JORDAN C*-ALGEBRAS

J. D. Maitland Wright

INTRODUCTION

In his final lecture to the 1976 St. Andrews Colloquium of the Edinburg Mathematical Society, Professor Kaplansky introduced the concept of a Jordan C*-algebra (see below for definitions), pointed out its potential importance, and made the following conjecture. Let $\mathscr{A}_1$, $\mathscr{A}_2$ be unital Jordan C*-algebras and let $\phi\colon \mathscr{A}_1 \to \mathscr{A}_2$ be a surjective isometry with $\phi 1 = 1$; then ϕ is a Jordan *-isomorphism. In verifying this conjecture [15], extensive use was made of the deep results of Alfsen, Schultz, and Störmer [2] on JB-algebras.

It is easy to see that the self-adjoint part of a Jordan C*-algebra is a JB-algebra. The main part of this paper, Section 2, is devoted to establishing a converse result. *Each* JB-*algebra is the self-adjoint part of a unique Jordan* C*-*algebra.* First we establish the result for finite-dimensional algebras. This is not entirely straightforward and seems to require quite delicate arguments. Once this is accomplished; in particular, when we know of the existence of an exceptional Jordan C*-algebra, $\mathscr{M}_3^8$, whose self-adjoint part is M_3^8 (the exceptional Jordan algebra discovered by von Neumann, Jordan, and Wigner [6]), then the general result can be obtained quite quickly.

In the final section we consider ideals and quotients of Jordan C*-algebras and, applying the results of Section 2 and the main theorem of [2], show that for each Jordan C*-algebra $\mathscr{A}$ there exists a unique *-ideal $\mathscr{J}$ such that (i) $\mathscr{A}/\mathscr{J}$ can be isometrically *-isomorphically embedded into the special Jordan *-algebra of bounded operators on a complex Hilbert space and (ii) each 'factorial' representation of $\mathscr{A}$ which does not annihilate $\mathscr{J}$ is onto $\mathscr{M}_3^8$.

I would like to draw the attention of the reader to an interesting recent paper by Bonsall [3] in which he obtains a generalization of the Vidav-Palmer Theorem to special Jordan *-algebras.

1. BASIC PROPERTIES OF JORDAN C*-ALGEBRAS

Definition (Kaplansky). Let $\mathscr{A}$ be a complex Banach space and a complex Jordan algebra equipped with an involution *. Then $\mathscr{A}$ is a *Jordan* C*-*algebra* if the following four conditions are satisfied.

(i) $\|x \circ y\| \leq \|x\|\,\|y\|$ for all x and y in $\mathscr{A}$.

(ii) $\|z\| = \|z^*\|$ for all z in $\mathscr{A}$.

(iii) $\|\{zz^*z\}\| = \|z\|^3$ for all z in $\mathscr{A}$.

(Here $\{abc\}$ is the Jordan triple product as defined on page 36 [5].)

(iv) Each norm-closed, associative *-subalgebra of $\mathscr{A}$ is a C*-algebra

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We can dispense with condition (iv) because, if z and z^* lie in a norm-closed, associative $*$-subalgebra of $\mathscr{A}$, then the Jordan triple product $\{zz^*z\}$ reduces to $(z \circ z^*) \circ z$. Hence, by (i) and (iii), $\|z\|^3 \leq \|z \circ z^*\| \, \|z\|$. But, by (i) and (ii), $\|z\|^2 = \|z\| \, \|z^*\| \geq \|z \circ z^*\|$. Thus $\|z \circ z^*\| = \|z\|^2$.

A Jordan C*-algebra is said to be a JC*-*algebra* if it is isometrically *-isomorphic to a norm-closed, Jordan *-subalgebra of the Jordan *-algebra of all bounded operators on a complex Hilbert space; *e.g.*, the three-dimensional Jordan *-algebra of 2×2 symmetric matrices over $\mathbb{C}$. Clearly, every JC*-algebra is a special Jordan algebra but, as we shall see presently, there exist Jordan C*-algebras which are exceptional and so are not JC*-algebras. In all that follows we shall only consider Jordan C*-algebras which are *unital;* that is, possess a multiplicative identity.

Definition (Alfsen-Schultz-Störmer [2]). Let A be a real Banach space and a real Jordan algebra equipped with a multiplicative unit 1. Then A is a JB-*algebra* if the following conditions are satisfied.

(i) $\|x \circ y\| \leq \|x\| \, \|y\|$ for all x and y in A.

(ii) For each $a \in A$, the norm-closed, associative subalgebra of A generated by 1 and a is isometrically isomorphic to the self-adjoint part of a commutative C*-algebra.

Let $\mathscr{A}$ be a Jordan C*-algebra and let $A = \{x \in \mathscr{A} : x = x^*\}$. Since $\|z\| = \|z^*\|$ for all z in $\mathscr{A}$, A is a closed (real) subspace of $\mathscr{A}$. It is straightforward to verify that $\mathscr{A} = A \oplus iA$ and that A is a JB-algebra.

A *Jordan *-algebra* is a complex (unital) Jordan algebra equipped with an involution *.

