{pmatrix}$ by the rule $\begin{pmatrix} \delta & 0 \\ m & \gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \delta x \\ g(m \otimes x) + \gamma y \end{pmatrix}$. The correspondence $(X,Y,g) \to \begin{pmatrix} X \\ Y \end{pmatrix}$ then embeds $\mathscr{C}$ in mod T , as is easily verified. For the reverse correspondence, let $e_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in T, and identify Δ with $e_R T e_R$, Γ with $e_S T e_S$. Let W be a T-module. Then there is a map

$$g: M \otimes_{\Delta} e_R W \to e_S W,$$

induced from the action of $\begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}$ on W. If W is such that $e_R W \in \mathbf{P}(\Delta)$ and $e_S W \in \mathbf{P}(\Gamma)$ then W gives rise to the object $(e_R W, e_S W, g) \in \mathscr{C}(\Delta, \Gamma; M)$, and this is the inverse of the embedding $\mathscr{C} \to$ mod T described above. Unfortunately, we have not been able to use this embedding in any significant manner.

Our final interpretation of the category $\mathscr{C}(\Delta,\Gamma;M)$ is by means of matrices, and for this we shall restrict our attention to the category $\mathscr{B}(\Delta,\Gamma,\phi)$ defined earlier. Let us consider the case in which Δ,Γ arise from the orders Λ,Λ' defined as in (1.11), so we assume Λ is local and commutative, and Λ' maximal. Then I is a proper ideal of Λ, so $I \subset \operatorname{rad} \Lambda$, and therefore $\Lambda/\operatorname{rad}\Lambda \cong \Delta/\operatorname{rad}\Delta = k$ (say). Thus Δ is also a local ring. Further, k is a field of finite dimension over the residue class field $\bar{R}$ of R.

Next, we may write $A = \coprod_{i=1}^{r} K_i$, where each K_i is a finite separable field extension of K. Let R_i denote the integral closure of R in K_i, so each R_i is also a complete discrete valuation ring, and K_i is its quotient field. Then (see [17, (10.5)]) the integral closure Λ' of R in A is the *unique* maximal R-order in A, and we have $\Lambda' = \coprod R_i$. Let P_i be the prime ideal of R_i. The Λ'-ideal I may then be written as $I = \prod_1^r P_i^{e_i}$, and thus

$$(2.4) \qquad \Gamma = \coprod_1^r (R_i/P_i^{e_i}) = \coprod_1^r \Gamma_i,$$

where each Γ_i is a local principal ideal ring (sometimes called a *uniserial* ring).

Given any $(X,Y,f) \in \mathscr{B}(\Delta,\Gamma,\phi)$, we have $X \cong \Delta^{(m)}$ for some m; since over a local ring Δ every projective Δ-module is free. Likewise, we may write each $Y \in \mathbf{P}(\Gamma)$ as $Y = \coprod_1^r Y_i$, where Y_i is Γ_i-projective and hence Γ_i-free, say $Y_i \cong \Gamma_i^{(n_i)}$. The Δ-map $f: X \to Y$ then gives Δ-maps $f_i : \Delta^{(m)} \to \Gamma_i^{(n_i)}$, $1 \le i \le r$, so we may represent (X,Y,f) by an r-tuple of matrices $(f_1,\ldots,f_r)$, where f_i is an $m \times n_i$ matrix over Γ_i. Any $\mathscr{B}$-endomorphism of (X,Y,f) then corresponds to an $m \times m$ matrix α over Δ, together with an r-tuple $(\beta_1,\ldots,\beta_r)$ where β_i is an $n_i \times n_i$ matrix over Γ_i, such that

$$(2.5) \qquad \alpha f_i \beta_i = f_i, \qquad 1 \le i \le r.$$

In particular, the object (X,Y,f) is decomposable in $\mathscr{B}$ if and only if there exist matrices $\alpha \in GL(m,\Delta)$, $\beta_i \in GL(n_i,\Gamma_i)$, $1 \le i \le r$, such that

$$(2.6) \qquad (\alpha f_1 \beta_1, \ldots, \alpha f_r \beta_r) = (f'_1, \ldots, f'_r) \oplus (f''_1, \ldots, f''_r)$$

for some matrices $\{f'_i, f''_i\}$.

We note that $\Gamma/\Delta \cong \Lambda'/\Lambda$, $\mathrm{rad}\,\Delta = (\mathrm{rad}\,\Lambda)/I$, the latter because $\mathrm{rad}\,\Lambda \supset I$. If we put $r = \mathrm{rad}\,\Delta$, then $\Delta/r \cong \Lambda/\mathrm{rad}\,\Lambda = k$, and

$$\mathrm{rad}_\Delta(\Gamma/\Delta) = r(\Gamma/\Delta) = (r\Gamma + \Delta)/\Delta.$$

Thus we have $\mu_\Lambda(\Lambda'/\Lambda) = \mu_\Delta(\Gamma/\Delta)$, $\mu_\Lambda(\mathrm{rad}_\Lambda(\Lambda'/\Lambda)) = \mu_\Delta(\mathrm{rad}_\Delta(\Gamma/\Delta))$, and $\mu_\Delta(\mathrm{rad}_\Delta(\Gamma/\Delta)) = \dim_k (r\Gamma + \Delta)/(r^2\Gamma + \Delta)$. As in the proof of (1.13), we obtain $\dim_k(\Gamma/r\Gamma) = 1 + \mu_\Delta(\Gamma/\Delta)$. Thus, in order to prove (1.11) and the Jacobinski-Drozd-Roiter Theorem, we must establish the following:

(2.7) THEOREM. *Let Δ,Γ be as above. Then $\mathscr{A}(\Delta,\Gamma,\phi)$ has only a finite number of non-isomorphic indecomposable objects if and only if*

$$(2.8) \qquad \dim_k \Gamma/r\Gamma \le 3 \quad \text{and} \quad \dim_k (r\Gamma + \Delta)/(r^2\Gamma + \Delta) \le 2.$$

3. CHANGE OF GROUND RING

Let $\Lambda \subset \Lambda'$ be as in (1.11), but for the moment we drop the hypothesis that Λ be local. Both $\Lambda/\text{rad}\,\Lambda$ and $\Lambda'/\text{rad}\,\Lambda'$ are direct sums of fields, and for later calculations it would be most convenient to have these fields all coincide. We shall show how to reduce the original problem to this case, by using a technique due to Jacobinski.

Let K′ be a finite unramified extension of K, to be chosen later, and let R′ be its valuation ring. We set $\Delta' = R' \otimes \Delta$, $\Gamma' = R' \otimes \Gamma$, where $\otimes$ means $\otimes_R$. We claim to begin with that

$$\text{(3.1)} \qquad \text{rad}\,\Delta' = R' \otimes (\text{rad}\,\Delta).$$

Certainly rad Δ' contains $R' \otimes \text{rad}\,\Delta$, and to prove equality it suffices to show that $\Delta'/(R' \otimes \text{rad}\,\Delta)$ is semisimple. This quotient is isomorphic to $R' \otimes (\Delta/\text{rad}\,\Delta)$, and hence to $(R'/PR') \otimes (\Delta/\text{rad}\,\Delta)$, since $P\Delta \subset \text{rad}\,\Delta$. (Here, P is the maximal ideal of R.) Since R′ is unramified over R, R'/PR' is the residue class field of R′, and we have $(R'/PR') \otimes_R (\Delta/\text{rad}\,\Delta) = (R'/PR') \otimes_{R/P} (\Delta/\text{rad}\,\Delta)$. Since $\Delta/\text{rad}\,\Delta$ is semisimple, and R'/PR' is separable over R/P, it follows that this latter tensor product is also semisimple. This completes the proof of (3.1); an analogous formula holds with Δ', Δ replaced by Γ', Γ, respectively. Furthermore,

$$\Gamma'/\Delta' \cong [R' \otimes (\Gamma/\Delta)], \ \text{rad}_{\Delta'} (\Gamma'/\Delta') \cong R' \otimes \text{rad}_\Delta (\Gamma/\Delta),$$

which gives

$$\text{(3.2)} \qquad \mu_{\Delta'}(\Gamma'/\Delta') = \mu_\Delta(\Gamma/\Delta), \qquad \mu_{\Delta'}(\text{rad}_{\Delta'}(\Gamma'/\Delta')) = \mu_\Delta(\text{rad}_\Delta(\Gamma/\Delta)).$$

We note also that $\Delta'/\text{rad}\,\Delta' \cong (R'/PR') \otimes_{R/P} (\Delta/\text{rad}\,\Delta)$, and likewise for Δ replaced by Γ.

Next, we claim that Γ' is a direct sum of local principal ideal rings. Now Γ is a direct sum of rings of the type $S/\mathfrak{P}^n$, where S is the valuation ring in some finite extension L of K, and $\mathfrak{P}$ is the maximal ideal of S. To prove our claim about Γ', it suffices to consider $R' \otimes (S/\mathfrak{P}^n)$, where $\otimes$ is $\otimes_R$. Of course, $R' \otimes (S/\mathfrak{P}^n) = (R' \otimes S)/(R' \otimes \mathfrak{P}^n)$, so let us turn our attention to $R' \otimes S$. We have $K' \otimes_K L \cong \coprod L_i$, a direct sum of fields L_i, each of which is a finite unramified extension of L. Now any R-basis of R′ is also an S-basis of $R' \otimes S$, so the discriminant of $R' \otimes S$ relative to S equals that of R′ relative to R, and hence is a unit in R. This proves that $R' \otimes S \cong \coprod S_i$, where S_i is the valuation ring of L_i. (This incidentally shows that $R' \otimes_R \Lambda'$ is a maximal R′-order in $K' \otimes_K A$.) Further, for each i we know that S_i is unramified over S, so $\mathfrak{P} S_i$ is its maximal ideal. Finally, we have $(R' \otimes S)/(R' \otimes \mathfrak{P}^n) \cong \coprod S_i/(\mathfrak{P} S_i)^n$, and thus Γ' is a ring of the same type as Γ.

We now prove that the category $\mathscr{A}(\Delta,\Gamma,\phi)$ is of finite type if and only if $\mathscr{A}(\Delta',\Gamma',\phi')$ is of finite type. Here, "finite type" means "has finitely many isomorphism classes of indecomposable objects." Let us write $\mathscr{A}$, $\mathscr{A}'$ for these categories, for brevity. Each $(X,Y,f) \in \mathscr{A}$ gives rise to an object $(X',Y',f') \in \mathscr{A}'$, where $X' = R' \otimes X$, etc. On the other hand, R' is R-free, say of rank m. Thence $\Delta' \cong \Delta^{(m)}$ as Δ-modules, etc., so that in $\mathscr{A}$ we have $(X', Y', f') \cong (X, Y, f)^{(m)}$. Since the Krull-Schmidt Theorem is valid in the category $\mathscr{A}$, it follows at once from the above that if $\mathscr{A}'$ is of finite type, then so is $\mathscr{A}$. This part of the argument does not depend on the fact that R' is unramified over R. This hypothesis is needed for the proof of the converse, however.

