# GENERALIZED HOMOLOGY THEORIES ON COMPACT METRIC SPACES

Daniel S. Kahn, Jerome Kaminker, and Claude Schochet

## 1. INTRODUCTION

This paper is devoted to developing useful and tractable homology theories on the category $\mathscr{CM}$ of based compact metrizable spaces, and doing this, moreover, within the context of classical algebraic topology. In the introduction we explain why this is desirable and we then state our main results. Let $\mathscr{A}$ be the category of abelian groups.

DEFINITION 1.1 [28]. A *Steenrod homology theory* $h_*$ on $\mathscr{CM}$ is a sequence of covariant, homotopy-invariant functors $h_n$: $\mathscr{CM} \to \mathscr{A}$ such that the following axioms hold for all n and for all X in $\mathscr{CM}$:

*Exactness.* If A is a closed subset of X then the sequence

$$h_n(A) \to h_n(X) \to h_n(X/A)$$

is exact.

*Suspension.* There is a natural equivalence $h_n(X) \xrightarrow{\sigma} h_{n+1}(SX)$.

*Strong Wedge.* Suppose $X_j$ is in $\mathscr{CM}$, j = 1, 2, $\cdots$, . Then the natural map

$$h_n(\varprojlim_k(X_1 \vee \cdots \vee X_k)) \to \prod_j h_n(X_j)$$

is an isomorphism.

Classical (ordinary) Steenrod homology theory is denoted ${}^sH_*$. It was invented by Steenrod [40] and axiomatized by Milnor [34]. The theory ${}^sH_*$ is very well-behaved on $\mathscr{CM}$. Steenrod showed that it is related to Čech homology by the sequence

$$(1.2) \qquad 0 \to \varprojlim{}^1 H_{n+1}(X_j) \to {}^sH_n(X) \to \check{H}_n(X) \to 0 ,$$

where $X = \varprojlim X_j$ and the $X_j$ are finite complexes. This is a special case of the $\lim^1$ sequence of Milnor [34]

$$(1.3) \qquad 0 \to \varprojlim{}^1 h_{n+1}(X_j) \to h_n(X) \to \varprojlim h_n(X_j) \to 0 ,$$

which holds for any Steenrod homology theory.

---

Received March 2, 1976. Revision received January 10, 1977.
The research of the first and third authors was partially supported by NSF grants, and that of the second author by an IUPUI faculty fellowship.

Michigan Math. J. 24 (1977).

In [28] we showed that many other general properties of $h_*$ follow from the axioms. For example, if X is in $\mathcal{FCM}$ (finite-dimensional compact metrizable spaces), then there is a spectral sequence of Atiyah-Hirzebruch type converging to $h_*(X)$, with $E^2_{p,q} = {}^sH_p(X; h_q(S^0))$.

In [16, 18], L. G. Brown, R. G. Douglas, and P. Fillmore (abbreviated BDF) established the existence of a Steenrod homology theory $\mathscr{E}_*$ on $\mathscr{CM}$. Let $\mathfrak{A}$ be the Calkin algebra, the quotient of the C*-algebra of bounded operators on a separable complex Hilbert space by the compact operators. An *extension* is a unital C*-algebra injection $\tau: C(X) \to \mathfrak{A}$, where C(X) is the C*-algebra of continuous, complex-valued functions on $X \in \mathscr{CM}$. The set of unitary equivalence classes of extensions is denoted $\mathscr{E}xt(X)$.

BDF show that $\mathscr{E}xt(X)$ is a covariant, homotopy-invariant functor to $\mathscr{A}$ and that $\mathscr{E}xt(S^2X)$ is naturally equivalent to $\mathscr{E}xt(X)$. Define

$$\mathscr{E}_n(X) = \begin{cases} \mathscr{E}xt(X) & \text{if } n \text{ is odd,} \\ \mathscr{E}xt(SX) & \text{if } n \text{ is even.} \end{cases}$$

Then $\mathscr{E}_*$ is a Steenrod homology theory on $\mathscr{CM}$.

The homology theory $\mathscr{E}_*$ resembles homology K-theory $K_*$. There is a natural homomorphism $\gamma_\infty$: $\mathscr{E}_1(X) \to \hom(K^{-1}(X), Z)$. Here is its definition. Let $\tau: C(X) \to \mathfrak{A}$ and let $v: X \to \mathscr{U}(N)$ represent an element of $K^{-1}(X)$. Then $\gamma_\infty(\tau)(v) = \operatorname{index}((\tau \otimes 1_N)v)$. The map $\gamma_\infty$ is an isomorphism on spheres. More generally, L. G. Brown proved the following Universal Coefficient Theorem: for any X in $\mathscr{CM}$ there is a natural split short exact sequence

$$(1.4) \qquad 0 \longrightarrow \operatorname{Ext}(K^{n+1}(X), Z) \longrightarrow \mathscr{E}_n(X) \xrightarrow{\gamma_\infty} \hom(K^n(X), Z) \longrightarrow 0 .$$

Brown's proof uses fairly sophisticated algebraic K-theory. Of course, (1.4) resembles the Universal Coefficient Theorem for $K^*$ and $K_*$ of D. W. Anderson; in particular its splitting implies that the groups $K_n(X)$ and $\mathscr{E}_n(X)$ are abstractly isomorphic for all n and all X in $\mathscr{W}$, the category of finite CW-complexes.

BDF announced [16] that there is a natural isomorphism $\mathscr{E}_0(X) \cong K_0(X)$ on $\mathscr{W}$. But $\mathscr{E}_*$ arises naturally on $\mathscr{CM}$, not on $\mathscr{W}$. In fact, from an analysis point of view, "compact metrizable" is a much more reasonable restriction than "finite CW-complex."

The question was, then, is there a Steenrod homology theory ${}^sE_*$ corresponding to any cohomology theory $E^*$ which is constructed in a concrete way, which extends $E_*$ on $\mathscr{W}$, and which in the case of K-theory corresponds to $\mathscr{E}_*$? The answer provided here is yes.

