APPROXIMATING DISKS IN 4-SPACE

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1. INTRODUCTION

In this paper we show that certain topological embeddings of the $(n-2)$-disk into a PL n-manifold can be approximated by locally flat PL embeddings. The problem of approximating topological embeddings has been studied extensively and approximation theorems are already known in all codimensions other than two: Miller [8] proved that all topological embeddings of manifolds can be PL approximated in codimensions greater than or equal to 3, and Ancel and Cannon [1] have recently used a technique of Stankŏ to prove a locally flat approximation theorem for manifolds in codimension 1. Our main theorem applies to 2-disks embedded in a 4-manifold.

THEOREM 1. *If* $D: I^2 \to M^4$ *is a topological embedding of a disk into a* PL 4-*manifold, then* D *can be* ε-*approximated by a locally flat* PL *embedding* $E: I^2 \to M^4$ *for every* $\varepsilon > 0$. *Furthermore, if* $D|\partial I^2$ *is* PL *and* $D(\partial I^2) \subset \operatorname{Int} M^4$, *then* E *can be chosen so that* $E|\partial I^2 = D|\partial I^2$.

If $D(\partial I^2) \subset \partial M^4$, we cannot have $E|\partial I^2 = D|\partial I^2$. For example, if D is the cone from the center of the 4-ball B^4 to a trefoil knot on ∂B^4, then ∂D does not bound a locally flat PL disk in B^4 [6]. However, it is still unknown whether it is possible to have $D(\partial I^2) \subset \partial M^4$ in this case.

It is natural to ask whether Theorem 1 is true when I^2 is replaced by some other 2-manifold, since in codimension 3 an approximation theorem for manifolds follows from one for disks. In general it is not; an example is given in [5] of a topological embedding of the 2-torus $S^1 \times S^1$ into the 4-sphere S^4 which cannot be approximated arbitrarily closely by PL embeddings (not even by PL embeddings with non-locally flat points). The answer is unknown for embeddings of S^2 in S^4. The following theorem, which can be proved in exactly the same fashion as Theorem 1, gives a positive answer for certain 2-complexes.

THEOREM 2. *If* K *is a finite* 1-*complex and* $h: K \times I \to M^4$ *is a topological embedding, then* h *can be approximated by* PL *embeddings.*

Any topological disk in a topological 4-manifold has a neighborhood which can be immersed in $\mathbb{R}^4$ [7]. This neighborhood inherits a PL structure from $\mathbb{R}^4$ and so Theorem 1 can be used to find an approximation which is locally flat and PL with respect to the inherited structure. Since local flatness is a topological property, the following theorem is a consequence of Theorem 1.

THEOREM 3. *Any topological embedding of the* 2-*disk into a topological* 4-*manifold can be* ε-*approximated by locally flat embeddings for every* $\varepsilon > 0$.

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In high dimensions we can prove the following theorem.

THEOREM 4. *Suppose* M^n *is a* PL n-*manifold and* $D: I^{n-2} \to M^n$ *is a topological embedding. If* $D|I^{n-3} \times [0, 1/2]$ *can be* ε-*approximated by locally flat* PL *embeddings for every* $\varepsilon > 0$, *then* D *has the same property.*

COROLLARY 1. *If* $D: I^{n-2} \to M^n$ *is a topological embedding and if there exists an open set* $U \subset I^{n-2}$ *such that* $D|U$ *can be* ε-*approximated by a locally flat* PL *embedding for every* $\varepsilon > 0$, *then* D *has the same property.*

COROLLARY 2. *If* $D: I^{n-2} \to M^n$ *is a* PL *embedding, then* D *can be* ε-*approximated by a locally flat* PL *embedding for every* $\varepsilon > 0$.

Because of Corollary 2, the part of the hypothesis of Theorem 4 concerning local flatness is unnecessary; *i.e.*, it is enough to assume that $D|I^{n-3} \times [0, 1/2]$ can be approximated by PL embeddings. A simpler proof of Corollary 2 is given in [9].

The rest of this paper is organized as follows. All definitions and notation are listed together in Section 2. In Section 3 an inductive lemma is stated and the proof of Theorem 1 is reduced to this lemma. Two further lemmas are stated in Section 4 and Lemma 1 is proved assuming them. Lemmas 2 and 3 are proved in Section 5, and in Section 6 the modifications necessary to prove Theorem 4 are explained.