Given any object $(U,V,g) \in \mathscr{A}'$, we may view it as an object in $\mathscr{A}$, and then form $(R' \otimes U, R' \otimes V, 1 \otimes g) \in \mathscr{A}'$. We shall show that (U,V,g) is a direct summand (in $\mathscr{A}'$) of $(R' \otimes U, R' \otimes V, 1 \otimes g)$. Once this is done, it will follow at once that if $\mathscr{A}$ is of finite type, so is $\mathscr{A}'$.

Let G be the Galois group of K' over K, and consider the Δ'-module U. Since $U \in \mathbf{P}(\Delta')$ and Δ' is Δ-free, also $U \in \mathbf{P}(\Delta)$. We shall investigate the Δ'-module $R' \otimes_R U$, formed by viewing U as a Δ-module and tensoring with R'. For each $\sigma \in G$, we form a Δ'-module U^σ having the same elements as U, but where the action of Δ' is given by

$$\underbrace{(a \otimes \delta) * u}_{\text{in } U^\sigma} = \underbrace{(a^\sigma \otimes \delta) u}_{\text{in } U}, \qquad a \in R', \delta \in \Delta, u \in U.$$

We shall establish that

$$R' \otimes_R U \cong \coprod_{\sigma \in G} U^\sigma \text{ as left } \Delta'\text{-modules.}$$

We begin by defining a map $T : R' \otimes U \rightarrow \coprod U^\sigma$ by

$$T(a \otimes u) = \coprod_{\sigma \in G} a^\sigma u, \qquad a \in R', u \in U.$$

(i). T is a Δ'-homomorphism. For if $b \otimes \delta \in \Delta'$, where $b \in R'$, $\delta \in \Delta$, then

$$T((b \otimes \delta)(a \otimes u)) = T(ba \otimes \delta u) = \coprod b^\sigma a^\sigma \cdot \delta u = (b \otimes \delta) \coprod a^\alpha u = (b \otimes \delta) T(a \otimes u).$$

(ii). T is an isomorphism. For let $R' = \coprod_1^m Ra_i$, $G = \{\sigma_1, \ldots, \sigma_m\}$. Given elements $u_1, \ldots, u_m \in U$, we must find $w_1, \ldots, w_m \in U$ such that $T\left(\sum_1^m a_i \otimes w_i\right) = \coprod u_j$, where $u_j \in U^{\sigma_j}$. This condition gives $\sum_{i=1}^m a_i^{\sigma_j} w_i = u_j$, $1 \le j \le m$. Since R' is unramified over R, $\det(a_i^{\sigma_j})_{1 \le i,j \le m}$ is a unit in R (see Weiss [20]). Hence the equations can be solved uniquely for the w's in U.

(iii). We started with a triple $(U,V,g) \in \mathscr{A}'$. We have just shown the existence of Δ'- and Γ'-isomorphisms

$$T_1 : R' \otimes U \cong \coprod U^{\sigma}, \qquad T_2 : R' \otimes V \cong \coprod V^{\sigma}.$$

We define $g_\sigma : U^\sigma \to V^\sigma$ by $g_\sigma(u) = gu$, $u \in U^\sigma$. We claim that g_σ is a Δ'-homomorphism. Indeed, we have

$$(a^\sigma \otimes \delta)\, g(u) = g_\sigma((a^\sigma \otimes \delta)\, u) = g_\sigma((a \otimes \delta) * u) = (a \otimes \delta) * g_\sigma(u) = (a^\sigma \otimes \delta)\, g(u).$$

Thus we obtain $(R' \otimes U, R' \otimes V, 1 \otimes g) \cong \coprod_{\sigma \in G} (U^\sigma, V^\sigma, g_\sigma)$ in $\mathscr{A}'$. Note that $V^\sigma = \Gamma' \cdot g_\sigma(U^\sigma)$ because $V = \Gamma' \cdot g(U)$; further, $\ker g_\sigma = \ker g \subset (\operatorname{rad} \Delta')U^\sigma$. Thus each $(U^\sigma, V^\sigma, g_\sigma)$ is an object in $\mathscr{A}'$. This completes the proof that $\mathscr{A}$ is of finite type if and only if $\mathscr{A}'$ is of finite type.

Remark. It is easily shown that if $(X,Y,f) \in \mathscr{A}$ is indecomposable, then $(R' \otimes X, R' \otimes Y, 1 \otimes f)$ splits into at most m indecomposable summands in $\mathscr{A}'$, where m is the R-rank of R$'$. Hence if $\mathscr{A}$ contains indecomposables of arbitrarily large size then so does $\mathscr{A}'$.

Now let us return to the original problem, in which $\Delta/\operatorname{rad}\Delta$ and $\Gamma/\operatorname{rad}\Gamma$ are direct sums of fields $\{k_i\}$, each a finite extension of the residue class field k of R. We may choose a finite extension k' of k containing all of the fields k_i; in that case, $k' \otimes_k k_i$ is a direct sum of $(k_i : k)$ copies of k'. Choose an unramified extension K' of K, whose valuation ring R' has residue class field k' (this can always be done; see Weiss [20]). Then $\Delta'/\operatorname{rad}\Delta' \cong k' \otimes_k (\Delta/\operatorname{rad}\Delta) \cong \coprod k'$, and likewise for $\Gamma'/\operatorname{rad}\Gamma'$. Replacing our original Δ,Γ by Δ',Γ' does not affect the hypothesis (in view of (3.2)), nor the conclusion (since $\mathscr{A}$ and $\mathscr{A}'$ are of the same type). Of course, if Δ is local, the ring Δ' need not be local; this causes no difficulty, since we may then decompose Δ' into a direct sum of local rings, with a corresponding decomposition of Γ'. Therefore, in proving Theorem 2.7, we may hereafter assume that Δ is local, that Γ is a direct sum of local principal ideal rings, and that $\Delta/\operatorname{rad}\Delta \cong k$, $\Gamma/\operatorname{rad}\Gamma \cong \coprod_1^r k$, where k is the residue class field of R.

4. INFINITE TYPE

Our aim here is to prove the "only if" part of Theorem 2.7. In the course of the proof, we shall establish several reduction theorems about the categories $\mathscr{A}(\Delta,\Gamma,\phi)$ which apply to somewhat more general situations.

As a preliminary step, let $\mathscr{V}$ be the category whose objects are pairs (V,θ), with V a finite dimensional space over a skewfield k, and $\theta \in \operatorname{End}_k V$. A morphism $\mu : (V,\theta) \to (V',\theta')$ is a map $\mu \in \operatorname{Hom}_k(V,V')$ for which $\mu\theta = \theta'\mu$. It is well known that $\mathscr{V}$ is of infinite type, as we see from the following argument: for each n, choose $V = k[x]/(x^n)$ on which θ acts as multiplication by x. Since V is indecomposable as a $k[x]$-module, it follows readily that (V,θ) is indecomposable in $\mathscr{V}$. Thus $\mathscr{V}$ contains indecomposables of arbitrarily large size, and hence is of infinite type. This construction lies at the heart of most proofs of infinite type.

Now let $\bar{\Delta} = \Delta/\mathfrak{a}$, $\bar{\Gamma} = \Gamma/\mathfrak{b}$, where $\mathfrak{a}$ is a two-sided ideal of Δ, and $\mathfrak{b}$ of Γ. Every $\bar{\Delta}$-module is also a Δ-module, whence if Δ is of finite representation type, so is $\bar{\Delta}$. Let us generalize this result to the case of categories $\mathscr{A}(\Delta,\Gamma,\phi)$:

(4.1) PROPOSITION. *Let $\phi : \Delta \rightarrow \Gamma$ be a homomorphism of artinian rings such that $\phi(\mathfrak{a}) \subset \mathfrak{b}$, and let $\bar{\phi} : \bar{\Delta} \rightarrow \bar{\Gamma}$ be the ring homomorphism induced by ϕ. Set $\mathscr{A} = \mathscr{A}(\Delta, \Gamma, \phi)$ and $\bar{\mathscr{A}} = \mathscr{A}(\bar{\Delta}, \bar{\Gamma}, \bar{\phi})$. Then if $\mathscr{A}$ is of finite type, so is $\bar{\mathscr{A}}$.*

Proof. For each object $\bar{A} \in \bar{\mathscr{A}}$ we shall find an object $F(\bar{A}) \in \mathscr{A}$, in such a manner that

(i). If $\bar{A}$ is indecomposable, then so is $F(\bar{A})$, and

(ii). If $F(\bar{A}) \cong F(\bar{B})$, then $\bar{A} \cong \bar{B}$.

This clearly implies the desired result.

Given $\bar{A} = (\bar{X},\bar{Y},\bar{f}) \in \bar{\mathscr{A}}$, let $\bar{W} = \bar{f}(\bar{X}) \subset \bar{Y}$ and let $\rho: Y \rightarrow \bar{Y}$ be the Γ-projective cover of $\bar{Y}$. Since $\bar{Y}$ is projective, it follows readily that $\ker \rho = \mathfrak{b}Y \subset (\mathrm{rad}\,\Gamma)Y$. Now define W as a pullback

$$\begin{array}{ccccc} & & W & \overset{\psi}{\dashrightarrow} & Y \\ & & {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\rho} \\ \bar{X} & \xrightarrow{\bar{f}} & \bar{W} & \longrightarrow & \bar{Y} \end{array}.$$

Then ψ is monic, and $\ker \sigma \cong \ker \rho$. Let $\tau: X \rightarrow W$ be a Δ-projective cover of W, and set $F(\bar{A}) = (X, Y, \psi\tau)$. We now verify that $F(\bar{A}) \in \mathscr{A}$. First of all,

$$\ker \psi\tau = \ker \tau \subset (\mathrm{rad}\,\Delta)X,$$

since ψ is monic and τ gives a projective cover. Secondly, from $\bar{Y} = \overline{\Gamma W}$ we obtain $Y = \Gamma\psi\tau(X) + \mathfrak{b}Y$, whence $Y = \Gamma\psi\tau(X)$. Thus $F(\bar{A}) \in \mathscr{A}$, as claimed.