THEOREM A. *Let* $E^*$ *be a cohomology theory given by the spectrum* $\underline{E}$. *Then there is a Steenrod homology theory* ${}^sE_*$ *on* $\mathscr{CM}$ *which is naturally equivalent to* $E_*$ *on* $\mathscr{W}$.

We work in the Boardman-Vogt category of spectra as expounded by Adams [2, 42].

The theory ${}^sE_*$ is defined by setting ${}^sE_k(X) = E^{-k}(\underline{F}(X))$, where $\underline{F}(X)$ is a CW-approximation to the function spectrum given by the function spaces $F(X, S^n)$. Thus Theorem A is a direct generalization of the Spanier-Whitehead duality constructions of [38, 39].

The theories ${}^sE_*$ are particularly well-behaved on $\mathcal{FCM}$. For example, the Steenrod duality theorem holds in this generality. (This was proved by Steenrod for ${}^sH_*$.)

THEOREM B. *Let* X *be a closed subset of* $S^{n+1}$. *Then*

$$ {}^sE_k(X) \cong E^{n-k}(S^{n+1} \setminus X), $$

*and the isomorphism is natural for inclusions* $X \subset Y \subset S^{n+1}$.

Finally, we relate our general construction to the BDF theory.

THEOREM C. *There is a natural equivalence of Steenrod homology theories* $\Gamma_*$: $\mathcal{E}_* \cong {}^sK_*$ *on* $\mathcal{FCM}$.

Theorem C yields the assertions of BDF and generalizes their theorem [18, 7.7] from $\mathcal{W}$ to $\mathcal{FCM}$. As corollaries we obtain easy proofs of two theorems of L. G. Brown: a Universal Coefficient Theorem (1.4) (in a somewhat more general form (7.4) but only on $\mathcal{FCM}$), and the strong homotopy property for $\mathcal{E}_*$ (7.5) (again only on $\mathcal{FCM}$). Theorem 7.5 implies that the BDF theory is equivalent to an analogous theory announced by Kasparov [30] on $\mathcal{FCM}$.

The map $\Gamma_*$ is constructed via a slant pairing introduced by Atiyah and BDF which generalizes $\gamma_\infty$. Alternately, it may be described in a more analytic fashion using Clifford algebras, as observed by M. F. Atiyah and G. B. Segal [10].

Edwards and Hastings [23] have defined Steenrod homology theories using pro-categories and have verified Theorems A and B for their theories.

The paper is organized as follows. Section 2 deals with function spectra; the necessary technical information is accumulated to formulate the definition of ${}^sE_*$. In Section 3 the theory ${}^sE_*$ is defined and Theorem A is proved. Section 4 is devoted to Steenrod duality in general and to Theorem B in particular. In Section 5 the natural transformation from $\mathcal{E}_*$ to ${}^sK_*$ is defined—in Section 6 it is shown to be a natural equivalence. Section 7 contains the proofs of L. G. Brown's two theorems and assorted comments.

We wish to thank M. G. Barratt, L. G. Brown, R. G. Douglas, P. Fillmore, and I. M. Singer for their encouragement and for illuminating comments. Special thanks are due to Graeme Segal, who suggested using function spectra. We particularly wish to acknowledge the pervasive influence of M. F. Atiyah, who via discussions and his papers (particularly [8]) strongly affected our work. We are most grateful.

## 2. FUNCTION SPECTRA

In this paper all spaces are based, functions preserve basepoints, homotopies respect basepoints, and homology and cohomology theories are reduced. Spectra are denoted $\underline{A}$, $\underline{B}$, $\cdots$. If X is a space, then its suspension spectrum is denoted $\underline{S}(X)$. A *function* $\alpha\colon \underline{A} \to \underline{B}$ is a sequence of continuous functions $\alpha_n\colon A_n \to B_n$ respecting the structure maps. A CW *spectrum* is a spectrum $\underline{A}$ where each $A_n$ is a CW complex and $SA_n \to A_{n+1}$ identifies the complex $SA_n = A_n \wedge S^1$ with a subcomplex of $A_{n+1}$. A *map* $\underline{A} \to \underline{B}$ is an equivalence class of functions $\underline{A}' \to \underline{B}$, where $\underline{A}'$ is a cofinal CW-subspectrum of the CW-spectrum $\underline{A}$. Homotopy classes of maps are called *morphisms* $\underline{A} \to \underline{B}$. The set of morphisms from $\underline{A}$ to $\underline{B}$ will be denoted by $[\underline{A}, \underline{B}]$. A function $\alpha\colon \underline{A} \to \underline{B}$ is a *weak equivalence* if $\alpha_*\colon \pi_*(\underline{A}) \to \pi_*(\underline{B})$ is an isomorphism. In particular, this is true if each $(\alpha_n)_*\colon \pi_i(A_n) \to \pi_i(B_n)$ is an isomorphism. An *equivalence* $\underline{A} \to \underline{B}$ is an invertible morphism. (See [42] for details.)

Let X be in $\mathscr{CM}$. The *function spectrum* $F(X, \underline{S})$ for X is defined as follows. Let $F(X, S^n)$ be the space of based maps $X \to S^n$ with the compact-open topology. Let $\lambda_n$: $SF(X, S^n) \to F(X, S^{n+1})$ be defined by $\lambda_n(f \wedge t)(x) = f(x) \wedge t$. Then $F(X, \underline{S}) = \{F(X, S^n); \lambda_n\}$. This is not, in general, a CW-spectrum, and so $[F(X, \underline{S}), \underline{A}]$ is not defined. This essentially technical difficulty is dealt with in the remainder of this section.

Given a spectrum $\underline{A}$, there is a CW-spectrum $\underline{A}'$ weakly equivalent to it. The fact that any space is weakly equivalent to a CW-complex has been well known for 25 years (*cf.* [25]). The situation for spectra is essentially the same—we follow [22].

DEFINITION 2.1. A CW-*substitute* for a spectrum $\underline{A}$ is a CW-spectrum $\underline{A}'$ and a function $\alpha$: $\underline{A}' \to \underline{A}$ which is a weak equivalence.

We now show that CW-substitutes exist, that they are essentially unique, and that they are natural in an appropriate sense.