LEMMA 1.1. *Let $\mathscr{A}$ be a Jordan *-algebra over $\mathbb{C}$. Let $\|\ \|$ be any norm for $\mathscr{A}$ such that $\|\{zz^*z\}\| = \|z\|^3$ whenever $z \in \mathscr{A}$. Let $\|\ \|_1$ be an equivalent norm such that $\|\{zz^*z\}\|_1 \leq \|z\|_1^3$ for all $z \in \mathscr{A}$. Then, for every $w \in \mathscr{A}$, $\|w\|_1 \geq \|w\|$.*

Proof. Suppose that, for some w, $\|w\|_1 < \|w\|$. Since we could replace w by $w/\|w\|$, we lose no generality by requiring that $\|w\| = 1$. Let us construct a sequence (z_n) $(n = 1, 2, \cdots)$ by setting $z_0 = w$ and $z_{n+1} = \{z_n z_n^* z_n\}$ for $n = 0, 1, 2, \cdots$. Then $\|z_n\|_1 \leq \|w\|_1^{3^n}$. Thus $\|z_n\|_1 \to 0$ as $n \to \infty$. Since $\|\ \|_1$, and $\|\ \|$ are equivalent, it follows that $\|z_n\| \to 0$ as $n \to \infty$. But this is impossible, because $\|z_n\| = \|w\|^{3^n} = 1$ for each n. Thus $\|w\|_1 \geq \|w\|$ for all w in $\mathscr{A}$.

LEMMA 1.2. *Let $\mathscr{A}$ be a Jordan *-algebra over $\mathbb{C}$. Let $\|\ \|$ be a norm for $\mathscr{A}$ and $\|\ \|_1$ a seminorm for $\mathscr{A}$ such that, for each $z \in \mathscr{A}$, $\|z\| = \|z^*\|$ and $\|z\|_1 = \|z^*\|_1$. Furthermore, let $\|x\| = \|x\|_1$ whenever $x = x^*$. Then $\|\ \|$ and $\|\ \|_1$ are equivalent norms and, whenever x and y are self-adjoint,*

$$\max(\|x\|, \|y\|) \leq \|x + iy\|_1 \leq \|x\| + \|y\| .$$

Proof. For $x, y \in A$ we have,

$$\|x\| = \frac{1}{2}\|(x + iy) + (x - iy)\|_1 \leq \frac{1}{2}(\|x + iy\|_1 + \|x - iy\|_1) = \|x + iy\|_1 .$$

So

$$\max(\|x\|, \|y\|) \leq \|x+iy\|_1 \leq \|x\|_1 + \|y\|_1 = \|x\| + \|y\| .$$

Let $(x_n + iy_n)$ $(n = 1, 2, \cdots)$ be a sequence in $\mathscr{A}$ with x_n and y_n self-adjoint for each n. Then $\|x_n + iy_n\|_1 \to 0$ if, and only if, $\|x_n\| \to 0$ and $\|y_n\| \to 0$, that is, if, and only if, $\|x_n + iy_n\| \to 0$. Hence $\|\ \|$ and $\|\ \|_1$ are equivalent norms.

PROPOSITION 1.3. *Let $\mathscr{A}$ be a Jordan* C*-*algebra with respect to the norm* $\|\ \|$. *Let* $\|\ \|_1$ *be another norm for $\mathscr{A}$ such that the following conditions hold.*

(i) $\|\{zz^*z\}\|_1 = \|z\|_1^3$ *for all* $z \in \mathscr{A}$.

(ii) $\|z\|_1 = \|z^*\|_1$ *for all* $z \in \mathscr{A}$.

(iii) $\|z \circ w\|_1 \leq \|z\|_1 \ \|w\|_1$, *whenever the *-subalgebra of $\mathscr{A}$ generated by* z, w *and* 1 *is associative. Then* $\|w\| = \|w\|_1$ *for all* $w \in \mathscr{A}$.

Proof. Let $a \in A$ and let $\mathbb{C}(a)$ be the $\|\ \|$-closed, associative *-subalgebra of $\mathscr{A}$ generated by 1 and a. Then $\mathbb{C}(a)$ is a commutative C*-algebra with respect to the norm $\|\ \|$. By (iii), $\|\ \|_1$ is an algebra norm on $\mathbb{C}(a)$. Furthermore, (i) implies that $\|z \circ z^*\|_1 = \|z\|_1^2$ for all $z \in \mathbb{C}(a)$. Hence by a well-known theorem of Kaplansky [7] (see Theorem 1.2.4 and Corollary 1.2.5 of Sakai [13]) we have $\|a\| = \|a\|_1$. Hence by Lemma 1.2, $\|\ \|$ and $\|\ \|_1$ are equivalent. So, by Lemma 1.1, for any $w \in \mathscr{A}$, $\|w\| \leq \|w\|_1 \leq \|w\|$.

COROLLARY 1.4. *Let $\mathscr{A}$ and $\mathscr{B}$ be Jordan* C*-*algebras and let* $\Phi: \mathscr{A} \to \mathscr{B}$ *be an injective Jordan *-homomorphism. Then* Φ *is an isometry.*

COROLLARY 1.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be Jordan* C*-*algebras and let* $\Phi: \mathscr{A} \to \mathscr{B}$ *be any Jordan *-homomorphism. Then* Φ *is a contraction.*

Proof. Let us consider the algebraic direct sum $\mathscr{A} \oplus \mathscr{B}$ and let a norm be defined on $\mathscr{A} \oplus \mathscr{B}$ by setting $\|a \oplus b\| = \max(\|a\|, \|b\|)$. Then $\mathscr{A} \oplus \mathscr{B}$ is a Jordan C*-algebra. The map $a \to a \oplus \Phi(a)$ is an injective *-homomorphism of $\mathscr{A}$ into $\mathscr{A} \oplus \mathscr{B}$ and hence, by Corollary 1.4, is an isometry. So $\|\Phi(a)\| \leq \|a\|$.