2. NOTATION

Throughout, M^n will denote a piecewise linear (PL) manifold of dimension n. The metric on M is ρ, and if $X \subset M$ and $\varepsilon > 0$, then $N_\varepsilon(X) = \{x \in M | \rho(x, X) < \varepsilon\}$. The interval [0, 1] is denoted by I, and D denotes a topological disk. The same notation is used for both the embedding $D: I^{n-2} \to M^n$ and for the image $D = D(I^{n-2}) \subset M^n$. We say that E is an ε-*approximation* of D if

$$\rho(D(x), E(x)) < \varepsilon \qquad \text{for every } x \in I^{n-2}.$$

If we wish to approximate D, we can first push D into the interior of M and then do the approximation there. Thus we will always assume that this has been done and that we are working in the manifold-without-boundary int M. To say that an isotopy h_t of M moves points ε-*parallel to fibers of* D means that for every $p \in M$, either $h_t(p) = p$ for every t or there exists one $x \in I^{n-3}$ such that $h_t(p) \in N_\varepsilon(D(x \times I))$ for every t.

3. AN INDUCTIVE LEMMA

In Sections 3–5, M^4 will denote a PL 4-manifold and $D: I^2 \to M^4$ a topological embedding.

LEMMA 1. *Suppose* $0 = a_k < a_{k-1} < \ldots < a_1 = a_0 = 1$ *is a partition of* [0, 1] *and* $0 \le j \le k$. *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* $E: I^2 \to M^4$ *is a* *locally flat* PL *embedding satisfying*

(1) $E(I \times [a_i, a_{i-1}]) \subset N_\delta(D(I \times [a_i, a_{i-1}]))$ *for all* $i \leq j$,

(2) $E(I \times [0, a_i]) \subset N_\delta(D(I \times [0, a_i]))$ *for all* i, *and*

(3) $E(x \times I) \subset N_\delta(D(x \times I))$ *for all* $x \in I$,

then E *can be replaced by a locally flat* PL *embedding* E′ *satisfying* (1)-(3) *with* j *replaced by* j + 1 *and* δ *replaced by* ε.

Proof that Lemma 1 *implies Theorem* 1. Choose $\varepsilon_1 > 0$ and a partition

$$0 = a_k < a_{k-1} < \ldots < a_1 = a_0 = 1$$

of [0, 1] such that if $E: I^2 \to M^4$ is any embedding satisfying

$$E(I \times [a_i, a_{i-1}]) \subset N_{\varepsilon_1}(D(I \times [a_i, a_{i-1}])) \qquad \text{for } i = 1, \ldots, k \quad \text{and}$$

$$E(x \times I) \subset N_{\varepsilon_1}(D(x \times I)) \qquad \text{for every } x \in I,$$

then $\rho(D, E) < \varepsilon$. Inductively apply Lemma 1 with $j = k - i$ and $\varepsilon = \varepsilon_{i-1}$ for $i = 2, 3, \ldots, k - 1$ to find $\varepsilon_i > 0$ corresponding to the δ of the conclusion. Now we need only find E satisfying (1)-(3) with $\delta = \varepsilon_{k-1}$ and $j = 1$, since then our choices of ε_i will go to work for us and eventually produce an E′ satisfying (1)-(3) with $j = k - 1$ and $\delta = \varepsilon_1$. This is enough by the choice of ε_1.

Choose $\varepsilon_k > 0$ such that any path lying in the union of the ε_k-neighborhoods of 3 fibers of D lies in the ε_{k-1}-neighborhood of any one of these fibers. (Here a *fiber* of D is $D(x \times I)$ for some $x \in I$.) Such an ε_k can be found using the uniform continuity of D and D^{-1}.