LEMMA 2.2. *Let* X *be a* CW-*complex,* Y *a space, and* $f: X \to Y$. *Then there exist a* CW-*complex* Y' *containing* X *as a subcomplex and a map* $f': Y' \to Y$ *extending* f *such that* $f'_*: \pi_i(Y') \to \pi_i(Y)$ *is an isomorphism for* $i \geq 1$.

*Proof.* See [22, p. 143].

PROPOSITION 2.3. *Every spectrum* $\underline{A}$ *has a* CW-*substitute.*

*Proof.* Apply (2.2) with $X = \{x_0\}$, $Y = A_0$ to obtain $\alpha_0 : A_0' \to A_0$. Inductively suppose that $\alpha_j$: $A_j' \to A_j$ exist for $j < n$ with $\alpha_{j_*} : \pi_i(A_j', a_j') \to \pi_i(A_j, a_j)$ an isomorphism for $j < n$ and $i \geq 1$ and suppose given commutative diagrams for $j < n$

$$\begin{array}{ccc} SA'_{j-1} & \xrightarrow{S\alpha_{j-1}} & SA_{j-1} \\ \varepsilon'_j \downarrow & & \downarrow \varepsilon_j \\ A'_j & \xrightarrow{\alpha_j} & A_j \end{array}$$

with $\varepsilon'_j$ an inclusion of a subcomplex. Apply (2.2) with $X = SA'_{n-1}$, $Y = A_n$ and $X \to Y$ the map

$$SA'_{n-1} \xrightarrow{S\alpha_{n-1}} SA_{n-1} \xrightarrow{\varepsilon_{n-1}} A_n$$

to obtain $A'_n$, $\alpha_n$: $A'_n \to A_n$, and $\varepsilon'_{n-1}$: $SA'_{n-1} \to A'_n$. This yields a CW-spectrum $\underline{A}' = \{A'_n, \varepsilon'_n\}$, and a weak equivalence $\alpha$: $\underline{A}' \to \underline{A}$ as desired.

PROPOSITION 2.4. *Suppose* $\alpha$: $\underline{A}' \to \underline{A}$ *and* $\bar{\alpha}$: $\underline{A}'' \to \underline{A}$ *are* CW-*substitutes. Then there is a unique morphism* $[f]$: $\underline{A}' \to \underline{A}''$ *such that* $[\alpha] = [\bar{\alpha}f]$, *and* $[f]$ *is an equivalence.*

*Proof.* The induced map $\bar{\alpha}_*$: $[\underline{A}', \underline{A}''] \to [\underline{A}', \underline{A}]$ is an isomorphism, by [2, Theorem 3.4], hence $[\alpha] = [\bar{\alpha}f]$ uniquely. By symmetry, f is an equivalence.

PROPOSITION 2.5. *Suppose* $\alpha$: $\underline{A}' \to \underline{A}$ *and* $\beta$: $\underline{B}' \to \underline{B}$ *are* CW-*substitutes and* f: $\underline{A} \to \underline{B}$ *is a function. Then there is a unique morphism* $[f']$: $\underline{A}' \to \underline{B}'$ *such that* $[f\alpha] = [\beta f']$.

*Proof.* Use [2, Theorem 3.4] again.

*Remark.* It is possible to construct CW-substitutes functorially. Given a spectrum $\underline{A}$, canonically associate a spectrum $\underline{A}'$ to it (via a telescope construction) such that $SA'_n \subset A'_{n+1}$ and a weak equivalence $\underline{A}' \to \underline{A}$ (*cf.* [42], Prop. 8.3). Let $K_n$ be the realization of the singular complex of $A'_n$; then the composite $\underline{K} \to \underline{A}' \to \underline{A}$ is a functorial substitute. However, (2.4) and (2.5) suffice for our purposes.

Note that in the future we confuse maps with morphisms by deleting brackets when this causes no harm.

The next order of business is the determination of the relationship of function spectra to wedges. Let $X_j$ be a sequence of spaces. Fix the following notation;

$$\overline{\bigvee_j} X_j = \varinjlim_k \left( \bigvee_1^k X_j \right), \quad \text{the weak wedge;}$$

$$\overline{\prod_j} X_j = \varinjlim_k \left( \prod_1^k X_j \right), \quad \text{the weak product;}$$

$$\bigvee_j X_j = \varprojlim_k \left( \bigvee_1^k X_j \right), \quad \text{the strong wedge;}$$

$$\prod_j X_j = \varprojlim_k \left( \prod_1^k X_j \right), \quad \text{the strong product;}$$

and similarly for spectra. Note that $\overline{\prod}_j X_j$ is the natural Cartesian product (for spaces) and $\overline{\bigvee}_j X_j$ is the wedge used in the CW-category (for spaces and for spectra).

PROPOSITION 2.6. *Let* $\underline{A}^{(j)}$, $j = 1, 2, \cdots$, *be a sequence of spectra, and suppose that each* $A_n^{(j)}$ *is of the homotopy type of a* CW-*complex. Then the natural function* $\iota : \overline{\bigvee}_j \underline{A}^{(j)} \to \overline{\prod}_j \underline{A}^{(j)}$ *is a weak equivalence.*

*Proof.* Suppose first that each $\underline{A}^{(j)}$ is a CW-spectrum. Then the natural function $\iota : \bigvee_1^k \underline{A}^{(j)} \to \prod_1^k \underline{A}^{(j)}$ induces an equivalence, by [2, Prop. 3.14]. Since direct limits commute with $\pi_*$, the map $\iota_* : \pi_*\left(\overline{\bigvee}_j \underline{A}^{(j)}\right) \to \pi_*\left(\overline{\prod}_j \underline{A}^{(j)}\right)$ is an isomorphism. In the general case, choose CW-substitutes $\gamma_j : \underline{C}^{(j)} \to \underline{A}^{(j)}$ such that each $\gamma_j : C_n^{(j)} \to A_n^{(j)}$ is a weak equivalence. Then each $\gamma_j$ is an equivalence, since $A_n^{(j)}$ is of the homotopy type of a CW-complex. Thus $\overline{\bigvee}_j \underline{C}^{(j)} \to \overline{\bigvee}_j \underline{A}^{(j)}$ is a weak equivalence. Furthermore, $\overline{\prod}_j \underline{C}^{(j)} \to \overline{\prod}_j \underline{A}^{(j)}$ is a weak equivalence, and so is $\iota'$, by the first part of the proof. Since the diagram

$$\begin{array}{ccc} \overline{\bigvee_j} \underline{C}^{(j)} & \xrightarrow{\iota'} & \overline{\prod_j} \underline{C}^{(j)} \\ \downarrow & & \downarrow \\ \overline{\bigvee_j} \underline{A}^{(j)} & \xrightarrow{\iota} & \overline{\prod_j} \underline{A}^{(j)} \end{array}$$

commutes, $\iota$ is a weak equivalence.