Suppose now that $(X, Y, \psi\tau) = (X_1, Y_1, g_1) \oplus (X_2, Y_2, g_2)$ in $\mathscr{A}$. Then we have $\bar{W} = W/\mathfrak{b}Y = \bar{W}_1 \oplus \bar{W}_2$, where $\bar{W}_i = \sigma\tau(X_i)$, and correspondingly

$$\bar{Y} = Y/\mathfrak{b}Y = \bar{Y}_1 \oplus \bar{Y}_2,$$

with $\bar{W}_i \subset \bar{Y}_i$. If $\bar{V}_i$ is a $\bar{\Delta}$-projective cover of $\bar{W}_i$, $i = 1,2$, then we find at once that $\bar{A} = (\bar{X}, \bar{Y}, \bar{f}) \cong (\bar{V}_1, \bar{Y}_1, g_1) \oplus (\bar{V}_2, \bar{Y}_2, g_2)$ for some maps g_1, g_2. This shows that if $F(\bar{A})$ decomposes, then so does $\bar{A}$. Property (ii) is an easy consequence of the type of reasoning given in Section 2.

At the end of this section, we shall give an example in which $\bar{\mathscr{A}}$ is of finite type, but $\mathscr{A}$ of infinite type. Thus, the converse of (4.1) need not hold in general. Nevertheless, we may establish:–

(4.2) COROLLARY. *Let* $\phi: \Delta \to \Gamma$ *be a homomorphism of artinian rings with kernel* $\mathfrak{a}$, *and let* $\bar{\phi}: \Delta/\mathfrak{a} \to \Gamma$ *be the induced inclusion. Then* $\mathscr{A}(\Delta,\Gamma,\phi)$ *is of finite type if and only if* $\mathscr{A}(\Delta/\mathfrak{a}, \Gamma, \bar{\phi})$ *is of finite type.*

Proof. Taking $\mathfrak{b} = 0$, $\mathfrak{a} = \ker \phi$ in (4.1), we see that if $\mathscr{A}$ is of finite type, then so is $\bar{\mathscr{A}}$. To prove the converse, we define a functor $G: \mathscr{A} \to \bar{\mathscr{A}}$ as follows: given an object $A = (X, Y, f) \in \mathscr{A}$, we have $f(\mathfrak{a}X) = \phi(\mathfrak{a})f(X) = 0$, so f induces a $\bar{\Delta}$-map $\bar{f}: \bar{X} \to Y$, where $\bar{X} = X/\mathfrak{a}X$. We set $G(A) = (\bar{X}, Y, \bar{f})$ and verify that $G(A) \in \bar{\mathscr{A}}$. It follows from the equality $Y = \Gamma f(X)$ that also $Y = \Gamma \bar{f}(\bar{X})$, so it remains for us to show that $\ker \bar{f} \subset (\operatorname{rad} \bar{\Delta})\bar{X}$. But

$$\begin{aligned} \ker \bar{f} &= (\mathfrak{a}X + \ker f)/\mathfrak{a}X \subset (\mathfrak{a}X + (\operatorname{rad}\Delta)X)/\mathfrak{a}X \\ &\subset \{(\mathfrak{a} + \operatorname{rad}\Delta)/\mathfrak{a}\}\,\{X/\mathfrak{a}X\} \subset (\operatorname{rad}\bar{\Delta})\bar{X} \end{aligned}$$

as desired. It is clear that G is a well defined additive functor.

Furthermore, let $F: \bar{\mathscr{A}} \to \mathscr{A}$ be the set map defined in (4.1). Then we have $A \cong FG(A)$ for each $A \in \mathscr{A}$; hence non-isomorphic A's give non-isomorphic G(A)'s in $\bar{\mathscr{A}}$, and if A is indecomposable, so is G(A). Thus, if $\mathscr{A}$ is of infinite type, then so is $\bar{\mathscr{A}}$. This completes the proof.

We may observe that the categories $\mathscr{A}$ and $\bar{\mathscr{A}}$ of (4.2) are representation equivalent. Hence, in discussing the category $\mathscr{A}(\Delta,\Gamma,\phi)$, there is no loss of generality in assuming that ϕ is monic. We shall restrict our attention to this case hereafter, and omit the symbol ϕ. The above methods of proof (compare section 2) also yield

(4.3) COROLLARY. *Let* $\Delta \subset \Gamma$, *and let* $\mathfrak{b}$ *be the conductor of* Γ *in* Δ, *that is, the largest two-sided* Γ*-ideal in* Δ. *Then* $\mathscr{A}(\Delta,\Gamma)$ *is representation equivalent to* $\mathscr{A}(\Delta/\mathfrak{b}, \Gamma/\mathfrak{b})$.

Turning now to specific calculations, for the remainder of this section we put

$$\Gamma = \prod_1^s \Gamma_i, \quad r_i = \operatorname{rad}\Gamma_i, \quad N = \operatorname{rad}\Gamma = \prod_1^s r_i, \quad \Gamma_i = \Gamma e_i, \qquad (4.4)$$
$$r = \operatorname{rad}\Delta, \quad \Delta/r = k,$$

and assume that Δ is local (so k is a skewfield). The next result is essentially due to Dade [3]):

(4.5) PROPOSITION. $\mathscr{A}(\Delta,\Gamma)$ *is of infinite representation type if* $s \geq 4$.

Proof. Replacing $\Gamma_4 \times \ldots \times \Gamma_s$ by a new Γ'_4 and changing notation, we may hereafter assume that $s = 4$. Put $L_i = \Gamma_i/r_i$, $1 \leq i \leq 4$. Since Δ is local, we have $r = N \cap \Delta$; choosing $\mathfrak{a} = r$, $\mathfrak{b} = N$ in (4.1), we see that in order to prove that $\mathscr{A}(\Delta,\Gamma)$ is of infinite type, it suffices to show that $\mathscr{A}(k, L_1 \times L_2 \times L_3 \times L_4)$ is of infinite type. We may write $L = \prod L_i$, $L_i = L\bar{e}_i$, with the $\{\bar{e}_i\}$ central idempotents.

Now let $\mathscr{V}$ be the category defined earlier in this section. For each indecomposable object $(V,\theta) \in \mathscr{V}$, we shall construct an indecomposable object $F(V, \theta) \in \mathscr{A}(k, L)$.

Since $\mathscr{V}$ has arbitrarily large indecomposables, the same will then be true for $\mathscr{A}(k,L)$.

We choose $F(V,\theta) = (V\oplus V, L\otimes_k V, \psi)$, where

$$\psi(v, v') = \bar{e}_1 \otimes v + \bar{e}_2 \otimes v' + \bar{e}_3 \otimes (v + v') + \bar{e}_4 \otimes (v + \theta v'), \qquad v, v' \in V.$$

Each $\mathscr{A}$-endomorphism of $F(V,\theta)$ is given by a pair (α,β) such that

$$\text{(4.6)} \qquad \alpha \in \mathrm{End}_k(V \oplus V), \quad \beta \in \mathrm{End}_L(L \otimes_k V), \qquad \psi\alpha = \beta\psi.$$

Write $\alpha = (\alpha_{ij})^{2\times 2}$, with each $\alpha_{ij} \in \mathrm{End}_k V$. We may also write

$$\beta(\bar{e}_i \otimes v) = \bar{e}_i \otimes \beta_i v \qquad \text{for some } \beta_i \in \mathrm{End}_k V,\ 1 \leq i \leq 4.$$

The condition $\psi\alpha(v, 0) = \beta\psi(v, 0)$ then becomes

$$\begin{aligned} &\bar{e}_1 \otimes \alpha_{11} v + \bar{e}_2 \otimes \alpha_{21} v + \bar{e}_3 \otimes (\alpha_{11} v + \alpha_{21} v) + \bar{e}_4 \otimes (\alpha_{11} v + \theta\alpha_{21} v) \\ &\quad = \bar{e}_1 \otimes \beta_1 v + \bar{e}_3 \otimes \beta_3 v + \bar{e}_4 \otimes \beta_4 v \qquad \text{for all } v \in V. \end{aligned}$$

This gives $\alpha_{11} = \beta_1, \alpha_{21} = 0, \alpha_{11} = \beta_3, \alpha_{11} = \beta_4$. On the other hand, from the condition $\psi\alpha(0, v) = \beta\psi(0, v)$ we obtain $\alpha_{12} = 0, \alpha_{22} = \beta_2 = \beta_3, \theta\alpha_{22} = \beta_4\theta$. Hence, $\alpha = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}$, $\beta = 1 \otimes \gamma$, where $\gamma \in \mathrm{End}_k V$ and $\theta\gamma = \gamma\theta$. But then γ is a $\mathscr{V}$-endomorphism of the indecomposable object (V,θ). Hence γ cannot be a non-trivial idempotent, whence neither can α. This shows that $F(V,\theta)$ is indecomposable in $\mathscr{A}$, and completes the proof.

(4.7) COROLLARY. *Let* R *be a complete discrete valuation ring with quotient field* K *and let* Λ *be a local* R*-order in a* K*-algebra having at least* 4 *idempotents. Then* $\mathscr{L}(\Lambda)$ *is of infinite type.*

In the same vein, we prove

(4.8) LEMMA. *Let* Γ *be a finite dimensional algebra over a field* k, *and let* $N = \mathrm{rad}\,\Gamma$ *be such that* $\dim_k N/N^2 \geq 2$. *Then* $\mathscr{A}(k,\Gamma)$ *is of infinite type.*

Proof. By taking $\bar{\mathfrak{a}} = 0$ and $\bar{\mathfrak{b}} = N^2$ in (4.1), it is clear that we need only test the case where $N^2 = 0$. For this case, let $\Gamma = M \oplus N$ as k-spaces, where

$$M = \coprod_1^m k y_i, \qquad N = \coprod_1^n k x_j;$$

we may assume that $y_1 = 1$. Then $n \geq 2$ by hypothesis.

Now define $G: \mathscr{V} \to \mathscr{A}(k,\Gamma)$ by $G(V, \theta) = (V \oplus V, \Gamma \otimes_k V, \psi)$, where

$$\psi(v, v') = 1 \otimes v + x_1 \otimes v' + x_2 \otimes \theta v', \qquad v, v' \in V.$$

Note that ψ is monic, and that $\Gamma \cdot \mathrm{im}\,\psi = \Gamma \otimes V$, so $G(V,\theta)$ is indeed an object in $\mathscr{A}(k,\Gamma)$. The $\mathscr{A}$-endomorphism ring of $G(V,\theta)$ consists of pairs (α,β) satisfying (4.6).