There exists a neighborhood U of $D(I \times 0)$ which strong $\varepsilon_k/2$-deformation retracts to $D(I \times 0)$ in M. Let $E: I \times 0 \to U$ be a PL $\varepsilon_k/2$-approximation of $D|I \times 0$. Extend E to $E: I \times I \to U$ with the property that

$$E(x \times I) \subset N_{\varepsilon_k/2}(D(x \times I)) \qquad \text{for every } x \in I.$$

(E will have very short fibers.) E can also be chosen so that E is locally flat. By the choice of U, there is a homotopy of $E(I \times 1)$ first down to $D(I \times 0)$ and then along D to $D(I \times 1)$. For each $x \in I$, the path of $E(x, 1)$ under this homotopy will lie in the ε_k-neighborhood of one fiber of D. By general position, it may be assumed that the track of the homotopy misses $E(I \times 0)$. By [2, Theorem 4.1] there is a radial engulfing isotopy h_t which pushes $E(I \times 1)$ down near $D(I \times 1)$ and does not move $E(I \times 0)$. Furthermore, any point of M which is moved by h_t stays ε_k-close to 2 fibers of D. Thus for any $(x, y) \in I \times I$, $h_t E(x, y)$ lies in $N_{\varepsilon_k}(D(x \times I))$ union the ε_k-neighborhoods of two other fibers of D. The choice of ε_k therefore implies that $h_t E(x, y) \in N_{\varepsilon_{k-1}}(D(x \times I))$.

Finally, reparametrize $I \times I$ so that $h_1 E(I \times [0, a_2]) \subset N_{\varepsilon_{k-1}}(D(I \times 0))$. Then $h_1 E$ is the locally flat PL embedding needed to complete the proof. If $D|\partial I^2$ is PL, then $D|\partial I^2$ and $E|\partial I^2$ are close PL embeddings, so there is a small ambient isotopy pushing $E|\partial I^2$ to $D|\partial I^2$ [3]. Thus we may assume that $D|\partial I^2 = E|\partial I^2$ in this case.

4. PROOF OF LEMMA 1

LEMMA 2. *Suppose that* $0 < a \leq 1$. *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* P *and* Q *are disjoint, finite* 1-*polyhedra in* $N_\delta(D(I \times [0, a]))$, *then there is a* PL *isotopy* h_t *of* M^4 *such that*

(i) $h_0 = \mathrm{id}$,

(ii) $h_t = \mathrm{id}$ *on* $N_\delta(D(I \times [a, 1])) \cup Q$ *and outside of* $N_\varepsilon(D(I \times [0, a]))$,

(iii) $h_1(N_\varepsilon(D(I \times [a, 1]))) \supset P$, *and*

(iv) h_t *moves* ε-*parallel to fibers of* D.

LEMMA 3. *Suppose that* $0 = a_k < a_{k-1} < \dots < a_1 = 1$ *is a partition of* $[0, 1]$ *and* $1 \leq j \leq k$. *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* $E\colon I^2 \to M^4$ *is a locally flat* PL *embedding satisfying* (1)–(3) *and* L^2 *is a finite 2-polyhedron in* M^4 *then* E *can be replaced by* E′ *satisfying* (1)–(3) *with* δ *replaced by* ε *and having the additional properties that* $E'(I \times [a_{j+1}, a_j]) \cap L^2 \subset N_\varepsilon(D(I \times [a_{j+1}, 1]))$ *and* $E'|I \times 0 = E|I \times 0$.

Proof of Lemma 1. Choose $\varepsilon_1 > 0$ such that $\varepsilon_1 \leq \varepsilon$ and any path lying in the union of the $2\varepsilon_1$-neighborhoods of two fibers of D lies in the ε-neighborhood of either one of them and such that

$$N_{\varepsilon_1}(D(I \times [0, a_{j+1}])) \cap N_{\varepsilon_1}(D(I \times [a_j, 1])) = \emptyset; \text{ and}$$
$$N_{\varepsilon_1}(D(I \times 0)) \cap N_{\varepsilon_1}(D(I \times [a_{j+1}, 1])) = \emptyset.$$

Let $\delta_1 > 0$ $(\delta_1 \leq \varepsilon_1)$ be the δ of the conclusion of Lemma 2 corresponding to $\varepsilon = \varepsilon_1$ and $a = a_{j+1}$. Let V be a PL manifold neighborhood of $D(I \times [0, a_{j+1}])$ lying inside $N_{\delta_1}(D(I \times [0, a_{j+1}]))$. Choose δ_2 $(0 < \delta_2 < \min\{\delta_1, \varepsilon_1\})$ such that

$$N_{\delta_2}(D(I \times [0, a_{j+1}])) \subset V.$$

Let $\delta > 0$ be the δ of the conclusion of Lemma 3 corresponding to $\varepsilon = \delta_2$. Suppose E satisfies (1)–(3) for this choice of δ.