PROPOSITION 2.7. *Let* $X = \varprojlim X_i$, *where each* $X_i$ *is in* $\mathscr{C}\mathscr{M}$. *Then the natural map* $\varinjlim_i \pi_j(F(X_i, S^n)) \to \pi_j(F(X, S^n))$ *is a bijection for all* $j \geq 0$.

*Proof.* For $j = 0$, this is the continuity property for cohomotopy [42, p. 327]. In general, note that $X \wedge S^n \cong \varprojlim (X_i \wedge S^n)$ and that there is a natural isomorphism $\pi_j(F(Y, S^n)) \cong \pi_0(F(Y \wedge S^j, S^n))$. The lemma follows.

We now prove the technical result which will eventually yield the wedge axiom for ${}^sE_*$. Recall from [33] that if $Y$ is a compact metric space, then $F(Y, S^n)$ is of the homotopy type of a countable CW complex. Let $\underline{F}(X)$ be a CW-substitute for the function spectrum $F(X, \underline{S})$.

PROPOSITION 2.8. *Suppose* $\{X_j\}$ *is a sequence in* $\mathscr{C}\mathscr{M}$. *The natural function* $\rho\colon \overline{\bigvee}_j \underline{F}(X_j) \to \underline{F}\left(\bigvee_j X_j\right)$ *is an equivalence. Thus* $\underline{F}$ *converts strong wedges into weak wedges.*

*Proof.* The function $\rho$ is the composite of $\iota\colon \overline{\bigvee}_j \underline{F}(X_j) \to \overline{\prod}_j \underline{F}(X_j)$ (which is a weak equivalence by (2.6)) and

$$\overline{\prod_j} \underline{F}(X_j) \equiv \varinjlim_k \underline{F}\left(\bigvee_1^k X_j\right) \to \underline{F}\left(\bigvee_j X_j\right),$$

which is a weak equivalence by the spectrum version of (2.7). Weak equivalences of CW-spectra are equivalences [2, p. 150], completing the argument.

## 3. THE BASIC CONSTRUCTION

Suppose that $E^*$ is a cohomology theory represented by a spectrum $\underline{E}$. Then setting $E_n(X) = [\underline{S}, \underline{E} \wedge X]_n = \pi_n(\underline{E} \wedge X)$ yields a homology theory $E_*$ on $\mathscr{H}$, finite complexes. In this section we define a Steenrod homology theory ${}^sE_*$ on $\mathscr{C}\mathscr{M}$ which extends $E_*$. Note that if $h_*$ is a homology theory on $\mathscr{H}$, then $h_n(X) = \pi_n(\underline{E} \wedge X)$ for some spectrum $\underline{E}$, by [13,43,1], and so $h_*$ extends to some Steenrod homology theory ${}^sh_*$ on $\mathscr{C}\mathscr{M}$.

Recall from (1.1) that a Steenrod homology theory $h_*$ is a homology theory on $\mathscr{C}\mathscr{M}$ satisfying exactness (for any closed $A \subset X$), suspension, and wedge—which now takes the form

(3.1)
$$h_n\left(\bigvee_j X_j\right) \xrightarrow{\cong} \prod_j h_n(X_j)\ .$$

*Remark.* To obtain unreduced theories on compact pairs, set

$$h'_n(X, A) = h_n(X/A) .$$

Then $h'_*$ satisfies the Eilenberg-Steenrod axioms (except dimension) as well as strong excision: $h'_n(X, A) = h'_n(X/A), \text{pt})$ for all compact pairs. The Čech extension of a theory on $\mathscr{W}$ usually satisfies strong excision but not exactness. The singular extension satisfies exactness but not strong excision. Steenrod homology theories satisfy both!

If $h_*$ is a Steenrod homology theory with $h_j(S^0) = 0$ for $j \neq 0$, then

$$h_j(X) \cong {}^sH_j(X; h_0(S^0)),$$

by Milnor's uniqueness theorem [34]. The other basic example of a Steenrod homology theory is the Brown-Douglas-Fillmore theory $\mathscr{E}_*$ discussed in Section 1. For general information on Steenrod homology theories, the reader is referred to [34, 28].

Let $\underline{E}$ be a spectrum with associated cohomology theory $E^*$ on the Boardman-Vogt category of CW-spectra and on CW-complexes in the usual fashion. Define ${}^sE_k: \mathscr{C}\mathscr{M} \to \mathscr{A}$ by

$$(3.2) \qquad {}^sE_k(X) = E^{-k}(\underline{F}(X)),$$

where we recall that $\underline{F}(X)$ denotes a CW-substitute for the function spectrum of X. Propositions (2.4) and (2.5) imply that each ${}^sE_k$ is a covariant functor.

THEOREM A. *Let $\underline{E}$ be a spectrum. Then ${}^sE_*$ is a Steenrod homology theory on $\mathscr{C}\mathscr{M}$ and there is a natural equivalence ${}^sE_*(X) \cong \pi_*(\underline{E} \wedge X) \equiv E_*(X)$ on $\mathscr{W}$.*

*Proof.* The functors ${}^sE_n$ are homotopy-invariant by (2.5). The wedge axiom is verified as follows. If $X = \bigvee_j X_j$, then there is an equivalence

$$\rho: \overline{\bigvee_j} \underline{F}(X_j) \to \underline{F}(X),$$

by (2.8). Then, since $\rho^*$ is an isomorphism and $E^*$ is additive,

$${}^sE_n(X) = E^{-n}(\underline{F}X) \cong E^{-n}\left(\overline{\bigvee_j} \underline{F}(X_j)\right) \cong \prod_j E^{-n}(\underline{F}(X_j)) = \prod_j {}^sE_n(X_j).$$

It remains to define the suspension transformations and to verify the suspension and exactness axioms.