2. COMPLEXIFICATIONS OF JB-ALGEBRAS

Let A be any algebra over $\mathbb{R}$ (associative or not). Let us define the *complexification* of A to be a pair $(h, \mathscr{A})$ where $\mathscr{A}$ is an algebra over $\mathbb{C}$ equipped with an involution *, and $h: A \to \mathscr{A}$ is a monomorphism ($\mathscr{A}$ being regarded, at this point, as an algebra over $\mathbb{R}$) such that $h[A]$ is the self-adjoint part of $\mathscr{A}$. The existence of a complexification is trivial, for we may take $\mathscr{A}$ to be the set $A \times A$, equipped with addition, scalar multiplication, product and involution defined in the obvious way, and may take h(a) to be (a, 0) for each $a \in A$. When A comes equipped with a topology we shall suppose its complexification to be equipped with the product topology. When A is a Jordan algebra over $\mathbb{R}$ then $\mathscr{A}$ can be shown to be a Jordan *-algebra over $\mathbb{C}$ [5; Section 6, Chapter 1].

For the remainder of this section A shall be a JB-algebra. We shall identify A with the self-adjoint part of its complexification $\mathscr{A}$, and shall always suppose $\mathscr{A}$ equipped with the product topology induced by A. The object of this section is to prove that $\mathscr{A}$ can be equipped with a norm which organizes $\mathscr{A}$ as a Jordan C*-algebra.

PROPOSITION 2.1. *Let* A *be a* JB*-algebra. Let* E_0 *be the subalgebra of* A *generated by* 1, a *and* b; *let* E *be the norm closure of* E_0. *Then* E *is isometrically isomorphic to a Jordan algebra of self-adjoint operators on a complex Hilbert space.*

Proof. Clearly, E is, itself, a JB-algebra. Thus, by the Alfsen-Schultz-Störmer structure theorem [2], either E is isometrically isomorphic to a Jordan algebra of self-adjoint operators on a complex Hilbert space, or, there exists a surjective homomorphism h from E onto M_3^8, the exceptional Jordan algebra of hermitian 3×3 matrices over the Cayley numbers [6]. Let us suppose the latter possibility holds. Then, since M_3^8 is finite dimensional, $h[E_0]$ is a finite dimensional, and hence closed, subspace of the Banach space M_3^8. Thus $h^{-1}[h[E_0]]$ is a closed subspace of E which contains E_0. So $E = h^{-1}[h[E_0]]$. Hence

$$h[E_0] = h[E] = M_3^8.$$

Thus M_3^8 is generated by 1, h(a) and h(b). So, by the Shirshov-Cohn theorem [4], M_3^8 is special but, by Albert's theorem [1, 5], M_3^8 is exceptional. This contradiction establishes the proposition.

The following corollary will be very useful to us. Whenever $\mathscr{S}$ is a subset of $\mathscr{A}$, let Jord($\mathscr{S}$) be the smallest closed *-subalgebra of $\mathscr{A}$ which contains 1 and $\mathscr{S}$.

COROLLARY 2.2. *Let* $\mathscr{A}$ *be the complexification of a* JB*-algebra* A. *Let* a *and* b *be any self-adjoint elements of* $\mathscr{A}$. *Then* Jord(a, b) *is* *-*isomorphic and homeomorphic to a* JC^**-algebra.*

Unitaries. An element u of $\mathscr{A}$ is said to be *unitary* if $u \circ u^* = 1$ and $u^2 \circ u^* = u$, in other words, u^* is the inverse of u [5; Section 11, Chapter 1]. We have Jord(u) = Jord(u, u*) is *-generated by the self-adjoint elements $(u + u^*)/2$ and $(u - u^*)/2i$. Thus, by Corollary 2.2, there exists a complex Hilbert space H, a JC^*-algebra of operators on H, $\mathscr{D}$, and a *-isomorphism h from Jord(u) onto $\mathscr{D}$. Then, see [5; Section 11, Chapter 1] h(u) and h(u)* are inverses in $\mathscr{L}(H)$, that is, h(u) is a unitary operator. Hence $\mathscr{D}$ is a commutative C^*-algebra. In particular, when Jord(u) is finite-dimensional, finite-dimensional spectral theory shows that there exists a self-adjoint a in Jord(u) such that u = exp ia.

Throughout the rest of this section, $E = \{\exp ia : a \in A\}$ and U is the convex hull of E in $\mathscr{A}$.

LEMMA 2.3. (i) U *is absolutely convex.*

(ii) *Let* $a \in A$. *If* $\|a\| < 1$ *then* $a \in U$. *If* $a \in U$ *then* $\|a\| \leq 1$.

(iii) U *is absorbent.*

(iv) *There exists a seminorm* ρ *on* $\mathscr{A}$ *such that*

$$\{z \in \mathscr{A} : \rho(z) < 1\} \subset U \subset \{z \in \mathscr{A} : \rho(z) \leq 1\}.$$

(v) *Whenever* $a \in A$ *then* $\rho(a) = \|a\|$.

(vi) *For each* $z \in \mathscr{A}$, $\rho(z) = \rho(z^*)$.