Let T be a triangulation of V with mesh less than or equal to ε_1 such that no simplex of T intersects both $\mathrm{Cl}(N_{\delta_2}(D(I \times [a_{j+1}, a_j]))$ and the complement of $N_{\delta_1}(D(I \times [a_{j+1}, a_j]))$ and such that $E \cap V$ is a subcomplex of T. Let L^2 be the union of all 2-simplexes of T which do not intersect $N_{\delta_2}(D(I \times [a_{j+1}, a_j]))$. Let V_* be the dual skeleton of V. Then $\dim(V_* - N_{\delta_1}(D(I \times [a_{j+1}, a_j]))) = 1$.

By Lemma 3, there exists an E′ satisfying (1)–(3) with $\delta = \delta_2$ and having the property that $E'(I \times [a_{j+1}, a_j]) \cap L^2 \subset N_{\delta_2}(D(I \times [a_{j+1}, 1]))$. In fact, since

$$L^2 \subset V \subset N_{\delta_1}(D(I \times [0, a_{j+1}])),$$
$$E'(I \times [a_{j+1}, a_j]) \cap L^2 \subset N_\delta(D(I \times [a_{j+1}, 1])) \cap N_{\delta_1}(D(I \times [0, a_{j+1}]))$$
$$\subset N_{\delta_2}(D(I \times [a_{j+1}, a_j])).$$

Because $E'(I \times 0) = E(I \times 0) \subset L^2$, $E'(I \times 0) \cap V_* = \emptyset$.

Thus by Lemma 2 there exists an ambient PL isotopy h_t such that

$E'(I \times 0)$ is not moved by h_t,

$h_t | N_{\delta_1}(D(I \times [a_{j+1}, 1])) = id$,

$h_t = id$ outside $N_{\varepsilon_1}(D(I \times [0, a_{j+1}]))$,

h_t moves points ε_1-parallel to fibers of D, and

$V_* \subset h_1(N_\varepsilon(D(I \times [a_{j+1}, 1])))$.

Now $L^2 \subset [M - E'(I \times [a_{j+1}, a_j])] \cup N_{\delta_2}(D(I \times [a_{j+1}, a_j]))$, so let f_t be a push across the join structure between L^2 and V_* so that

$$V \subset f_1 h_1(N_\varepsilon(D(I \times [a_{j+1}, 1]))) \cup [M - E'(I \times [a_{j+1}, a_j])] \cup N_{\delta_2}(D(I \times [a_{j+1}, a_j])).$$

Note that f_t is an ε_1-push, since the mesh of T is that small, and that

$$f_t | N_{\delta_2}(D(I \times [a_{j+1}, a_j])) \cup L^2 = id.$$

The embedding $E'' = h_1^{-1} f_1^{-1} E'$ satisfies parts (1) and (3) of the conclusion of Lemma 1. E'' also satisfies (2) for $i \leq j + 1$. Since $E''(I \times 0) = I(I \times 0)$, a simple reparametrization of $I \times [0, a_{j+1}]$ gives an E''' which satisfies (2) for all i.

5. PROOFS OF LEMMAS 2 AND 3

The proof of Lemma 2 is much like the proof of [4, Proposition 4.1], and so only a brief outline is included here.

Proof of Lemma 2. As in [4], choose $\delta > 0$ so that $N_\delta(D(I \times [0, a]))$ can be homotoped in $N_\varepsilon(D(I \times [0, a]))$ to $N_\varepsilon(D(I \times [a, 1]))$ moving only points of

$$N_\delta(D(I \times [0, a])) - N_\delta(D(I \times [a, 1]))$$

and moving close to fibers of D. By general position, it may be assumed that the track of the homotopy misses Q. Apply [2, Theorem 4.1] to get the desired isotopy.

Proof of Lemma 3. Choose $\varepsilon_1 > 0$ such that any path which lies in the union of the ε_1-neighborhoods of three fibers of D lies in the ε-neighborhood of any one of them. Choose $c_2, c_3, \ldots, c_j$ such that $c_i > a_i$ and

$$D(I \times [a_{i+1}, c_i]) \subset N_\varepsilon(D(I \times [a_{i+1}, a_i]))$$

and let $c_{j+1} = a_{j+1}$. Choose $\varepsilon_2 > 0$ such that the collection

$$\{N_{\varepsilon_2}(D(I \times [c_{i+1}, a_i]))\}_{i=1}^{j}$$

is pairwise disjoint. Choose $\varepsilon_3 > 0$ such that any loop in $N_{\varepsilon_3}(D(x \times [c_{i+1}, a_i]))$ for some $x \in I$ and some $i \leq j$ bounds a singular disk in $N_{\varepsilon_2}(D(x \times [c_{i+1}, a_i]))$