Since $\Gamma \otimes_k V = \coprod y_i \otimes V \oplus \coprod x_j \otimes V$, we may write

$$\beta(1 \otimes v) = \sum y_i \otimes \beta_i v + x_j \otimes \gamma_j v, \qquad v \in V,$$

with each $\beta_i, \gamma_j \in \mathrm{End}_k V$. The condition $\psi\alpha(v, 0) = \beta\psi(v, 0)$ then gives

$$\alpha_{11} = \beta_1, \quad 0 = \beta_2 = \dots = \beta_m, \quad \alpha_{21} = \gamma_1, \quad \theta\alpha_{21} = \gamma_2, \quad \gamma_3 = \dots = \gamma_n = 0.$$

On the other hand, from $\psi\alpha(0, v) = \beta\psi(0, v)$ we obtain

$$\alpha_{21} = 0, \quad \alpha_{22} = \beta_1, \quad \beta_1\theta = \theta\alpha_{22}.$$

Thus $\alpha = \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & \alpha_{11} \end{pmatrix}$, $\alpha_{11}\theta = \theta\alpha_{11}$. Suppose now that (V,θ) is indecomposable in $\mathscr{V}$, and let (α,β) be idempotent. Then α_{11} is idempotent, hence $\alpha_{11} = 0$ or 1. Replacing (α,β) by $1 - (\alpha,\beta)$ if need be, we may assume that $\alpha_{11} = 0$. But then the equation $\alpha^2 = \alpha$ shows that $\alpha_{21} = 0$, so $\alpha = 0$ and $\beta = 0$. This shows that $G(V,\theta)$ is indecomposable, and completes the proof of the lemma.

From now on we restrict our attention to the case where Δ and Γ are commutative, and Δ is local. Then $\Delta \cap r\Gamma = r$, and hence

$$(r\Gamma + \Delta)/(r^2\Gamma + \Delta) = r\Gamma/(r + r^2\Gamma). \tag{4.9}$$

We are now ready to establish the "only if" part of Theorem 2.7. Keeping the notation of (4.4), we have seen in Section 3 that it suffices to treat the case where $\Gamma_i/r_i \cong k$ for each i, and we assume this to be so hereafter. We must show that if either $\dim_k \Gamma/r\Gamma > 3$ or $\dim_k r\Gamma/(r + r^2\Gamma) > 1$, then $\mathscr{A}(\Delta,\Gamma)$ is of infinite type.

Case 1. Suppose first that $\dim \Gamma/r\Gamma > 3$; by (4.1), it suffices to prove that $\mathscr{A}(\Delta/r, \Gamma/r\Gamma)$ is of infinite type. Changing notation, we must now show that $\mathscr{A}(k,\Gamma)$ is of infinite type whenever $\dim \Gamma > 3$. By (4.5) and (4.8), we already know that $\mathscr{A}(k,\Gamma)$ is of infinite type whenever $s > 3$ or $\dim N/N^2 > 1$. Thus, we may assume that $s \le 3$ and $\dim N/N^2 \le 1$ from now on. We keep the notation of (4.4).

Since $\dim N/N^2 \le 1$, it follows that at most one r_i is nonzero. If each $r_i = 0$, then $\dim \Gamma = s \le 3$, contradicting our hypothesis. Hence exactly one $r_i \ne 0$, so suppose that $r_1 \ne 0$. We are assuming that each Γ_i is a principal ideal ring, so we may choose a generator π_1 of r_1. Let $b \ge 1$ be minimal such that $\pi_1^b = 0$; then $\dim \Gamma = b + s - 1$, so there are exactly three possible choices for Γ, namely:

$$\Gamma = \Gamma_1 \times k \times k, \quad b > 1; \qquad \Gamma = \Gamma_1 \times k, \quad b > 2; \qquad \Gamma = \Gamma_1, \quad b > 3.$$

Now define $G: \mathscr{V} \to \mathscr{A}(k,\Gamma)$ by $G(V, \theta) = (V \oplus V, \Gamma \otimes_k V, \psi)$, where

$$\psi(v, v') = x_1 \otimes v + x_2 \otimes v' + x_3 \otimes (v + v') + x_4 \otimes (v + \theta v'), \qquad v, v' \in V.$$

The $\{x_i\}$ are to be chosen as follows, according to the possible Γ's:

$$x_1 = e_1, \quad x_2 = e_2, \quad x_3 = e_3, \quad x_4 = \pi_1;$$
$$x_1 = e_1, \quad x_2 = e_2, \quad x_3 = \pi_1, \quad x_4 = \pi_1^2;$$
$$x_1 = 1, \quad x_2 = \pi_1, \quad x_3 = \pi_1^2, \quad x_4 = \pi_1^3.$$

It is easily checked in each case that if (V,θ) is indecomposable, then so is G(V,θ). Thus $\mathscr{A}(k,\Gamma)$ is of infinite type in this case, as claimed.

Case 2. Suppose now that $\dim \Gamma/r\Gamma \leq 3$, and that

$$\dim r\Gamma/(r + r^2\Gamma) \geq 2 \tag{4.10}$$

By (4.1), it suffices to show that $\mathscr{A}(\Delta/r^2, \Gamma/r^2\Gamma)$ is of infinite type. The hypotheses on dimensions remain valid if we replace Δ by Δ/r^2, and Γ by $\Gamma/r^2\Gamma$; doing so, and changing notation, we may hereafter assume that $r^2 = 0$. Let

$$r_i = \Gamma_i \pi_i, \qquad 1 \leq i \leq s,$$

and let b_i be minimal such that $\pi_i^{b_i} = 0$. If $r\Gamma = \prod_1^s r_i^{a_i}$, then each $a_i \geq 1$, and

$$3 \geq \dim \Gamma/r\Gamma = \sum_1^s a_i .$$

From the above inequality and condition (4.10), we find readily that the only possible choices for Γ are

$$\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3, \quad r\Gamma = r_1 \times r_2 \times r_3, \quad b_1 = b_2 = b_3 = 2;$$
$$\Gamma = \Gamma_1 \times \Gamma_2, \quad r\Gamma = r_1^2 \times r_2, \quad b_1 = 4, \quad b_2 = 2;$$
$$\Gamma = \Gamma_1, \quad r\Gamma = r_1^3, \quad b_1 = 6.$$

In each case, $r = \Delta x$ for some $x \in \Delta$.

Given $(V,\theta) \in \mathscr{V}$, let $V \cong k^{(n)}$ and choose $P = \Delta^{(n)}$. We now set

$$G(V, \theta) = (P \oplus P, \Gamma \otimes_\Delta P, \psi),$$

where $\psi(p, q) = y_1 \otimes p + xy_2 \otimes q + xy_3 \otimes \theta q$, $p, q \in P$. The $\{y_i\}$ are chosen as follows, according to the three cases described above:

$$y_1 = 1, \quad y_2 = e_2, \quad y_3 = e_3;$$
$$y_1 = 1, \quad y_2 = \pi_1, \quad y_3 = e_2;$$
$$y_1 = 1, \quad y_2 = \pi_1, \quad y_3 = \pi_1^2.$$

Again, it is easily checked that if (V,θ) is indecomposable, then so is G(V,θ).

Thus $\mathscr{A}(\Delta,\Gamma)$ is of infinite type, as claimed. This completes the proof of the "only if" part of Theorem 2.7.

To conclude this section, consider the following example: let k be a field, and set $\Delta = k[x]/(x^2), \Gamma = \Delta \times \Delta \times \Delta$, with Δ embedded diagonally in Γ. Then $\dim \Gamma/r\Gamma = 3$ and $\dim (r\Gamma + \Delta)/(r^2\Gamma + \Delta) = 2$, so by the above proof we know that $\mathscr{A}(\Delta,\Gamma)$ is of infinite type. On the other hand, $\mathscr{A}(\Delta/r, \Gamma/N) = \mathscr{A}(k, k \times k \times k)$, and the latter is of finite type (as follows from the calculations in Section 5, or the results of Gabriel [10] and Dlab-Ringel [4]).

5. FINITENESS THEOREMS

We are now ready to prove that $\mathscr{L}(\Lambda)$ is of finite type whenever Λ satisfies the hypotheses of Theorem 1.11. As shown in Section 2, the problem reduces to the study of the category $\mathscr{A} = \mathscr{A}(\Delta, \Gamma)$, where Δ and Γ are commutative artinian rings with Δ local, and $\Delta \subset \Gamma$. Here, Γ is a direct product $\prod_{i=1}^{s} \Gamma_i$ of local principal ideal rings Γ_i. We keep the notation of (4.4); by the results of Section 3, it suffices to treat the case where

$$\Gamma_i / r_i \cong k = \Delta / r, \qquad 1 \leq i \leq s. \tag{5.1}$$

At this stage in the proof, we can safely forget about the original order Λ, and simply start with a pair of rings Δ and Γ as above. By (4.9), the conditions on Δ and Γ may be stated thus:

$$\dim \Gamma/r\Gamma \leq 3, \qquad \dim r\Gamma/(r + r^2\Gamma) \leq 1, \tag{5.2}$$

where *dim* means k-dimension.

As we shall see, the proof that $\mathscr{A}$ is of finite type can be reduced to a small number of test cases, which we now list explicitly:

(I) $\Gamma = \Gamma_1$, and there exist elements $x,y \in r$ such that $\Gamma x = r_1^3$, $\Gamma y = r_1^4$ or r_1^5

(II) $\Gamma = \Gamma_1 \times \Gamma_2$, and there exist $x,y \in r$ such that $\Gamma x = r_1 \oplus r_2^2$, $\Gamma y = r_2^2$ or r_2^3,

(III) $\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3$, and there exist $x,y \in r$ such that

$$\Gamma x = r_1 \oplus r_2^b, \quad \Gamma y = r_2 \oplus r_3, \qquad \text{for some } b \leq 1.$$

Further, if b_i denotes the nilpotency index of r_i, then $b_1 \leq b_2 = b_3$.

Let us show how the general problem can be reduced to one of these cases. After factoring out the conductor as in (4.3), we may assume that Δ contains no nonzero Γ-ideal. If $\dim \Gamma/r\Gamma = 1$, then of course $s = 1$ and $r\Gamma_1 = r_1$. From (5.1) we obtain $r\Gamma_1 \subset r + r^2\Gamma_1$. If $r \neq 0$, let e be its nilpotency index; then

$$r^{e-1}\Gamma_1 \subset r^{e-1} + r^e\Gamma_1 = r^{e-1},$$

so r^{e-1} is a nonzero Γ-ideal in Δ, a contradiction. Therefore $r = 0$ and

$$\mathscr{A}(\Delta, \Gamma) = \mathscr{A}(\mathrm{k}, \mathrm{k}),$$

which is obviously of finite type (and is a "degenerate" case of (I)).