(vii) ρ *is a Banach space norm for* $\mathscr{A}$ *and the norm topology induced on* $\mathscr{A}$ *by* ρ *is the same as the product topology induced by* A. *Furthermore, for each* x *and* y *in* A, $\max(\|x\|, \|y\|) \leq \rho(x + iy) \leq \|x\| + \|y\|$.

Proof. (i) A typical element of the absolutely convex hull of E is

$$z = \sum_{j=1}^{n} \beta_j \exp(ia_j),$$

where $\sum_1^n |\beta_j| \leq 1$ and $a_j \in A$ $(j = 1, 2, \cdots, n)$. For each j, the complex number β_j is of the form $|\beta_j| e^{i\theta_j}$, so that $z = \sum_1^n |\beta_j| \exp i(a_j + \theta_j 1)$. Let

$$\sum_1^n |\beta_j| = \cos \phi .$$

Then, either $z = 0$ and so is certainly in U, or else we may put $\lambda_j = \dfrac{|\beta_j|}{\cos \phi}$ $(j = 1, 2, \cdots, n)$, so that $\sum_1^n \lambda_j = 1$ and $\lambda_j \geq 0$ $(j = 1, 2, \cdots, n)$. In the latter event, we see that z is a convex combination of $\exp i(a_j + \theta_j 1 \pm \phi 1)$ $(j = 1, 2, \cdots, n)$.

(ii) If $\|a\| < 1$ then $a \pm i(1 - a^2)^{1/2} = \exp i(\pm \arccos a)$ and so a is the average of two elements of E.

If $a \in U$ then $a = \sum_1^n \lambda_j \exp ia_j$, where $\sum_1^n \lambda_j = 1$ and $\lambda_j \geq 0$ $(j = 1, 2, \cdots, n)$. Since a is self-adjoint, $a = \frac{1}{2}(a + a^*) = \sum_1^n \lambda_j \cos a_j$. So

$$\|a\| \leq \sum_1^n \lambda_j \|\cos a_j\| \leq 1 .$$

(iii) Let a and b be any elements of A. Then, for large enough n, $\|a\| < n/2$ and $\|b\| < n/2$.

So, by (ii), $2a/n$ and $2b/n$ are in U. Hence $\frac{1}{2}\left(\frac{2}{n}a + i\frac{2}{n}b\right) \in U$, that is

$$\frac{1}{n}(a + ib) \in U.$$

Hence U is absorbent.

(iv) Let ρ be the Minkowski seminorm associated with the absolutely convex and absorbent set U. [11, Section 4, Chapter 1.]

(v) For any positive ε, $\rho\left(\frac{1}{\rho(a) + \varepsilon} a\right) < 1$. So, by (iv), $a/(\rho(a) + \varepsilon) \in U$. Thus, by (ii), $\|a\| \leq \rho(a) + \varepsilon$. Hence $\|a\| \leq \rho(a)$.

Conversely, for any positive ε, $\left\|\left(\frac{1}{\|a\| + \varepsilon}\right) a\right\| < 1$ and so, by (ii), $\rho\left(\frac{1}{\|a\| + \varepsilon} a\right) < 1$. Thus $\rho(a) \leq \|a\|$.

(vi) Clearly $w \in U$ if, and only if, $w^* \in U$. A routine argument now shows that $\rho(w) = \rho(w^*)$.

(vii) This follows from (v), (vi) and Lemma 1.2.

LEMMA 2.4. *Let* $\mathscr{B}$ *be a finite dimensional* JC*-*algebra of operators on a complex Hilbert space* H. *Let* z *be an invertible element of* $\mathscr{B}$ *with* $\|z\| \leq 1$. *Then there exist unitaries* w *and* v *in* $\mathscr{B}$ *such that* $z = \frac{1}{2}(w + v)$. *Furthermore, if* a *is an element of* $\mathscr{B}$ *with* $\|a\| < 1$, *then there exist unitaries* u_1, u_2, u_3, u_4 *in* $\mathscr{B}$ *such that* $a = \frac{1}{4}(u_1 + u_2 + u_3 + u_4)$.

Proof. Let r be a positive self-adjoint operator on H and u a partial isometry on H such that $z = ru$. Since z is invertible in the special Jordan algebra $\mathscr{B}$, it is invertible in $\mathscr{L}(H)$ [5, Section 11, Chapter 1]. Hence r is invertible and u is unitary. Moreover, $\|r\| = \|ru\| \leq 1$.

We have $r^3 u = \{zz^*z\} \in \mathscr{B}$. By repeating this argument we find that $r^{3^n} u \in \mathscr{B}$ for $n = 0, 1, 2, \cdots$. Since $\mathscr{A}$ is finite-dimensional $\sum_{j=0}^{n} c_j r^{3^j} u = 0$ for some constants (not all zero), $c_0, c_1, \cdots, c_n$. Multiplying on the right by u^*, we see that r satisfies a nontrivial polynomial identity and so r has finite spectrum. Thus there exist orthogonal projections $p_1, p_2, \cdots, p_n$ and positive real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ such that $r = \sum_1^n \lambda_j p_j$ and $\sum_1^n p_j = 1$. Since $\left(\frac{r}{\lambda_n}\right)^{3^k} u \in \mathscr{B}$ for $k = 0, 1, 2, \cdots$, it follows that $p_n u \in \mathscr{B}$. On replacing z by $z - \lambda_n p_n u$ and repeating the above argument we find that $p_{n-1} u \in \mathscr{B}$. Similarly $p_j u \in \mathscr{B}$ for each j. It now follows that

$$w = (r + i\sqrt{1 - r^2})u \quad \text{and} \quad v = (r - i\sqrt{1 - r^2})u$$

are both in $\mathscr{B}$. Being products of unitaries, w and v are unitaries. Clearly $z = \frac{1}{2}(w + v)$.