and such that $N_{\varepsilon_3/2}(D(I \times c_i)) \subset N_{\varepsilon_2}(D(I \times [c_i, a_{i-1}])) \cap N_{\varepsilon}(D(I \times [a_{i+1}, a_i]))$. Choose $\varepsilon_4 > 0$ such that any two points in $N_{\varepsilon_4}(D(I \times c_i)) \cap N_{\varepsilon_4}(D(x \times I))$ for some $x \in I$ and some $i \leq j + 1$ can be joined by an arc in $N_{\varepsilon_3/2}(D(x, c_i))$. Finally, choose $\delta > 0$ $(\delta \leq \varepsilon_4)$ such that

$$N_\delta(D(I \times [0, a_{j+1}])) \cap N_\delta(D(I \times [a_{j+1}, a_j])) \subset N_{\varepsilon_4}(D(I \times a_{j+1}))$$

and such that $N_{\varepsilon_4}(D(I \times c_i))$ separates $D(I \times a_i)$ from $D(I \times a_{i-1})$ in

$$N_\delta(D(I \times [a_i, a_{i-1}]))$$

for all $i \leq j$.

Suppose E and L are as in the lemma. Put $E(I \times [a_{j+1}, a_j])$ in general position with respect to L keeping $E(I \times 0)$ fixed. Then $E(I \times [a_{j+1}, a_j]) \cap L$ consists of a finite number of points, say $\{E(x_i, t_i)\}$. Consider the collection of arcs $\{A_i\}$ defined by $A_i = E(x_i \times [a_{j+1}, t_i])$. For each i, there is a finite collection $\{B_{i,\ell}\}$ of disjoint subarcs of A_i such that $A_i - \bigcup_\ell B_{i,\ell} \subset N_\delta(D(I \times [0, a_{j+1}]))$ and the ends of the arc $B_{i,\ell}$ can be joined by an arc $C_{i,\ell} \subset N_{\varepsilon_3/2}(D(x_i, c_{j+1}))$. Now $C_{i,\ell} \cup B_{i,\ell}$ bounds a singular disk $D_{i,\ell} \subset N_{\varepsilon_2}(D(x_i \times [c_{j+1}, a_j]))$.

Put all the disks $D_{i,\ell}$ in general position with respect to E, each other, and the arcs A_i. $D_\alpha \cap E(I \times [a_j, a_{j-1}])$ consists of a finite number of points for each pair α, say $D_\alpha \cap E(I \times [a_j, a_{j-1}]) = \{P_{\alpha,i}\}_i$. Let $A_{\alpha,i}$ denote the fiber of E from $P_{\alpha,i}$ to $E(I \times a_j)$. As before, there is a finite collection $\{B_{\alpha,i,\ell}\}_\ell$ of disjoint subarcs of $A_{\alpha,i}$ such that $A_{\alpha,i} - \bigcup_\ell B_{\alpha,i,\ell} \subset N_{\varepsilon_2}(D(I \times [a_{j+1}, a_j]))$. Join the ends of the arc $B_{\alpha,i,\ell}$ with an $\varepsilon_3/2$-arc $C_{\alpha,i,\ell}$. Now each $C_{\alpha,i,\ell} \cup B_{\alpha,i,\ell}$ bounds a singular disk

$$D_{\alpha,i,\ell} \subset N_{\varepsilon_2}(D(x \times [c_j, a_{j-1}]))$$

for some x.

Continue this process until a finite collection of singular disks is defined in each of the sets $N_{\varepsilon_2}(D(I \times [c_{i+1}, a_i]))$, $i = 1, \ldots, j$.

Consider one $D_\alpha \subset N_{\varepsilon_2}(D(I \times [c_2, a_1]))$. Then $\partial D_\alpha = B_\alpha \cup C_\alpha$. We wish to push B_α to C_α with an ambient isotopy that only moves points near D_α. This can easily be done using the techniques of [3]. We must be careful, however, since there will be one isotopy corresponding to each of the singular disks in $N_{\varepsilon_2}(D(I \times [c_2, a_1]))$, and we do not want any point of E to be moved by more than two of these isotopies.