PROPOSITION 3.3. *${}^sE_*$ satisfies the suspension axiom.*

*Proof.* Let $\mu_n: F(X, S^n) \to F(SX, S^{n+1})$ be defined by $\mu_n(f)(x \wedge t) = f(x) \wedge t$. For a loopspace $\Omega Y$, let $\chi: \Omega Y \to \Omega Y$ be the map which reverses the direction of each loop. Let $\zeta_n: F(SX, S^n) \to \Omega F(X, S^n)$ be the canonical homeomorphism defined by $\zeta_n(f)(t)(x) = f(x \wedge t)$.

LEMMA 3.4. *Let $\dot{\mu}_n = \zeta_{n+1}^{-1} \circ \chi^n \circ \zeta_{n+1} \circ \mu_n$. Then the diagrams*

$$\begin{array}{ccc} SF(X, S^n) & \xrightarrow{S\dot{\mu}_n} & SF(SX, S^{n+1}) \\ \downarrow{\lambda_n(X)} & & \downarrow{\lambda_{n+1}(SX)} \\ F(X, S^{n+1}) & \xrightarrow{\dot{\mu}_{n+1}} & F(SX, S^{n+2}) \end{array}$$

*homotopy commute, and the homotopies may be chosen to be natural with respect to maps* $X \to Y$.

*Proof.* By explicit computation, one verifies that

$$\lambda_{n+1}(SX) \circ S\dot{\mu}_n(f \wedge t)(x \wedge s) = \begin{cases} f(x) \wedge (1 - s) \wedge t, & n \text{ odd}; \\ f(x) \wedge s \wedge t, & n \text{ even}; \end{cases}$$

and

$$\dot{\mu}_{n+1} \circ \lambda_n(X)(f \wedge t)(x \wedge s) = \begin{cases} f(x) \wedge t \wedge s, & n \text{ odd}; \\ f(x) \wedge t \wedge (1 - s), & n \text{ even}. \end{cases}$$

Natural homotopies are then obtained by choosing once and for all the homotopies $\chi \wedge 1 \simeq T$ and $1 \simeq T \circ (\chi \wedge 1)$, where $T\colon S^2 = S^1 \wedge S^1 \to S^1 \wedge S^1$ is the map which switches factors.

Fix the homotopies of (3.4). The maps $\dot{\mu}_n\colon F(X, S^n) \to F(SX, S^{n+1})$ do not induce a function on spectra, since diagrams there must strictly commute. This is a technical problem which may be solved by the use of the telescope spectrum $\mathscr{T}el(\underline{F}(X))$ (*cf.* [42]). (Note that if the homotopies are functorial, then so is the telescope construction.) Let $\tau\colon \mathscr{T}el(\underline{F}(X)) \to \underline{F}(X)$ be the canonical weak equivalence. The homotopies of (3.4) fit together to define a function

$$\nu\colon \mathscr{T}el(\underline{F}(X)) \to \underline{F}(SX)$$

of degree -1. Then by [42, p. 252], there is a commutative diagram

$$\begin{array}{ccccc} \pi_{-k}(\underline{F}(X)) & \xrightarrow{\tau_*^{-1}} & \pi_{-k}(\mathscr{T}el(\underline{F}(X))) & \xrightarrow{\nu_*} & \pi_{-k-1}(\underline{F}(SX)) \\ \uparrow\cong & & & & \uparrow\cong \\ H^k(X; \underline{S}) & & \xrightarrow{\sigma} & & H^{k+1}(SX, \underline{S}) \end{array},$$

where $\underline{S}$ is the sphere spectrum and $\sigma$ is the suspension isomorphism of the theory $H^*(-; \underline{S})$. Thus $\nu_*$ is an isomorphism, so $\nu$ is a "weak equivalence" of degree -1.

The weak equivalences $\underline{F}(X) \xleftarrow{\tau} \mathscr{T}el(\underline{FX}) \xrightarrow{\nu} \underline{F}(SX)$ then induce the suspension isomorphism, for (with $T \to \mathscr{T}el(X)$ a CW-substitute) one has

$$\begin{aligned} {}^sE_n(X) &= E^{-n}(\underline{F}(X)) \cong E^{-n}(T) && (\text{via } \tau) \\ &\cong E^{-n-1}(\underline{F}(SX)) && (\text{via } \nu) \\ &= {}^sE_{n+1}(SX), \end{aligned}$$

yielding a natural equivalence $\sigma_n: {}^sE_n \to {}^sE_{n+1} \circ S$, and completing the proof of (3.3).

PROPOSITION 3.5. ${}^sE_*$ *satisfies the exactness axiom.*

*Proof.* Let A be a closed subset of X in $\mathscr{C}\mathscr{M}$. By [24], the sequence $F_0(X/A, S^n) \to F_0(X, S^n) \to F_0(A, S^n)$ is a fibration (where $F_0(Y, S^n)$ denotes the path component of the constant map). Thus the sequence

$$\underline{F}(X/A) \to \underline{F}(X) \to \underline{F}(A) \tag{3.6}$$

induces a long exact sequence in homotopy. As is well known, this implies that (3.6) is a cofibration in the Boardman-Vogt category. Since $E^*$ converts cofibrations to exact sequences, the sequence

$$\begin{array}{ccccc} E^{-n}(\underline{F}(A)) & \longrightarrow & E^{-n}(\underline{F}(X)) & \longrightarrow & E^{-n}(\underline{F}(X/A)) \\ \| & & \| & & \| \\ {}^sE_n(A) & \longrightarrow & {}^sE_n(X) & \longrightarrow & {}^sE_n(X/A) \end{array}$$

is exact. This proves (3.5).