Let C(a) be the smallest closed subalgebra of $\mathscr{B}$ which contains 1 and a. Then C(a) is a finite dimensional Banach algebra. Thus the spectrum of a, with respect to C(a), is finite. Hence, for large enough n, $a + \frac{1}{n}1$ and $a - \frac{1}{n}1$ are both invertible and, since $\|a\| < 1$, for large enough n are of norm less than 1. Since

$$a = \frac{1}{2}\left(\left(a + \frac{1}{n}1\right) + \left(a - \frac{1}{n}1\right)\right),$$

the required result follows from the first part of this lemma.

LEMMA 2.5. *Let* $\mathscr{A}$ *be finite dimensional. Then the following hold.*

(i) *Each unitary* u *in* $\mathscr{A}$ *is of the form* exp ia *for some* $a \in A$.

(ii) *Let* u *and* v *be unitaries in* $\mathscr{A}$. *Then* $\rho(u \circ v) \leq 1$. *Also* $\{uvu\}$ *is unitary and* $\{u^*\{uvu\}u^*\} = v$.

(iii) *For each unitary* u, $\rho(u) = 1$.

(iv) *For any* z *and* w *in* $\mathscr{A}$, $\rho(z \circ w) \leq \rho(z)\rho(w)$.

(v) *For any* $z \in \mathscr{A}$ *and any unitary* $u \in \mathscr{A}$, $\rho(\{uzu\}) = \rho(z)$.

(vi) *For any unitary* $V \in \mathscr{A}$; *and any* z *and* w *in* $\mathscr{A}$, $\rho(\{zv^2w\}) \leq \rho(z)\rho(w)$.

(vii) *For any* z_1, z_2, z_3 *in* $\mathscr{A}$, $\rho(\{z_1 z_2 z_3\}) \leq \rho(z_1)\rho(z_2)\rho(z_3)$.

Proof. (i) See the remarks after Corollary 2.2.

(ii) For some a and b in A, $u = \exp ia$ and $v = \exp ib$. Hence

$$\mathrm{Jord}(u, v) \subset \mathrm{Jord}(a, b).$$

By Corollary 2.2, there exist a JC*-algebra $\mathscr{E}$ of bounded operators on a Hilbert space H and a Jordan *-isomorphism h from Jord(a, b) onto $\mathscr{E}$. Choose any positive $\delta < 1$. Then $h(\delta u \circ v) = \delta h(u) \circ h(v) = \frac{\delta}{2}(h(u)h(v) + h(v)h(u))$. Thus

$$\|h(\delta u \circ v)\| < 1.$$

Thus, by Lemma 2.4, there exist unitaries u_1, u_2, u_3, u_4 in Jord(a, b) such that $\delta u \circ v = \frac{1}{4}(u_1 + u_2 + u_3 + u_4)$. Hence $\rho(u \circ v) \leq 1$.

We have $h\{uvu\} = \{h(u)h(v)h(u)\} = h(u)h(v)h(u)$, which is unitary. Hence $\{uvu\}$ is unitary. A similar argument shows that $\{u^*\{uvu\}u^*\} = v$.

(iii) $1 = \rho(u \circ u^*) \leq \rho(u)\rho(u^*) = \rho(u)^2$. Thus $1 \leq \rho(u)$.

But, since $u \in E$, we have $\rho(u) \leq 1$.

(iv) It suffices to show that when $\rho(z) < 1$ and $\rho(w) < 1$ then $\rho(z \circ w) \leq 1$.

We have $z = \sum_1^n \lambda_j u_j$ and $w = \sum_1^m \mu_k v_k$ where $\sum_1^n \lambda_j = \sum_1^m \mu_k = 1$, $\lambda_j \geq 0$ and $\mu_k \geq 0$ for each j and k, and u_j and v_k are unitaries for each j and k. Then $z \circ w = \sum_{j=1}^n \sum_{k=1}^m \lambda_j \mu_k u_j \circ v_k$. Thus

$$\rho(z \circ w) \leq \sum_{j=1}^{n} \sum_{k=1}^{m} \lambda_j \mu_k \rho(u_j \circ v_k) \leq \sum_{j=1}^{n} \sum_{k=1}^{m} \lambda_j \mu_k = 1.$$

(v) Choose $\varepsilon > 0$ and let $w = z/(\rho(z) + \varepsilon)$. So $w = \sum_1^n \lambda_j v_j$, where $\lambda_j \geq 0$ and $v_j \in E$ for $j = 1, 2, \cdots, n$ and $\sum_1^n \lambda_j = 1$. Thus $\{uwu\} = \sum_1^n \lambda_j \{uv_j u\}$. So $\rho(\{uwu\}) \leq 1$. Hence $\rho(\{uzu\}) \leq \rho(z)$. Hence $\rho(z) = \rho(\{u^*\{uzu\}u^*\}) \leq \rho(\{uzu\})$.