There are two kinds of points on E which are near to D_α and consequently must be moved by the isotopy: (a) points of E near a point of $E \cap \operatorname{int} D_\alpha$, and (b) points of E near B_α. D_α can be adjusted slightly so that no point of type (a) is moved into a disk D_β, $\beta \neq \alpha$, and thus these points will be moved only once. A neighborhood of D_α on E will also be moved and after the isotopy may intersect some of the disks D_β, $\beta \neq \alpha$. However, on D_β these points are now points of type (a) and thus will be moved at most once more.

After each of the arcs $B_\alpha \subset N_{\varepsilon_2}(D(I \times [c_2, a_1]))$ has been moved to the corresponding C_α, look at the disks $D_\beta \subset N_{\varepsilon_2}(D(I \times [c_3, a_2]))$. For each point of $D_\beta \cap E$, the fiber of E down to the a_2 level is now contained in $N_\varepsilon(D(I \times [a_3, a_2]))$, so by moving only points on the disk E itself, we can push the a_2 level through D_β and get that $D_\beta \cap E(I \times [a_2, a_1]) = \emptyset$. Do this for every intersection point of every D_β. Now get PL isotopies moving across the disks D_β as above.

If this process is continued back to the very first disks chosen, the end result will be a modified E which satisfies (1)–(3) with δ replaced by ε, agrees with the original in $I \times 0$, and has the property that the image of each arc A_i is contained in $N_\varepsilon(D(I \times [0, a_{j+1}]))$. Furthermore, all the modification has been done in

$$N_\varepsilon(D(I \times [a_j, 1])) .$$

The proof of the lemma is now completed by simply pushing the a_{j+1} level out over each of the points of $L \cap E(I \times [a_{j+1}, a_j])$ exactly as was done in clearing the intersection of D_β with E.

6. HIGH DIMENSIONS

We wish to use exactly the same techniques to prove Theorem 4 as were used to prove Theorem 1. A problem arises in replacing Lemma 2 with a codimension 3 engulfing lemma: it is not possible in general to engulf one codimension 3 set keeping another fixed. This difficulty is overcome using the additional hypothesis of Theorem 4 to separate the two sets. We will not repeat the proof of Theorem 1 here, but will merely state the corresponding n-dimensional lemmas and indicate the changes that must be made in the 4-dimensional proofs given earlier. The procedure is much like that in [10].

In each of the following lemmas, $D: I^{n-2} \to M^n$ is a topological embedding of an $(n-2)$-cell into a PL n-manifold.

LEMMA 1′. *Suppose* $0 = a_k < a_{k-1} < \dots < a_1 = a_0 = 1$ *is a partition of* $[0, 1]$ *and* $0 \le j \le k$. *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* $E: I^{n-2} \to M^n$ *is a locally flat* PL *embedding satisfying*

(1′) $E(I^{n-3} \times [a_i, a_{i-1}]) \subset N_\delta(D(I^{n-3} \times [a_i, a_{i-1}]))$ *for all* $i \le j$,

(2′) $E(I^{n-3} \times [a_i, a_{i-1}]) \subset N_\delta(D(I^{n-3} \times [(1/2)a_i, (1/2)a_{i-1}]))$ *for* $i \ge j+2$,

(3′) $E(I^{n-3} \times [a_{j+1}, a_j]) \subset N_\delta(D(I^{n-3} \times [(1/2)a_{j+1}, a_j]))$, *and*

(4′) $E(x \times I) \subset N_\delta(D(x \times I))$ *for all* $x \in I^{n-3}$,

then E *can be replaced by a locally flat* PL *embedding* E′ *satisfying* (1′)–(4′) *with* j *replaced by* j + 1 *and* δ *replaced by* ε.

LEMMA 2′. *Suppose that* $0 \le a < b \le 1$. *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* P^k $(k \le n-3)$ *is a finite* k*-dimensional polyhedron in*

$$N_\delta(D(I^{n-3} \times [a, b])) ,$$

then there exists a PL *isotopy* h_t *of* M^n *such that*

(i) $h_0 = \mathrm{id}$,

(ii) $h_t = \mathrm{id}$ on $N_\delta(D(I^{n-3} \times [b, 1]))$ and outside of $N_\varepsilon(D(I^{n-3} \times [a, b]))$,

(iii) $h_1(N_\varepsilon(D(I^{n-3} \times [b, 1]))) \supset P^k$, *and*

(iv) h_t *moves points* ε*-parallel to fibers of* D.