Suppose next that $\dim \Gamma/r\Gamma = 2$; define $\Gamma^* = \Gamma \times \mathrm{k}$, and let $\Delta \to \Gamma^*$ be given by $\mathrm{x} \to (\mathrm{x}, \bar{\mathrm{x}})$, $\mathrm{x} \in \Delta$, where the bar denotes image in k. Then $r\Gamma^* = r\Gamma$, and so $\dim \Gamma^*/r\Gamma^* = 3$, $r\Gamma^*/(r + r^2\Gamma^*) \cong r\Gamma/(r + r^2\Gamma)$. Thus the pair Δ, Γ^* also satisfies hypotheses (5.2), but now $\dim \Gamma^*/r\Gamma^* = 3$. But choosing $\mathfrak{b} = 0 \times \mathrm{k}$ in Γ^*, we have $\Gamma = \Gamma^*/\mathfrak{b}$ and $\Delta \cap \mathfrak{b} = 0$. Hence by (4.1), if $\mathscr{A}(\Delta,\Gamma^*)$ is of finite type, then so is $\mathscr{A}(\Delta,\Gamma)$.

We are thus reduced to treating the case in which $\dim \Gamma/r\Gamma = 3$. Suppose for the moment that $r\Gamma = r + r^2\Gamma$; if $r \neq 0$, then as before we obtain a nonzero conductor of Γ in Δ. Thus r must be 0 and $\dim \Gamma = 3$. If $\mathrm{s} = 1$, then $r_1^3 = 0$ and we have a special case of (I). Likewise $\mathrm{s} = 2$ or 3 gives special cases of (II) and (III), repectively.

We turn finally to the cases of most interest and greatest difficulty, namely those for which

$$(5.3) \qquad \dim \Gamma/r\Gamma = 3, \qquad \dim r\Gamma/(r + r^2\Gamma) = 1.$$

Of course $\mathrm{s} \leq 3$, and we consider successively the cases $\mathrm{s} = 1,2,3$. When $\mathrm{s} = 1$ and $\Gamma = \Gamma_1$, we have $r\Gamma = r_1^3$, so there exists an $\mathrm{x} \in r$ such that $\mathrm{x}\Gamma = r_1^3$. We shall say that x has "degree" 3. Furthermore, the second condition in (5.3) becomes $\dim r_1^3/(r + r_1^6) = 1$. It follows from this that r must contain an element y of degree 4 or 5, so we are in case (I). When $\mathrm{s} = 2$ we must have $\Gamma r = r_1 \oplus r_2^2$, so there exist $\mathrm{x}_1, \mathrm{x}_2 \in r$ such that $\mathrm{x}_1\Gamma_1 = r_1$, $\mathrm{x}_2\Gamma_2 = r_2^2$. If $\mathrm{x}_1\Gamma_2 = r_2^2$, use x_1 as the desired x for case (II). Likewise, if $\mathrm{x}_2\Gamma_1 = r_1$ we can choose $\mathrm{x} = \mathrm{x}_2$. However, if $\mathrm{x}_1\Gamma_2 \neq r_2^2$, and $\mathrm{x}_2\Gamma_1 \neq r_1$, then we need only take $\mathrm{x} = \mathrm{x}_1 + \mathrm{x}_2$. Furthermore, from $\dim (r_1 \oplus r_2^2)/(r + r_1^2 \oplus r_2^4) = 1$ it follows easily that there exists an element $\mathrm{y} \in r$ for which $\Gamma_2\mathrm{y} = r_2^2$ or r_2^3. If $\Gamma_2\mathrm{y} = r_2^3$, then r contains $\Gamma_2 r_2^5$; hence $r_2^5 = 0$, since otherwise Γr_2^5 is a nonzero Γ-ideal in Δ. Thus, when $\mathrm{s} = 2$ we are in case (II). An analogous argument shows that case (III) must occur when $\mathrm{s} = 3$.

We now prove that $\mathscr{A} = \mathscr{A}(\Delta,\Gamma)$ is of finite type in cases (I)–(III), and we shall begin with case (I), the hardest of all. The methods introduced here will be useful for the other cases as well. Since our results differ somewhat from those of Jacobinski [14] for this case, it seems desirable to give a fairly detailed calculation for case (I).

(5.4) THEOREM. *$\mathscr{A}(\Delta,\Gamma)$ is of finite type in case* (I).

Proof. For convenience of notation, we write π and Γ in place of π_1 and Γ_1. If $\mathrm{z} \in \Gamma$ is such that $\mathrm{z}\Gamma = \pi^{\mathrm{m}}\Gamma$, we shall call z of *degree* m. We note that x has degree 3, y degree 4 or 5, from which it follows that $r \supset \Gamma\pi^6$ or $\Gamma\pi^8$, respectively. Since the conductor of Γ in Δ is assumed to be 0, we have $\pi^6 = 0$ or $\pi^8 = 0$, depending on the degree of y. Of course, there may be degenerate cases in which a lower power of π vanishes; as we shall see, the proof applies equally well to these cases, but some of the modules listed below become decomposable.

Given $(X,Y,f) \in \mathscr{A}$, we may write

$$X = \coprod_1^m \Delta x_i, \qquad Y = \coprod_1^n \Gamma y_j,$$

and

$$f(x_i) = \sum_{j=1}^n \gamma_{ji} y_j, \qquad 1 \le i \le m.$$

We put $F = [\gamma_{ij}]$, an $n \times m$ matrix over Γ. Since we wish to classify the object (X,Y,f) up to isomorphism in $\mathscr{A}$, we may make basis changes in X and Y. This has the effect of replacing F by UFV, where $U \in GL(n,\Gamma)$, $V \in GL(m,\Delta)$; we write $F \sim UFV$ for convenience. Note that $\Delta/r \cong \Gamma/\pi\Gamma \cong k$, and that there are surjections $GL(n,\Gamma) \to GL(n,k)$, $GL(m,\Delta) \to GL(m,k)$.

Since $(X,Y,f) \in \mathscr{A}$, we have $Y = \Gamma f(X)$, which implies that $\bar{F}$ has rank n; as always, bars denote images in the field k. Choose U,V so that $\bar{U}\bar{F}\bar{V} = [\bar{I}\ 0]$ over k; then $F \sim [I + \pi F_1, \pi F_2]$ for some F_1, F_2. But $I + \pi F_1 \in GL(\Gamma)$, so

$$F \sim (I + \pi F_1)^{-1}[I + \pi F_1, \pi F_2] = [I, \pi F_3] \qquad \text{for some } F_3.$$

Furthermore,

$[I, \pi F_3] \sim V_1 [V_1^{-1}, \pi F_3 V_2] = [I, \pi V_1 F_3 V_2]$ for any V_1, $V_2 \in GL(\Delta)$. Choosing the V's suitably, we can diagonalize $\bar{F}_3$, and we obtain

$$F \sim \begin{bmatrix} I & 0 & \pi I + \pi^2 G_1 & \pi^2 G_2 \\ 0 & I & \pi^2 G_3 & \pi^2 G_4 \end{bmatrix} \tag{5.5}$$

for some matrices $\{G_i\}$. For brevity, we write this as $F \sim \begin{bmatrix} I & 0 & 1^* & 2 \\ 0 & I & 2 & 2 \end{bmatrix}$, where the symbol *2* denotes a matrix of the form $\pi^2 G$, and 1^* a matrix of the form $\pi(I + \pi G)$; likewise, 2^* will denote a matrix of the form $\pi^2(I + \pi G)$, and so on. We shall now apply to the matrix (5.5) a certain number of standard types of operations, which will be used repeatedly below. For convenience, let C_j denote the j^{th} set of columns in a partitioned matrix, and R_i the i^{th} set of rows.

Type (i): Starting with (5.5), choose a matrix G_0 over Δ such that $\bar{G}_0 = \bar{G}_1$ over k. Add G_0 times C_3 to C_1, then multiply F on the left by $(I + \pi G_0 + \pi^2 G_1 G_0)^{-1}$. This yields $F \sim \begin{bmatrix} I & 0 & \pi I + \pi^3 H & 2 \\ \pi^2 G_3 G_0 & I & 2 & 2 \end{bmatrix}$ for some H. Adding a suitable multiple of R_1 to R_2, we can eliminate the (2,1) block.

By hypothesis, Δ contains an element x of degree 3; without loss of generality, we may assume that $x \equiv \pi^3 \pmod{\pi^4}$. Now add x-multiples of C_1 to C_3, so as to change the $\pi^3 H$ term to one of the form $\pi^4 H_1$. The new (1,3) block is of the form $\pi(I + \pi^3 H_1)$; choose H_0 over Δ so that $\bar{H}_0 = \bar{H}_1$, and then multiply C_3 by $(I + xH_0)^{-1}$. This changes the (3,1) block to $\pi(I + \pi^4 H_2)$ for some H_2. Now let $y \in \Delta$ have degree 4 or 5; in the first case, the same operation as above can be used to change the (3,1) block to $\pi(I + \pi^5 H_3)$; on the other hand, when y has order 5, we can accomplish the same result by subtracting y-multiples of C_1 from C_3. Continuing in this manner, we eventually obtain

$$F \sim \begin{bmatrix} I & 0 & \pi I & 2 \\ 0 & I & 2 & 2 \end{bmatrix}. \tag{5.6}$$

Type (ii): Having performed the type (i) operation on F, we now start with the matrix (5.6), and begin by diagonalizing the (2,4) block over k; this can always be done by multiplying the new F on the left by a block diagonal matrix in $GL(\Gamma)$ and on the right by a block diagonal matrix in $GL(\Delta)$. Thus we obtain

$$F \sim \begin{bmatrix} I & 0 & 0 & \pi I & 2 & 2 \\ & I & 0 & 2 & 2^* & 3 \\ & & I & 2 & 3 & 3 \end{bmatrix},$$

where $2^* = \pi^2(I + \pi G)$ for some G. We now use very strongly the fact that

$$\Gamma = \Delta + r_1,$$

so every degree zero element of Γ is congruent mod r_1 to a degree zero element of Δ. Subtract degree zero Δ-multiples of R_2 from R_1, and add the corresponding multiples of C_1 to C_2, so as to change the (1,5)-entry from *2* to *3*. Likewise, the (2,4)-entry can be changed from a *2* to *3;* this procedure replaces πI by 1^*, but type (i) operations bring it back to the form πI.