(vi) The identity $\{v\{zv^2w\}v\} = \{vzv\} \circ \{vwv\}$ is valid in any special Jordan algebra and hence, by Macdonald's theorem [8], [4], in every Jordan algebra.

$$\rho(\{zv^2w\}) = \rho(\{v\{zv^2w\}v\}) = \rho(\{vzv\} \circ \{vwv\})$$

$$\leq \rho(\{vzv\})\rho(\{vwv\}) \leq \rho(z)\rho(w).$$

(vii) Choose $\varepsilon > 0$, let $\delta = (\rho(z_2) + \varepsilon)^{-1}$. Thus $\rho(\delta z_2) < 1$.

Thus $\delta z_2 = \sum_1^n \lambda_j v_j^2$, where $\lambda_j \geq 0$ $(j = 1, 2, \cdots, n)$, $\sum_1^n \lambda_j = 1$, and $v_j = \exp(ia_j/2)$, for self-adjoint $a_1, a_2, \cdots, a_n$. Thus $\{z_1(\delta z_2)z_3\} = \sum_1^n \lambda_j \{z_1 v_j^2 z_3\}$.

Thus $\rho(\{z_1(\delta z_2)z_3\}) \leq \sum_1^n \lambda_j \rho\{z_1 v^2 z_3\} \leq \rho(z_1)\rho(z_3)$.

Hence $\rho(\{z_1 z_2 z_3\}) \leq \rho(z_1)(\rho(z_2)+\varepsilon)\rho(z_3)$.

Since this holds for all $\varepsilon > 0$, the lemma is established.

THEOREM 2.6. *Let $\mathscr{A}$ be the complexification of a finite dimensional* JB-*algebra* A. *Then $\mathscr{A}$ is a Jordan* C^*-*algebra with respect to the norm* ρ.

Proof. Let z be any element of $\mathscr{A}$ and let $\|\ \|_1$ be the norm ρ restricted to Jord(z). By Corollary 2.2, there exists a norm $\|\ \|$ for Jord(z) which organizes Jord(z) as a JC^*-algebra.

Let $w \in$ Jord(z) such that $\|w\| < 1$. Then, by Lemma 2.4,

$$w = \frac{1}{4}(u_1 + u_2 + u_3 + u_4),$$

where u_1, u_2, u_3, u_4 are unitary in Jord(z). This implies that $\rho(w) \leq 1$, that is, $\|w\|_1 \leq 1$. Hence, for any $y \in$ Jord(z), $\|y\|_1 \leq \|y\|$.

By Lemma 1.2, $\|\ \|_1$ and $\|\ \|$ are equivalent norms for Jord(z). Hence, by Lemma 1.1 and Lemma 2.5, (vii), $\|w\|_1 \geq \|w\|$ for all $w \in$ Jord(z). Thus $\|w\|_1 = \|w\|$ for all $w \in$ Jord(z). Hence, in particular, $\rho(\{zz^*z\}) = \rho(z)^3$. Then Lemma 2.5, (iv), and Lemma 2.3, (vi) and (vii), show that when $\mathscr{A}$ is equipped with the norm ρ it becomes a Jordan C^*-algebra.

In [2] it is pointed out that the exceptional (real) Jordan algebra M_3^8 of von Neumann, Jordan, and Wigner [6], is a JB-algebra.

COROLLARY 2.7. *Let $\mathscr{M}_3^8$ be the complexification of M_3^8. Then there is a unique norm for $\mathscr{M}_3^8$ which organizes it as an (exceptional) Jordan* C^*-*algebra.*

From this point onwards $\mathscr{M}_3^8$ shall be the unique Jordan C^*-algebra whose self-adjoint part is M_3^8.

Let $(\mathscr{A}_\lambda)$ $(\lambda \in \Lambda)$ be a family of Jordan C^*-algebras. The direct sum $\bigoplus_{\lambda\in\Lambda} \mathscr{A}_\lambda$ is constructed as follows. First, the elements of $\bigoplus_{\lambda\in\Lambda} \mathscr{A}_\lambda$ are all families (a_λ) $(\lambda \in \Lambda)$ in $\prod_{\lambda\in\Lambda} \mathscr{A}_\lambda$ for which $\sup_\lambda \|a_\lambda\| < +\infty$. Secondly, the algebraic operations and involution are defined pointwise and $\|(a_\lambda)\|$ is defined to be $\sup_{\lambda\in\Lambda} \|a_\lambda\|$. A routine verification shows that $\bigoplus_{\lambda\in\Lambda} \mathscr{A}_\lambda$ is a Jordan C^*-algebra.

THEOREM 2.8. *Let* A *by any* JB-*algebra and let $\mathscr{A}$ be its complexification. Then there exists a unique norm on $\mathscr{A}$ which organizes $\mathscr{A}$ as a Jordan* C^*-*algebra.*

Proof. The uniqueness of such a norm follows from Corollary 1.4.

By Corollary 5.7 and Theorems 8.6 and 9.5 of [2], there exists a family $\mathscr{F}$ of homomorphism ϕ: A $\to$ B such that (i) for each non-zero a in A there can be found a ϕ in $\mathscr{F}$ such that $\phi(a) \neq 0$ and (ii) for each ϕ in $\mathscr{F}$, $B_\phi = M_3^8$ or B_ϕ is a norm-closed Jordan algebra of self-adjoint operators on a complex Hilbert space. For each ϕ in $\mathscr{F}$, let $\mathscr{B}_\phi$ be the complexification of B_ϕ. Then either, by Corollary 2.7, we can identify $\mathscr{B}_\phi$ with $\mathscr{M}_3^8$ or else $\mathscr{B}_\phi$ is a JC^*-algebra.