We defer the proof that ${}^sE_*(X)$ is naturally equivalent to $\pi_*(\underline{E} \wedge X)$ on $\mathscr{W}$ until Section 4. It follows the proof of Theorem B.

## 4. STEENROD DUALITY

If X is a closed subset of $S^{n+1}$, then Steenrod [40] has shown

$${}^sH_k(X) \cong H^{n-k}(S^{n+1} \setminus X). \tag{4.1}$$

This section is devoted to generalizing the Steenrod duality theorem (4.1) to the Steenrod homology theories created by Theorem A. (In fact, our results imply (4.1) as well.) Our method is to use the fact that if X embeds in $S^{n+1}$, then

$$\Sigma^{-n}\ \underline{S}(S^{n+1} \setminus X)$$

is a CW-substitute for $F(X, \underline{S})$. (Here $\Sigma^k$ is the translation suspension functor on spectra; $\Sigma^{-n}$ means "formally desuspend n times".)

Note the difference between (4.1) and Alexander duality isomorphism

$$\check{H}^k(X) \cong H_{n-k}(S^{n+1} \setminus X),$$

where $H_*$ is singular homology.

The following theorem of J. C. Moore provides the starting place for our investigation.

THEOREM 4.2 [35]. *Let* X *be a compact metric space of dimension* $d < \infty$. *Then there is a natural homomorphism*

$$\mu_F/: H_q(F(X, S^n)) \to \check{H}^{n-q}(X)$$

*which is an isomorphism if* $q < 2(n - d)$.

The element $\mu_F \in \check{H}^n(F(X, S^n) \wedge X)$ is defined as follows. Let

$$e_F\colon F(X, S^n) \wedge X \to S^n$$

be the evaluation map. Then $\mu_F = e_F^* \iota_n$, where $\iota_n \in H^n S^n$ is the generator.

Now assume that X is a closed subset of $S^{n+1}$ which contains the south pole but misses the north pole. Define $e_D\colon (S^{n+1} \setminus X) \wedge X \to S^n$ to be the composite

$$\begin{array}{ccc} (S^{n+1} \setminus X) \wedge X & \xrightarrow{\text{incl.}} & (S^{n+1} \wedge S^{n+1}) \setminus D \cong S^{2(n+1)} \setminus D \\ & & \downarrow \\ & & S^n, \end{array}$$

where (north pole, south pole) is the basepoint of $S^{n+1} \wedge S^{n+1}$, $D \simeq S^{n+1}$ is the diagonal, and $S^{2(n+1)} \setminus D \to S^n$ is the canonical deformation retraction. Define $\mu_D = e_D^* \iota_n$. Let $\hat{e}_D\colon S^{n+1} \setminus X \to F(X, S^n)$ be adjoint to $e_D$. Then an easy check shows that

$$\mu_F / \hat{e}_{D*}(w) = ((e_D \wedge 1)^* \mu_F)/w = \mu_D / w,$$

so that the diagram

$$\begin{array}{ccc} H_q(S^{n+1} \setminus X) & \xrightarrow{\hat{e}_{D*}} & H_q(F(X, S^n)) \\ & {\scriptstyle \mu_D/} \searrow \quad \swarrow {\scriptstyle \mu_F/} & \\ & \check{H}^{n-q}(X) & \end{array} \tag{4.3}$$

commutes for all q.

LEMMA 4.4. $\hat{e}_{D*}\colon H_q(S^{n+1} \setminus X) \to H_q(F(X, S^n))$ *is an isomorphism for* $q < 2(n - d)$, $d = \dim(X)$.

*Proof.* The map $\mu_D/$ is essentially Alexander duality (*cf.* [42]), so (4.2) and (4.3) imply the lemma.

THEOREM 4.5. *Let* X *be a compact metric space of dimension* $d < \infty$, *and suppose that* X *embeds as a closed subset of* $S^{n+1}$. *Then* $\hat{e}_D$ *induces a morphism*

$$\hat{e}\colon \Sigma^{-n} \underline{S}(S^{n+1} \setminus X) \to F(X, \underline{S}) \tag{4.6}$$

*which is a* CW-*substitute.*

*Proof.* There are maps $f_{n+k}\colon S^k(S^{n+1} \setminus X) \to F(X, S^{n+k})$ defined via

$$S^k(S^{n+1} \setminus X) \simeq S^{n+k+1} \setminus X \xrightarrow{\hat{e}_D} F(X, S^{n+k}).$$

Since $\underline{S}(S^{n+1} \setminus X)$ is a CW-spectrum, these maps may be deformed to a morphism of spectra

$$\hat{e}_n : \Sigma^{-n} \underline{S}(S^{n+1} \setminus X) \to F(X, \underline{S}) .$$

The maps $f_{n+k}$ induce homology isomorphisms in the stable range, by (4.4), and hence homotopy isomorphisms. Thus $\hat{e}_n$ is a weak equivalence.

*Remark* 4.7. Propositions (2.4), (2.5) guarantee the naturality of $\hat{e}_n$. As a digression, however, we consider the question directly. If $\underline{A}$ is a CW-spectrum and $\underline{B}$ is an arbitrary spectrum, then a *Whitehead map* $f: \underline{A} \to \underline{B}$ is a sequence of maps $f_n: A_n \to B_n$ which respect the structure maps up to homotopy. A Whitehead map is (Whitehead) homotopic to a function $g: \underline{A} \to \underline{B}$, but in general $g$ may not be unique (see [42]). The function $g$ *is* unique when $\underset{\overleftarrow{k}}{\lim}{}^1 B^{k-1}(A_k) = 0$. In the case at hand, with $\underline{A} = \Sigma^{-n} \underline{S}(S^{n+1} \setminus X)$ and $\underline{B} = F(X, \underline{S})$, we have (for $k$ large),

$$\begin{aligned} B^{k-1}(A_k) &= [\underline{S}(S^{k-n}(S^{n+1} \setminus X)),\ F(X, \underline{S})]_{1-k} \\ &= [S^{k-n}(S^{n+1} \setminus X),\ F(X, S^{k-1})] = [S^{k+1} \setminus X,\ F(X, S^{k-1})] . \end{aligned}$$

The maps in the inverse sequence

$$[S^{k+1} \setminus X,\ F(X, S^{k-1})] \leftarrow [S^{k+2} \setminus X,\ F(X, S^{k})]$$

are isomorphisms for $k$ large, since the fiber of the map $SF(X, S^{k-1}) \to F(X, S^k)$ is $2k - \dim(X)$ connected. Hence $\underset{\overleftarrow{k}}{\lim}{}^1 B^{k-1}(A_k) = 0$.