Having brought the (2,4) block to the form *3,* we can now add r-multiples of R_1 to R_2 and r-multiples of C_2 to C_4, so as to eliminate the (2,4) block. This procedure introduces r entries in the (2,1) position, and these can be eliminated by adding suitable r-multiples of C_2 to C_1. In a similar manner, adding r-multiples of C_1 and C_2 to C_5, we can eliminate the (1,5) block. Note that this elimination process depends on having a block *3* rather than *2.*

Type (iii): We wish to convert the (2,5) entry 2^* to $\pi^2 I$. First subtract x-multiples of C_1 from C_5, so as to make 2^* have the form $\pi^2(I + \pi^2 G)$ for some G. Then choose G_0 over Δ so that $\bar{G}_0 = \bar{G}$, and add G_0 times C_5 to C_2; then multiply R_2 on the left by $(I + \pi^2 G_0 + \pi^4 GG_0)^{-1}$, which changes the (2,5) block to

$$\pi^2(I + \pi^3 H) \qquad \text{for some H.}$$

Continuing in this way, as with type (i) operations, we can eventually bring the (2,5) block into the desired form $\pi^2 I$. During this procedure, the extraneous *2*'s introduced into the (1,2) and (3,2)-positions can be eliminated, and the πI in position (1,4) can be kept in that form by additional type (i) operations. Thus we obtain

$$F \sim \begin{bmatrix} I & & & \pi I & 3 & 2 \\ & I & & 3 & \pi^2 I & 3 \\ & & I & 2 & 3 & 3 \end{bmatrix}.$$

Our next step is to add r-multiples of C_2 and C_3 to C_6, so as to change both *3*'s to *4*'s. Likewise, the (3,5) entry can be changed to *4*. Now "diagonalize" the *2* occurring in position (3,4); then using operations of types (i)—(iii), we may bring this block into the form $\begin{pmatrix} \pi^2 I & 0 \\ 0 & 0 \end{pmatrix}$. This partitioning of the (3,4) block induces corresponding partitions of blocks (1,4) and (1,6), and we may bring the upper half of the (1,6) block into the form $\begin{pmatrix} \pi^2 I & 0 \\ 0 & 0 \end{pmatrix}$, in the same manner as above. Thus we obtain

$$(5.7)\quad F \sim \left[\begin{array}{ccccccc|ccc|c|cc} I & & & & & & & \pi I & & & & \pi^2 I & 0 \\ & I & & & & & & & \pi I & & 0 & 0 & 0 \\ & & I & & & & & & & \pi I & & 2 & 2 \\ \hline & & & I & & & & & 0 & & \pi^2 I & 4 & 4 \\ \hline & & & & I & & & \pi^2 I & & & 4 & 4 & 4 \\ & & & & & I & & & \pi^2 I & & 4 & 4 & 4 \\ & & & & & & I & & & 0 & 4 & 4 & 4 \end{array}\right]_{7\times 13}.$$

Now eliminate the (3,12) block by subtracting multiples of R_1 from R_3, and adding r-multiples of C_3 and C_{10} to C_{12}. The (3,13) block may then be brought into the form $\begin{pmatrix} \pi^2 I & 0 \\ 0 & 0 \end{pmatrix}$, and the zero rows split off. These yield indecomposables of the form $[1,\pi]$, and we are left with a new F in which the (3,12) block is zero, and the (3,13) block is $[\pi^2 I, 0]$.

Let us at once settle the easier case where y has degree 4. In the above array, we may change each *4* to a *5*, and then eliminate all *5*'s except that appearing in the lower right corner. But then we may diagonalize it, and F splits into a direct sum of

$$[1],\ [1\ \pi],\ [1\ \pi^2],\ [1\ \pi^5],\ [1\ \pi\ \pi^2],\ \begin{bmatrix} 1 & 0 & \pi \\ 0 & 1 & \pi^2 \end{bmatrix},\ \begin{bmatrix} 1 & 0 & \pi & \pi^2 \\ 0 & 1 & \pi^2 & 0 \end{bmatrix}.$$

Thus $\mathscr{L}(\Lambda)$ is of finite type in this case.

Now suppose that y has degree 5, and that r contains no element of order 4. Let us write down the matrix in (5.7), with the (3,12) block 0, and the (3,13) block $[\pi^2 I, 0]$. We have then (blanks denoting zeros)

$$F \sim \left[\begin{array}{ccccccc|ccc|c|ccc} I & & & & & & & \pi I & & & & \pi^2 I & 0 & 0 \\ & I & & & & & & & \pi I & & & 0 & 0 & 0 \\ & & I & & & & & & & \pi I & & 0 & \pi^2 I & 0 \\ \hline & & & I & & & & & & & \pi^2 I & 4 & 4 & 4 \\ \hline & & & & I & & & \pi^2 I & & & 4 & 4 & 4 & 4 \\ & & & & & I & & & \pi^2 I & & 4 & 4 & 4 & 4 \\ & & & & & & I & & & 0 & 4 & 4 & 4 & 4 \end{array}\right]_{7\times 14}.$$

We can replace the (7,14) entry by $\begin{bmatrix} \pi^4 I & 0 \\ 0 & 7 \end{bmatrix}$, and use the $\pi^4 I$ to eliminate all terms above it and to its left (except for the 1's on the extreme left). Then F splits into a sum of copies of $[1\ \pi^4]$, and a new F whose (7,14) block is 7. This block can be put in the form $\begin{bmatrix} \pi^7 I & 0 \\ 0 & 0 \end{bmatrix}$. Consider the upper left π^7 entry; if any π^4 term lies in the column above it, this π^4 can be used to eliminate π^7. The same argument applies if a π^4 term lies to the left of π^7. On the other hand, if no such π^4 terms occur, then we may use π^7 (and other allowable operations) to sweep out the row and column containing π^7; thus copies of $[1\ \pi^7]$ split off from F, and the new F has 0 in position (7,14).

Next, replace the (6,14) block by $\begin{bmatrix} \pi^4 I & 0 \\ 0 & 0 \end{bmatrix}$, and split off indecomposables of type $\begin{bmatrix} 1 & 0 & \pi & 0 \\ 0 & 1 & \pi^2 & \pi^4 \end{bmatrix}$. The new F has 0 in position (6,14). Likewise, we may replace the (7,13) block by 0, after splitting off copies of $\begin{bmatrix} 1 & 0 & \pi & \pi^2 \\ 0 & 1 & 0 & \pi^4 \end{bmatrix}$. It now remains for us to introduce a new operation:

Type (iv): Suppose that the (4,12) block is congruent to $\pi^4 T \pmod{\pi^5}$, where T is a matrix over R. Then add T times C_{11} to C_1, and subtract $\pi^2 T$ times R_1 from R_4. This eliminates the π^4 terms from the (4,12) block, and other allowable operations then reduce the block to 0, after some obvious steps to eliminate some extraneous π^4 and π^5 terms introduced by this procedure. Analogously, if the (5,12) block is $\pi^4 U \pmod{\pi^5}$, with U over R, then add U times C_8 to C_1 and subtract $\pi^2 U$ times R_1 from R_5; in this manner, we can make the (5,12) block eventually vanish. Continuing thus, we can eliminate the blocks (5,11),(6,11), (6,12),(4,13) and (5,13).

Once these changes have been made, it is clear that F splits into the direct sum of F_1 and F_2, where

$$F_1 = \left[\begin{array}{ccc|ccc} I & & & \pi I & 0 & 0 \\ & I & & 0 & \pi I & \pi^2 I \\ & & I & \pi^2 I & 0 & 4 \end{array}\right] 3\times 6, \qquad F_2 = \left[\begin{array}{cccc|cccc} I & & & & \pi I & 0 & \pi^2 I & 0 \\ & I & & & 0 & \pi^2 I & 0 & 4 \\ & & I & & \pi^2 I & 0 & 0 & 4 \\ & & & I & 0 & 4 & 4 & 0 \end{array}\right] 4\times 8.$$

After diagonalizing the (3,6) block of F_1, it is easily seen that F_1 decomposes into

$$[1\ \ \pi\ \ \pi^2], \quad \begin{bmatrix} 1 & 0 & \pi \\ 0 & 1 & \pi^2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & \pi & 0 & 0 \\ 0 & 1 & 0 & 0 & \pi & \pi^2 \\ 0 & 0 & 1 & \pi^2 & 0 & \pi^4 \end{bmatrix}.$$

Diagonalizing the (3,8) block of F_2, and then the left hand part of the partitioned (4,7) block, we find eventually that F_2 splits into

$$[1\ \ \pi^2],\quad [1\ \ \pi^2\ \ \pi^4],\quad \begin{bmatrix} 1 & 0 & \pi^2 \\ 0 & 1 & \pi^4 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & \pi & \pi^2 & 0 \\ 0 & 1 & \pi^2 & 0 & \pi^4 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & \pi & \pi^2 \\ 0 & 1 & \pi^2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & \pi^2 & \pi^4 \\ 0 & 1 & \pi^4 & 0 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0 & \pi & \pi^2 \\ 0 & 1 & 0 & \pi^2 & 0 \\ 0 & 0 & 1 & 0 & \pi^4 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0 & \pi & \pi^2 & 0 \\ 0 & 1 & 0 & \pi^2 & 0 & \pi^4 \\ 0 & 0 & 1 & 0 & \pi^4 & 0 \end{bmatrix}.$$

This completes the proof of Theorem 5.4.

It may console the reader to know that these same techniques work quite efficiently, and with less complication, in cases II and III below. We shall give some indication of this below, but first let us translate the results of Theorem 5.4 into corresponding results about lattices over orders.

(5.8) COROLLARY. *Let* R *be a complete discrete valuation ring with quotient field* K, *and let* Λ *be a local* R*-order in a finite extension field* A *of* K. *Let* Λ' *be the maximal* R*-order in* A, *that is, the valuation ring of* A. *Let* π *be a prime element of* Λ', *and suppose that* $\Lambda/\text{rad}\,\Lambda \cong \Lambda'/\pi\Lambda'$. *Suppose that* Λ *contains elements* x,y *such that* $\Lambda' x = \pi^3\Lambda'$, $\Lambda' y = \pi^4\Lambda'$. *Then every indecomposable* Λ*-lattice is isomorphic to one of the following:*

$$\Lambda,\ \Lambda + \pi\Lambda,\ \Lambda + \pi^2\Lambda,\ \Lambda + \pi^5\Lambda,\ \Lambda + \pi\Lambda + \pi^2\Lambda,\quad \text{or}$$

$$\Lambda u_1 + \Lambda u_2 + \Lambda(\pi u_1 + \pi^2 u_2),\ \Lambda u_1 + \Lambda u_2 + \Lambda(\pi u_1 + \pi^2 u_2) + \Lambda\pi^2 u_1,$$

where $\Lambda' u_1 \oplus \Lambda' u_2$ *is a free* Λ'*-module on two generators.*

Of course, not all of the above need be indecomposable; this depends on what other elements lie in rad Λ. For example, in the extreme case where $\Lambda = \Lambda'$, there is only one indecomposable Λ-lattice, namely Λ itself. On the other hand, if Λ contains no elements of degree 2 or 5, then the lattices listed in (5.8) are indecomposable, and no two of them are isomorphic. If Λ contains no element of degree 2, but does have one of degree 5, the same holds true with the exception that $\Lambda + \pi^5\Lambda$ coincides with Λ.