Consider the direct sum $\bigoplus_{\phi \in \mathscr{F}} \mathscr{B}_\phi$. Then we can construct a Jordan *-homomorphism $H: \mathscr{A} \to \bigoplus_{\phi \in \mathscr{F}} \mathscr{B}_\phi$, by, for each a, b in A, letting $H(a + ib)$ be $(\phi(a) + i\phi(b))$ $(\phi \in \mathscr{F})$. By property (i) of $\mathscr{F}$ we see that H is injective. Then, by Lemma 9.3 [2], H is an isometry on A. Let us norm $\mathscr{A}$ by setting $\|z\| = \|Hz\|$. Then, by Lemma 1.2, $\mathscr{A}$ is a Banach space. The other conditions for $\mathscr{A}$ to be a Jordan C*-algebra are clearly satisfied since $\bigoplus_{\phi \in \mathscr{F}} \mathscr{B}_\phi$ is a Jordan C*-algebra.

3. IDEALS AND QUOTIENTS OF JORDAN C*-ALGEBRAS

Let $\mathscr{A}$ be a Jordan C*-algebra with self-adjoint part A. A Jordan ideal $\mathscr{J}$ of $\mathscr{A}$ is said to be a *-ideal if, whenever $z \in \mathscr{J}$ then $z^* \in \mathscr{J}$. Let J be the self-adjoint part of a norm-closed *-ideal $\mathscr{J}$ of $\mathscr{A}$, then $\mathscr{J} = J + iJ$ and J is a norm-closed ideal of A.

Since a quotient of a special Jordan algebra may be an exceptional algebra, the following lemma may have some mild independent interest.

LEMMA 3.1. *Let $\mathscr{A}$ be a* JC**-algebra and let $\mathscr{J}$ be a proper norm-closed *-ideal in $\mathscr{A}$. Then $\mathscr{A}/\mathscr{J}$ is a* JC**-algebra.*

Proof. Let $\tilde{\mathscr{A}}$ be a C*-algebra in which $\mathscr{A}$ is embedded and which is generated by $\mathscr{A}$; let $\tilde{\mathscr{A}}''$ be the second dual of $\tilde{\mathscr{A}}$ and let us identify $\tilde{\mathscr{A}}''$ with the von Neumann envelope of $\tilde{\mathscr{A}}$ on the universal representation space of $\tilde{\mathscr{A}}$.

Since J (the self-adjoint part of $\mathscr{J}$) is a norm-closed Jordan ideal of the JB-algebra A, there exists an upward directed net $\langle u_\lambda \rangle_{\lambda \in \Lambda}$ of elements of J with $0 \le u_\lambda \le 1$ and such that $\|(1 - u_\lambda)a(1 - u_\lambda)\| \to 0$ for each $a \in J$ [2; Lemma 9.1]. Hence, for each $b \in J$, $\|(1 - u_\lambda)b^2(1 - u_\lambda)\| \to 0$ and so $\|b(1 - u_\lambda)\| \to 0$. Let f be the strong limit in $\tilde{\mathscr{A}}''$ of the increasing net $\langle u_\lambda \rangle$. Thus $0 \le f \le 1$. Now, for any $b \in J$, $b(1 - u_\lambda) \to b(1 - f)$ (strongly). Thus $b(1 - f) = 0$ and, on taking adjoints, $bf = b = b^* = fb$.

Let $\bar{J}$ be the weak closure of J in $\tilde{\mathscr{A}}''$ and let $\bar{A}$ be the weak closure of A in $\tilde{\mathscr{A}}''$. Then $\bar{J}$ is a weakly closed Jordan ideal of the JW-algebra $\bar{A}$. Thus, see Topping [14; Propositions 3 and 5], there exists a central projection e in $\bar{A}$ such that $\bar{J} = \bar{A}e$.

Let x be any element of $\bar{J}$, then there exists a net $\langle x_\gamma \rangle$ in J which converges strongly to x. Thus $xf = x = fx$. In particular, $f^2 = f$, that is, f is a projection. Also, putting $e = x$, $ef = e = fe$. But, since $\bar{J} = \bar{A}e$ and f is in $\bar{J}$, $f = fe$. Thus $f = e$.

Let $\mathscr{M} = \{a \in \tilde{\mathscr{A}}: a(1 - e) = 0\}$. Since e is central, $\mathscr{M}$ is a norm-closed two-sided ideal of the C*-algebra $\tilde{\mathscr{A}}$. Clearly $\mathscr{J} \subset \mathscr{M}$.

Let $v_\lambda = 1 - u_\lambda$. Then, on identifying the self-adjoint part of $\tilde{\mathscr{A}}$ with the space of real valued, affine, continuous functions on the state space X of $\tilde{\mathscr{A}}$, the net $\langle v_\lambda \rangle$ decreases pointwise to $1 - e$. Thus, for any $z \in \mathscr{M}$, $\langle z^* v_\lambda z \rangle$ decreases pointwise to $z^*(1 - e)z = 0$. Hence, by Dini's theorem $\|z^* v_\lambda z\| \to 0$. So $\|v_\lambda^{1/2} z\| \to 0$. But $\|v_\lambda z\| \le \|v_\lambda^{1/2}\| \, \|v_\lambda^{1/2} z\| \le \|v_\lambda^{1/2} z\| \to 0$. On replacing z by z^* and taking adjoints we see that $\|z v_\lambda\| \to 0$.