The duality theorem now follows easily.

THEOREM B. *Let* $X$ *be a closed subset of* $S^{n+1}$. *Then*

$${}^sE_k(X) \cong E^{n-k}(S^{n+1} \setminus X) ,$$

*and the isomorphism is natural for inclusions* $X \subset Y \subset S^{n+1}$.

*Proof.* By definition and by (4.5),

$${}^sE_k(X) = E^{-k}(\underline{F}(X)) = E^{-k}\left(\Sigma^{-n} \underline{S}(S^{n+1} \setminus X)\right) = E^{n-k}(S^{n+1} \setminus X) .$$

To complete the proof of Theorem A, it remains to show that ${}^sE_*(X)$ is naturally equivalent to $\pi_*(\underline{E} \wedge X)$ on $\mathscr{W}$. Let $X \in \mathscr{W}$ and let $D_n(X)$ be a strong deformation retract of $S^{n+1} \setminus X$, with $n$ large. Then by Theorem B and by [43],

$${}^sE_k(X) \cong E^{n-k}(S^{n+1} \setminus X) \cong E^{n-k}(D_n(X)) \cong \pi_k(\underline{E} \wedge X) .$$

The isomorphisms are natural for $n$ large, and they yield the desired natural equivalence on $\mathscr{W}$.

We now indicate more precisely the relation between classical Alexander duality and Steenrod duality. Let $X \subset S^{n+1}$ be closed. Then there exist sequences of finite complexes $\{X_j\}$, $\{Y_j\}$ embedded in $S^{n+1}$ such that

(i) $X_j \supset X_{j+1} \supset \cdots \supset X$ and $X = \bigcap_j X_j$;

(ii) $Y_j \subset Y_{j+1}$ as a subcomplex and $S^{n+1} \setminus X = \bigcup_j Y_j$;

(iii) $X_j$ is a Spanier-Whitehead dual of $Y_j$ in $S^{n+1}$.

If X is a finite complex, then $S^{n+1} \setminus X$ is of the homotopy type of a finite complex; namely, $D_n X$, the Spanier-Whitehead dual, and ${}^sE_k(X) \cong E_k(X)$. In this event the Steenrod duality isomorphism of Theorem B (denoted by $\theta$) may be identified with Alexander duality (*cf.* [41])

$$\theta_j \colon E_k(X_j) \cong E^{n-k}(X_j), \tag{4.8}$$

and the following proposition is obtained.

PROPOSITION 4.9. *Let* X *be closed in* $S^{n+1}$ *and let* $\{X_j\}$, $\{Y_j\}$ *be chosen as above. Then there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim{}^1 E_{k+1}(X_j) & \longrightarrow & {}^sE_k(X) & \longrightarrow & \varprojlim E_k(X_j) & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle \varprojlim^1 \theta_j} & & \Big\downarrow {\scriptstyle \theta} & & \Big\downarrow {\scriptstyle \varprojlim \theta_j} & & \\ 0 & \longrightarrow & \varprojlim{}^1 E^{n-k-1}(Y_j) & \longrightarrow & E^{n-k}(S^{n+1} \setminus X) & \longrightarrow & \varprojlim E^{n-k}(Y_j) & \longrightarrow & 0 \end{array}$$

*relating Milnor's* $\lim^1$ *sequences, with vertical maps all isomorphisms.*

*Proof.* The upper row is the $\lim^1$ sequence (1.3) obtained from $X = \varprojlim X_j$. The lower row is the more familiar $\lim^1$ sequence obtained from the CW-complex $S^{n+1} \setminus X = \varinjlim Y_j$. The diagrams

$$\begin{array}{ccc} {}^sE_k(X) & \longrightarrow & E_k(Y_j) \\ \Big\downarrow {\scriptstyle \theta} & & \Big\downarrow {\scriptstyle \theta_j} \\ E^{n-k}(S^{n+1} \setminus X) & \longrightarrow & E^{n-k}(Y_j) \end{array}$$

commute for each j, and a look at the explicit constructions of the sequences completes the proof.

## 5. CONSTRUCTION OF $\Gamma_* \colon \mathscr{E}_* \to {}^sK_*$

This section and Section 6 are devoted to proving Theorem C.

THEOREM C. *There is a natural equivalence of Steenrod homology theories* $\Gamma_* \colon \mathscr{E}_* \to {}^sK_*$ *on* $\mathscr{F}\mathscr{C}\mathscr{M}$, *where* $\mathscr{E}_*$ *is the* BDF *theory and* ${}^sK_*$ *is the theory constructed by Theorem* A *corresponding to complex* K*-theory.*

In this section, $\Gamma_*$ is defined and shown to be a natural transformation of Steenrod homology theories on $\mathscr{F}\mathscr{C}\mathscr{M}$. The first step is the construction of certain maps

$$\mu_n^X / \colon \mathscr{E}_1(X) \to K^{-2}(F(X, S^n))$$

for n odd and $X \in \mathscr{C}\mathscr{M}$. Then an argument which requires $X \in \mathscr{F}\mathscr{C}\mathscr{M}$ yields $\Gamma_*$.

The following pairings will be needed:

(5.1) $\wedge : K^r(X) \otimes K^s(Y) \to K^{r+s}(X \wedge Y)$;

(5.2) $/ : K^r(X \wedge Y) \otimes \mathscr{E}_s(Y) \to K^{r-s}(X)$ (slant product);

(5.3) $\cap : K^r(X) \otimes \mathscr{E}_s(X) \to \mathscr{E}_{s-r}(X)$ (cap product).