We remark that (5.8) is an immediate consequence of (5.4). Each of the (possibly) indecomposable matrices given in (5.4) corresponds to an object $(X, Y, f) \in \mathscr{A}(\Delta, \Gamma)$, since it is easily verified that $Y = \Gamma f(X)$ and that $\ker f \subset rX$. Each such object then gives rise to a Λ-lattice as in Section 2. For example, the matrix $[1\ \ \pi^2]$ yields a Λ-lattice M on two generators, contained in a free Λ'-lattice on one generator u_1, namely, $M = \Lambda u_1 + \Lambda\pi^2 u_1$. Likewise, the matrix

$$\begin{bmatrix} 1 & 0 & \pi \\ 0 & 1 & \pi^2 \end{bmatrix}$$

yields the Λ-lattice $\Lambda u_1 + \Lambda u_2 + \Lambda(\pi u_1 + \pi^2 u_2)$, a sublattice of the free Λ'-lattice $\Lambda' u_1 \oplus \Lambda' u_2$ on two generators.

In the same manner, we obtain

(5.9) COROLLARY. *Keep the notation and hypotheses of* (5.8), *except that* $\Lambda' y = \pi^5 \Lambda'$ (*instead of* $\pi^4\Lambda'$). *Let* $\coprod_1^n \Lambda' u_i$ *denote a free* Λ'*-module on* n *generators. Then every indecomposable* Λ*-lattice is isomorphic to one of the following:*

$$\Lambda,\ \Lambda+\pi\Lambda,\ \Lambda+\pi^2\Lambda,\ \Lambda+\pi^4\Lambda,\ \Lambda+\pi^7\Lambda,\ \Lambda+\pi\Lambda+\pi^2\Lambda,\ \Lambda+\pi^2\Lambda+\pi^4\Lambda,$$
$$\Lambda u_1+\Lambda u_2+\Lambda(\pi u_1+\pi^2 u_2)=M_1 \text{ (say)},$$
$$M_1+\Lambda\pi^2 u_1,\ M_1+\Lambda\pi^4 u_2,\ M_1+\Lambda\pi^4 u_2+\Lambda\pi^2 u_1,$$
$$\Lambda u_1+\Lambda u_2+\Lambda(\pi^2 u_1+\pi^4 u_2)=M_2 \text{ (say)},\ M_2+\Lambda\pi^4 u_1,$$
$$\Lambda u_1+\Lambda u_2+\Lambda u_3+\Lambda(\pi u_1+\pi^2 u_3)+\Lambda\pi u_2+\Lambda(\pi^2 u_2+\pi^4 u_3),$$
$$\Lambda u_1+\Lambda u_2+\Lambda u_3+\Lambda(\pi u_1+\pi^2 u_2)+\Lambda(\pi^2 u_1+\pi^4 u_3)=M_3 \text{ (say)},$$
$$M_3+\Lambda\pi^4 u_2.$$

If Λ contains no elements of degrees 2,4 or 7, then all of the above lattices are indecomposable, and no two of them are isomorphic. The same holds if Λ contains an element of degree 7, but none of degrees 2 and 4, except that in this case $\Lambda+\pi^7\Lambda=\Lambda$. There is a similar list in Proposition 6 of Jacobinski's article. However, the lattices he lists in e) and f) turn out to be decomposable; specifically, his L_4 and L_5 correspond to matrices

$$\begin{bmatrix} 1 & 0 & 0 & \pi & \pi^2 & 0 \\ 0 & 1 & 0 & \pi^2 & 0 & \pi \\ 0 & 0 & 1 & 0 & 0 & \pi^2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & \pi & 0 & \pi \\ 0 & 1 & 0 & 0 & \pi^2 & \pi & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \pi^4 \\ 0 & 0 & 0 & 1 & 0 & \pi^2 & 0 \end{bmatrix},$$

and both of these can be decomposed by using operations of type (iv) described in the proof of Theorem 5.4. Similar remarks apply to his lattices

$$L_4+\Lambda\pi^4 u_3, \qquad L_5+\Lambda\pi^4 u_4.$$

Turning next to the easier alternative in case (II), we prove

(5.10) THEOREM. *Suppose that* $\Gamma=\Gamma_1\times\Gamma_2$, *and that there exist elements* $x,y\in r$ *such that* $\Gamma x = r_1\oplus r_2^2, \Gamma y = r_2^2$. *Then* $\mathscr{A}(\Delta,\Gamma)$ *is of finite type.*

Proof. Each object $(X,Y,f)\in\mathscr{A}$ determines a matrix $F=\left(\frac{A}{B}\right)$, where A has entries in Γ_1, B in Γ_2. Basis changes correspond to letting $GL(\Delta)$ act on the columns of F, while $GL(\Gamma_1)$ acts on the rows of A, and $GL(\Gamma_2)$ on the rows of B. As in the first part of the proof of (5.4), we may bring A into the form $[I \quad \pi_1 A_1]$, where π_1 generates r_1. Subtracting x-multiples of C_1 from C_2, we can change $\pi_1 A_1$ to $\pi_1^2 A_2$; then subtract x^2-multiples, and so on, until we obtain $F\sim\left[\begin{array}{cc} I & 0 \\ \hline B_1 & B_2\end{array}\right]$.

Next, we bring B_2 into the form $\begin{bmatrix} I & \pi B_3 \\ \pi B_4 & \pi B_5 \end{bmatrix}$, where π generates r_2. Using row operations over Γ_2, we can eliminate B_4, so now

$$F \sim \left[\begin{array}{ccc} I & 0 & 0 \\ \hline B_{11} & I & \pi B_3 \\ B_{21} & 0 & \pi B_5 \end{array}\right],$$

say. Subtract Δ-multiples of C_2 from C_1, to bring B_{11} into the form $\pi B'_{11}$. Now observe that the rank of the original matrix B must equal the number of rows of B, since otherwise $\Gamma_2 f(X) \neq \Gamma_2 Y$. Therefore when we "diagonalize" B_{21} as our next step, we can bring B_{21} into the form $[I \quad \pi B_6]$. Then "diagonalize" the entry πB_5 of F, getting a new matrix

$$F \sim \left[\begin{array}{ccc|ccc} & I & & & 0 & \\ \hline \pi D_1 & \pi D_2 & \pi D_3 & I & \pi B_7 & \pi B_8 \\ I & 0 & \pi D_4 & 0 & \pi I & \pi^2 B_9 \\ 0 & I & \pi D_5 & 0 & \pi^2 B_{10} & \pi^2 B_{11} \end{array}\right] \; 4\times 6 .$$

By subtracting Γ_2-multiples of R_3 and R_4 from R_2, we can eliminate D_1 and D_2. Further, adding y-multiples of C_1 and C_2 to C_5 and C_6 enables us to replace B_9, B_{10}, B_{11} by $\pi B'_9, \pi B'_{10}, \pi B'_{11}$, respectively. Next, we can add Δ-multiples of R_3 to R_2, and subtract the corresponding multiples of C_4 from C_1, so as to replace πB_7 by $\pi^2 B'_7$. It is then easy to eliminate this block altogether, so we obtain

$$F \sim \left[\begin{array}{ccc|ccc} & I & & & 0 & \\ \hline 0 & 0 & \pi D_6 & I & 0 & \pi E_1 \\ I & 0 & \pi D_4 & 0 & \pi I & \pi^3 E_2 \\ 0 & I & \pi D_5 & 0 & \pi^3 E_3 & \pi^3 E_4 \end{array}\right].$$

We next add y-multiples of C_5 to C_6, so as to replace $\pi^3 E_2$ by $\pi^4 E'_2$; then add y^2-multiples of C_1 to C_6, so as to make this entry a multiple of π^5. Continuing in this manner, we eventually make the (2,6) block vanish. In a similar way, we can eliminate the blocks πD_4 and $\pi^3 E_3$. Now bring πE_1 into the form $\begin{bmatrix} \pi I & \pi^2 E_{12} \\ \pi^2 E_{21} & \pi^2 E_{22} \end{bmatrix}$. It is easily seen that we can eliminate both off-diagonal blocks, and replace $\pi^2 E_{22}$ by $\pi^3 E'_{22}$. Changing notation, we now have

$$F \sim \left[\begin{array}{ccc|ccccc} & I & & & & 0 & & \\ \hline 0 & 0 & 1 & I & 0 & 0 & \pi I & 0 \\ 0 & 0 & 1 & 0 & I & 0 & 0 & 3 \\ I & 0 & 0 & 0 & 0 & \pi I & 0 & 0 \\ 0 & I & 1 & 0 & 0 & 0 & 3 & 3 \end{array}\right] \; 5\times 8 ,$$

where *1* represents a matrix of the form πG, and *3* represents one of the form $\pi^3 H$.

By subtracting Δ-multiples of C_4 and C_7 from C_3, we can eliminate the (2,3) block. Further, by using the (2,7) block πI, and the (5,2) block I, we can eliminate the (5,7) block altogether. But then F splits into a direct sum of copies of the matrices $\left[\begin{array}{cc} 1 & 0 \\ \hline 1 & \pi \end{array}\right]$, $\left[\begin{array}{cc} \multicolumn{2}{c}{*} \\ \hline 1 & \pi \end{array}\right]$, and a new matrix

$$F_1 = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{I} & \multicolumn{2}{c}{0} \\ \hline 0 & \pi G_1 & I & \pi^3 H_1 \\ I & \pi G_2 & 0 & \pi^3 H_2 \end{array}\right].$$

The matrix $\left[\begin{array}{cc} \multicolumn{2}{c}{*} \\ \hline 1 & \pi \end{array}\right]$ corresponds to the object $(X,Y,f) \in \mathscr{A}$ for which

$$X = \Delta x_1 \oplus \Delta x_2, \qquad Y = \Gamma_2 y_1,$$

and $f(x_1) = y_1$, $f(x_2) = \pi y_1$.