From [4; Proposition 1.8.2] we have that, for each $z \in \mathscr{A}$,

$$\inf_{a \in \mathscr{M}} \|z + a\| = \lim_{\lambda} \|z - u_\lambda z\| = \lim_{\lambda} \|z - zu_\lambda\| .$$

So, for any $z \in \mathscr{J}$,

$$\inf_{a \in \mathscr{J}} \|z + a\| \leq \underline{\lim} \|z - u_\lambda \circ z\| \leq \overline{\lim} \|z - u_\lambda \circ z\|$$

$$\leq \frac{1}{2} \lim (\|z - zu_\lambda\| + \|z - u_\lambda z\|) = \inf_{a \in \mathscr{M}} \|z + a\| .$$

Thus $\inf_{a \in \mathscr{J}} \|z + a\| = \lim \|z - u_\lambda \circ z\| = \inf_{a \in \mathscr{M}} \|z + a\|$.

It follows that $\mathscr{J} = \mathscr{A} \cap \mathscr{M}$ and that the natural map of $\mathscr{A}/\mathscr{J}$ into $\tilde{\mathscr{A}}/\mathscr{M}$ is an isometry. Hence $\mathscr{A}/\mathscr{J}$ is a JC*-algebra.

THEOREM 3.2. *Let* $\mathscr{A}$ *be a Jordan* C*-*algebra; let* $\mathscr{J}$ *be a closed* *-*ideal. Then* $\mathscr{A}/\mathscr{J}$, *when equipped with the quotient norm, is a Jordan* C*-*algebra. Furthermore, if* J *is the self-adjoint part of* $\mathscr{J}$, *then the self-adjoint part of* $\mathscr{A}/\mathscr{J}$ *is isometrically isomorphic to* A/J.

Proof. Trivially, $\mathscr{A}/\mathscr{J}$ is a Jordan *-algebra. For $x \in A$,

$$\|x + \mathscr{J}\| = \inf\{\|x + a + ib\| : a, b \text{ in } J\} \geq \inf\{\|x + a\| : a \in J\},$$

(see Lemma 1.2). But $\|x + J\| = \inf\{\|x + a\| : a \in J\} \geq \|x + \mathscr{J}\|$.

Thus the self-adjoint part of $\mathscr{A}/\mathscr{J}$ is isometrically isomorphic to A/J and hence is, see [2; Lemma 9.2], a JB-algebra. Thus, by Theorem 2.8, there exists a unique Jordan C*-norm ρ on $\mathscr{A}/\mathscr{J}$.

Let $H: \mathscr{A} \to \mathscr{A}/\mathscr{J}$ be the canonical quotient homomorphism. For any $z \in \mathscr{A}$, $Hz = H(z + a)$, for any $a \in \mathscr{J}$. Thus $\rho(Hz) = \rho(H(z + a)) \leq \|z + a\|$, by Corollary 1.5. Thus $\rho(Hz) \leq \inf_{a \in \mathscr{J}} \|z + a\| = \|Hz\|$.

Fix any $z \in \mathscr{A}$. Then, by Corollary 2.2, Jord(z) is a JC*-algebra. Hence $\mathscr{E} = \text{Jord}(z)/(\mathscr{J} \cap \text{Jord}(z))$ is a JC*-algebra, with respect to the quotient norm. Let us consider the natural map $h : \mathscr{E} \to \mathscr{A}/\mathscr{J}$ given by

$$h(w + \mathscr{J} \cap \text{Jord}(z)) = w + \mathscr{J} = Hz .$$

Clearly, h is a *-isomorphism of $\mathscr{E}$ into $\mathscr{A}/\mathscr{J}$ and so, by Corollary 1.4, $\rho(Hz) = \|z + \mathscr{J} \cap \text{Jord}(z)\| \geq \inf_{a \in \mathscr{J}} \|z + a\| = \|Hz\|$. Hence $\rho(Hz) = \|Hz\|$, as required.

A Jordan C*-algebra $\mathscr{B}$ is a *factor* if its self-adjoint part is a JB-factor [2]. A factorial representation of a Jordan C*-algebra $\mathscr{A}$ is a *-homomorphism $h: \mathscr{A} \to \mathscr{B}$, where $\mathscr{B}$ is a factor Jordan C*-algebra. It follows from Theorem 2.8 and [2; Corollary 5.7] that each Jordan C*-algebra has a faithful family of factor representations.

Theorem 3.2 and Corollary 2.7 enable us to obtain an analogue, for Jordan C*-algebras, of the main theorem of [2].

THEOREM 3.3. *Let $\mathcal{A}$ be any Jordan* C*-*algebra. Then there exists a unique norm-closed *-ideal $\mathcal{J}$ such that $\mathcal{A}/\mathcal{J}$ is a* JC*-*algebra and every factorial representation of $\mathcal{A}$ which does not annihilate $\mathcal{J}$ is onto $\mathcal{M}_3^8$.*

By [2; Theorem 9.5], there exists a unique closed JB-ideal J in A such that A/J is the self-adjoint part of a JC*-algebra and each factorial JB-representation of A which does not annihilate J is onto M_3^8. Let $\mathcal{J} = J + iJ$ and then apply Theorem 3.2 and Corollary 2.7.

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Department of Mathematics
The University
Reading, England