LEMMA 3′. *Suppose that* $0 = a_k < a_{k-1} < \dots < a_1 = 1$ *is a partition of* $[0, 1]$ *and that* $1 \le j \le k$. *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* $E: I^{n-2} \to M^n$ *is a locally flat* PL *embedding satisfying* (1′)-(4′) *and* L^2 *is any finite* 2*-polyhedron in* M^n, *then* E *can be replaced by a locally flat* PL *embedding* E′ *satisfying*

(a) $E'(I^{n-3} \times [a_i, a_{i-1}]) \subset N_\varepsilon(D(I^{n-3} \times [a_i, a_{i-1}]))$ *for all* $i \le j$,

(b) $E'(I^{n-3} \times [a_i, a_{i-1}]) \subset N_\varepsilon(D(I^{n-3} \times [(1/2)a_i, (1/2)a_{i-1}]))$ *for* $i \ge j + 3$,

(c) $E'(I^{n-3} \times [a_{i+1}, a_i]) \subset N_\varepsilon(D(I^{n-3} \times [(1/2)a_{i+1}, a_i]))$ *for* $i = j, j+1$,

(d) $E'(x \times I) \subset N_\varepsilon(D(x \times I))$ *for all* $x \in I^{n-3}$, *and*

(e) $E'(I^{n-3} \times [a_{j+1}, a_j]) \cap L^2 \subset N_\varepsilon(D(I^{n-3} \times [a_{j+1}, 1]))$.

The proof of Lemma 3′ is exactly like that of Lemma 3. In fact, the argument is a little easier for $n \ge 5$ since the singular disks D_α in the proof of Lemma 3 can be realized as locally flat, pairwise disjoint, embedded disks. Similarly, the proof of Lemma 2′ follows that of Lemma 2 and these two lemmas combine to prove Lemma 1′.

To prove Theorem 4 using Lemma 1′, proceed as follows. First choose a partition $0 = a_k < \dots < a_1 = 1$ as in the proof of Theorem 1. Next use the hypothesis of Theorem 4 to find a locally flat PL embedding $E: I^{n-2} \to M^n$ which satisfies (2′) for all i. Then [2, Theorem 4.1] can be used to pull $E(I^{n-3} \times 1)$ down near $D(I^{n-3} \times 1)$ keeping $E(I^{n-3} \times [0, a_2])$ fixed. After that, it is just a matter of applying Lemma 1′ inductively to position the approximation one slice at a time.

REFERENCES

1. F. D. Ancel and J. W. Cannon, *Any embedding of S^{n-1} in S^n ($n \ge 5$) can be approximated by locally flat embeddings.* Notices Amer. Math. Soc. 23(1976), A-308.

2. R. H. Bing, *Vertical general position.* Geometric Topology (Proc. Conf., Park City, Utah, 1974). Lecture Notes in Mathematics, Vol. 438, pp. 16–41. Springer, Berlin, 1975.

3. R. H. Bing and J. M. Kister, *Taming complexes in hyperplanes.* Duke Math. J. 31(1964), 491–511.

4. J. L. Bryant, *Euclidean n-space modulo an (n − 1)-cell.* Trans. Amer. Math. Soc. 179(1973), 181–192.

5. W. T. Eaton, C. P. Pixley and G. A. Venema, *A topological embedding which cannot be approximated by a piecewise linear embedding.* Notices Amer. Math. Soc. 24(1977), A-302.

6. R. H. Fox and J. W. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots.* Osaka J. Math. 3(1966), 257–267.

7. J. A. Lees, *Immersions and surgeries of topological manifolds.* Bull. Amer. Math. Soc. 75(1969), 529–534.

8. R. T. Miller, *Approximating codimension 3 embeddings.* Ann. of Math. (2) 95(1972), 406–416.

9. C. P. Pixley and G. A. Venema, *On locally flat approximation of piecewise linear embeddings in codimension two,* to appear.

10. T. M. Price and C. L. Seebeck III, *A codimension two taming theorem.* Geometric Topology (Proc. Conf., Park City, Utah, 1974). Lecture Notes in Mathematics, Vol. 438, pp.371–394. Springer, Berlin, 1975.

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