We then decompose F_1 further, first "diagonalizing" G_2, and so on. We eventually obtain the following list of possible indecomposables:

$$\left[\begin{array}{c} * \\ \hline 1 \end{array}\right], \left[\begin{array}{c} 1 \\ \hline * \end{array}\right], \left[\begin{array}{c} 1 \\ \hline 1 \end{array}\right], \left[\begin{array}{cc} \multicolumn{2}{c}{*} \\ \hline 1 & \pi^m \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ \hline 1 & \pi^m \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ \hline \pi^m & 1 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & \pi^m \end{array}\right],$$

where $m = 1,3,5,\ldots$, up to the first odd integer q such that $\pi^q \in r$.

(5.11) COROLLARY. *Let* R *be a complete discrete valuation ring with quotient field* K, *and let* K_i *be an extension field of* K *with valuation ring* Λ_i, *prime element* π_i, $i = 1,2$. *Let* $A = K_1 \times K_2$, $\Lambda' = \Lambda_1 \times \Lambda_2$, *and let* e_1, e_2 *be the central idempotents in* A. *Let* Λ *be a local* R-*order in* A *such that* $\Lambda/\operatorname{rad}\Lambda \cong \Lambda_i/\pi_i\Lambda_i$, $i = 1, 2$. *Suppose that* Λ *contains elements* x,y *such that*

$$\Lambda' x = \pi_1 \Lambda_1 \oplus \pi_2^2 \Lambda_2, \qquad \Lambda' y = \pi_2^2 \Lambda_2,$$

and let q *be the smallest odd integer for which* $\pi_2^q \in \operatorname{rad}\Lambda$. *Then every indecomposable* Λ-*lattice is isomorphic to one of the following:*

$$\Lambda, \Lambda e_1, \Lambda e_2, (\Lambda + \Lambda\pi_2^m)e_2, \Lambda + \pi_2^m \Lambda e_2, \Lambda(e_1 + \pi_2^m e_2) + \Lambda e_2,$$
$$\Lambda(u_1 + u_2) + \Lambda(v_1 + \pi_2^m u_2),$$

with $m = 1,3,\ldots,q-2$. *The last expression denotes a* Λ-*sublattice of the direct sum* $\Lambda_1 u_1 \oplus \Lambda_1 v_1 \oplus \Lambda_2 u_2$ *of two copies of* Λ_1 *and one copy of* Λ_2.

We may remark that the above result also includes information about a simpler version of (5.8). Suppose that, as in (5.8), Λ is a local order in A, and that Λ contains an element x such that $\Lambda' x = \pi^2 \Lambda'$. The corresponding matrix problem is precisely the classification of the matrices $\left(\begin{array}{c} * \\ \hline B \end{array}\right)$, in the notation of (5.10). Hence for this case, the indecomposable Λ-lattices are given by

$$\Lambda,\ \Lambda + \Lambda\pi^m, \qquad m = 1, 3, \ldots, q-2.$$

A slightly more involved calculation of the type given in (5.10) enables us to determine all possibly indecomposable matrices in case (II), for the alternative in which $\Gamma y = r_2^3$. We omit the details, only noting that the condition that $r_2^5 = 0$ is vital, and simply list the possible indecomposables:

$$\left[\begin{array}{c} * \\ \hline 1 \end{array}\right], \left[\begin{array}{c} 1 \\ \hline * \end{array}\right], \left[\begin{array}{c} 1 \\ \hline 1 \end{array}\right], \left[\begin{array}{cc} \multicolumn{2}{c}{*} \\ \hline 1 & \pi \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ \hline \pi & 1 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ \hline 1 & \pi \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ \hline 1 & \pi^2 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ \hline 1 & \pi^4 \end{array}\right],$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 1 & \pi & \pi^2 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & \pi \end{array}\right], \left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 0 & 1 & \pi \\ 1 & 0 & \pi^2 \end{array}\right], \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & \pi & \pi^2 \end{array}\right],$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & \pi \\ 0 & 1 & \pi^2 \end{array}\right], \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & \pi \\ 1 & \pi & 0 & \pi^2 \end{array}\right], \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & \pi & \pi^2 \\ 0 & 1 & \pi^2 & 0 \end{array}\right].$$

We leave it to the reader to write down the corresponding lattices.

To conclude this section, we sketch the calculation for case (III).

(5.12) THEOREM. *$\mathscr{A}(\Delta,\Gamma)$ is of finite type in case* (III).

Proof. Let π_i generate r_i, $i = 1,2,3$, and let $x,y \in r$ be such that

$$\Gamma x = r_1 \oplus r_2^b \oplus r_3^c, \qquad \Gamma y = r_2 \oplus r_3,$$

where $b \geq 1$, $c \geq 1$. The actual values of b,c are irrelevant. We must classify matrices

$$F = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix},$$

where A_i has entries in Γ_i. As in the first part of the proof of (5.10) we may bring A_1 into the form $[I \quad 0]$, and then A_2 into the form $\begin{bmatrix} D_1 & I & \pi_2 B_1 \\ D_2 & \pi_2 B_2 & \pi_2 B_3 \end{bmatrix}$, where the D's are below the matrix I which occurs in A_1. Using the element y and its powers, it is a simple matter to eliminate D_1, B_1, B_2, and to bring F into the form

$$\left[\begin{array}{cc:cc} \multicolumn{2}{c:}{I} & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ \hline E_1 & E_2 & E_3 & E_4 \end{array}\right]$$

with each E_i a matrix over Γ_3. Upon "diagonalizing" E_4, we may at once split off direct summands of the type

$$\left[\begin{array}{c} * \\ \hline * \\ \hline 1 \end{array}\right],$$

and are left with a new F in which E_4 is replaced by $\pi_3 E_4$. Diagonalizing E_2, we obtain a further splitting off of summand of the type

$$\left[\begin{array}{c} 1 \\ \hline * \\ \hline 1 \end{array}\right].$$

Continuing in this manner, we readily obtain the following full list of possibly indecomposable matrices:

$$\left[\begin{array}{c} * \\ \hline * \\ \hline 1 \end{array}\right], \left[\begin{array}{c} 1 \\ \hline * \\ \hline 1 \end{array}\right], \left[\begin{array}{c} * \\ \hline 1 \\ \hline * \end{array}\right], \left[\begin{array}{c} 1 \\ \hline 1 \\ \hline * \end{array}\right], \left[\begin{array}{cc} \multicolumn{2}{c}{*} \\ \hline 1 & 0 \\ \hline 1 & \pi_3 \end{array}\right],$$

$$\left[\begin{array}{cc} 1 & 0 \\ \hline 1 & 0 \\ \hline 1 & \pi_3 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 1 \\ \hline 1 & 1 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 1 \\ \hline \pi_3 & 1 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ \hline 1 & \pi_3 \end{array}\right].$$

As emphasized before, some of these matrices may be decomposable; this is certainly the case if $\pi_3 = 0$, for example, or if $\Lambda = \Lambda'$. We leave it to the reader to describe the analogues of (5.9) and (5.11) for the case where $A = K_1 \times K_2 \times K_3$, and to list the indecomposable Λ-lattices.

6. CONCLUDING REMARKS

In Sections 4 and 5, we gave necessary and sufficient conditions that $\mathscr{A}(\Delta,\Gamma)$ be of finite representation type, in the following situation: Δ and Γ are commutative artinian rings, $\Delta \subset \Gamma$, Δ local, and Γ is a product $\prod \Gamma_i$ of local principal ideal rings Γ_i, such that $\Delta/\operatorname{rad}\Delta \cong \Gamma_i/\operatorname{rad}\Gamma_i$ for each i. In this section we shall point out some directions for further research, and shall state a few relevant results without proof.

First of all, we may drop the condition that Δ and Γ be commutative, but still require Δ to be local. It would be of great interest to determine precisely when $\mathscr{A}(\Delta,\Gamma)$ is of finite type. The solution to this problem should enable us, by means of reduction techniques analogous to those of Section 2, to decide which noncommutative local orders Λ have finite type. In this connection, see Drozd-Kirichenko [7].

Secondly, we may seek to investigate the general commutative case, where Γ is not necessarily of the form $\prod \Gamma_i$ as above. For instance, suppose that

$$\Delta = k[x, y]/(x, y)^2 \subset \Gamma = k[x, y, z]/(x, y, z)^2.$$

It is easy to prove that $\mathscr{A}(\Delta,\Gamma)$ is of finite type in this case; note that Δ and Γ are themselves artinian rings of infinite representation type. Of course, conditions (2.8) are satisfied in this case.

We note further that for Δ an arbitrary left artinian ring, not necessarily commutative, the category $\mathscr{A}(\Delta,\Delta)$ is always of finite type. However, the category $\mathscr{B}(\Delta,\Delta)$ may be of infinite type. Indeed, if Γ is of infinite representation type, then so is $\mathscr{B}(\Delta,\Gamma)$ (for all Δ).

Finally, and perhaps of greatest interest, one may study more general diagrams. Let $\mathscr{M}$ be a *modulated graph* in the following sense: $\mathscr{M}$ consists of a finite directed graph, with a local artinian ring Δ_α assigned to each vertex α, and a Δ_α-Δ_β-bimodule ${}_\beta M_\alpha$ assigned to each arrow $\alpha \to \beta$. A *representation* of $\mathscr{M}$ assigns to each vertex α a left module $P_\alpha \in \mathbf{P}(\Delta_\alpha)$, and to each arrow from α to β, a Δ_β-homomorphism $f_{\alpha\beta} : {}_\alpha M_\beta \otimes_{\Delta_\alpha} P_\alpha \to P_\beta$. Defining morphisms in an obvious manner, we obtain a category $\mathscr{D}(\mathscr{M})$. The basic problem is to find necessary and sufficient conditions that $\mathscr{D}(\mathscr{M})$ be of finite representation type.

In the case where each $\Delta_\alpha = \Delta$, and each ${}_\beta M_\alpha = \Delta$, the study of the category $\mathscr{D}(\mathscr{M})$ may be regarded as a generalized matrix problem over Δ.

The most accessible case is that in which the underlying graph of $\mathscr{M}$ has the form $\circ \to \circ$. For such $\mathscr{M}$, we have already defined the category $\mathscr{C}(\Delta, \Gamma, {}_\Gamma M_\Delta)$ in Section 2. In this setting the category analogous to $\mathscr{A}(\Delta,\Gamma)$ is the full subcategory $\mathscr{A}^*(\Delta,\Gamma,M)$ of $\mathscr{C}(\Delta,\Gamma,M)$ whose objects (X,Y,f) have the property that f is surjective. What are necessary and sufficient conditions on Δ, Γ, and M so that $\mathscr{A}^*(\Delta,\Gamma,M)$ (respectively $\mathscr{C}(\Delta,\Gamma,M)$) be of finite type?

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