The pairing (5.1) exists since $\underline{K}$ is a ring spectrum. Let $a \in K^{-2}(S^0)$ be the Bott generator. Then $\wedge\, a: K^s(Y) \to K^{s-2}(Y) = K^s(S^2 Y)$ is (topological) Bott periodicity.

The pairings (5.2) and (5.3) are due to BDF [18]. We recall the definition of the slant product in the case $K^{-1}(X \wedge Y) \otimes \mathscr{E}_1(Y) \to K^{-2}(X)$. Let $\mathscr{U} = \varinjlim \mathscr{U}(n)$, so $K^{-1}(X \wedge Y) = [X \wedge Y, \mathscr{U}]$, and let $\mathfrak{A}^r$ denote the invertible elements of the Calkin algebra, so $K^0(X) = [X, \mathfrak{A}^r]$. Let $\alpha: X \wedge Y \to \mathscr{U}(N)$ and $\tau: C(Y) \to \mathfrak{A}$. Then the composite $X \xrightarrow{\tilde{\alpha}} \mathscr{U}(N)^Y \xrightarrow{\tau \otimes I_N} \mathfrak{A}^r$ gives an element in $K^0(X)$. Compose with periodicity to obtain $\alpha/\tau \in K^{-2}(X)$. (Note that if $\tau$ is deformed continuously *through extensions*, then the image in $K^{-2}(X)$ is left unchanged. This will be useful in connection with strong homotopy invariance in Section 7.) Take the adjoint of $/$, set $X = S^0$, $s = 1$, $r = -1$, and obtain $\gamma_\infty: \mathscr{E}_1(Y) \to \hom(K^{-1}(Y), Z)$. In fact, this is reversible; $\gamma_\infty$ yields $/$ by treating $Y$ as a parameter space following Atiyah-Singer [9; III].

The cap product is basic. It gives (for $r = 0$, $s = 1$) a $K^0(X)$-module structure to $\mathscr{E}_1(X)$. Its definition is due essentially to Atiyah [8]. Here we recall the BDF formulation briefly. Let $E \downarrow X$ be a vector bundle represented by a projection-valued function $p_E: X \to M_n$ [7, p. 31]. Let $\tau: C(X) \to \mathfrak{A}$ be an extension. Then define $E \cap \tau: M_n(X) \to \mathfrak{A}$ by $(E \cap \tau)(f) = (\tau \otimes 1)(p_E \cdot f)$, where

$$M_n(X) = C(X) \otimes M_n$$

and we use the fact that $\mathscr{E}xt(X)$ may also be defined by $C^*$-algebra injections $M_n(X) \to \mathfrak{A}$ [18].

*Note.* We shall use certain properties of the BDF pairings [18] which have been verified by L. G. Brown. These properties yield in essence the key steps to our proofs of (5.8) and (5.11).

The following maps will be needed:

(1) $\lambda_n: F(X, S^n) \wedge S^1 \to F(X, S^{n+1})$, defined by $\lambda_n(f \wedge t)(x) = f(x) \wedge t$;

(2) the evaluation $e_n: F(X, S^n) \wedge X \to S^n$;

(3) $\rho_n: F(X, S^{n-2}) \wedge S^2 \to F(X, S^n)$, defined by $\rho_n(f \wedge t)(x) = f(x) \wedge t$.

The following lemma is easily verified.

LEMMA 5.4. *The diagram*

$$\begin{array}{ccccc} F(X, S^{n-2}) \wedge X \wedge S^2 & \xrightarrow{1 \wedge T} & F(X, S^{n-2}) \wedge S^2 \wedge X & \xrightarrow{\rho_n \wedge 1} & F(X, S^n) \wedge X \\ \downarrow{\scriptstyle e_{n-2} \wedge 1} & & & & \downarrow{\scriptstyle e_n} \\ S^{n-2} \wedge S^2 & & \longrightarrow & & S^n \end{array}$$

*commutes, where* T *interchanges the last two factors.*

Let $d_1 \in K^{-1}(S^1)$ be represented by $S^1 = \mathcal{U}(1) \to \mathcal{U}$, and inductively define $d_n = a \wedge d_{n-2} \in K^{-1}(S^n)$ for n odd. Then $d_n$ is the Bott generator in $\pi_n(\mathcal{U})$ and may be concretely realized using Clifford algebras by a map $S^n \to \mathcal{U}(2^{(n-1)/2})$ (see 7.8).

Note that $\wedge$ is graded commutative, so $T^*(a \wedge x) = x \wedge a$. We use this implicitly in the following lemma.

LEMMA 5.5. $(1 \wedge T)^*(\rho_n \wedge 1)^* e_n^* d_n = a \wedge (e_{n-2}^* d_{n-2})$ *in the group*

$$K^{-1}(F(X, S^{n-2}) \wedge X \wedge S^2).$$

*Proof.* By (5.4),

$$(1 \wedge T)^*(\rho_n \wedge 1)^* e_n^* d_n = (e_{n-2} \wedge 1)^*(a \wedge d_{n-2}) = a \wedge (e_{n-2}^* d_{n-2}).$$

Now following Atiyah [8], define $\mu_n^X \in K^{-1}(F(X, S^n) \wedge X)$ for n odd by letting $\mu_n^X$ be the homotopy class of the composite

$$F(X, S^n) \wedge X \xrightarrow{e_n} S^n \xrightarrow{d_n} \mathcal{U}.$$

Then (5.5) immediately implies:

LEMMA 5.6. $(1 \wedge T)^*(\rho_n \wedge 1)^* \mu_n^X = a \wedge \mu_{n-2}^X$ *in* $K^{-1}(F(X, S^{n-2}) \wedge X \wedge S^2)$.

The class $\mu_n^X$ should be thought of as a K-theory Spanier-Whitehead duality class. The existence of $\mu_n^X$ immediately yields a homomorphism

$$\mu_n^X/ : \mathscr{E}_1(X) \to K^{-2}(F(X, S^n)) \tag{5.7}$$

which is clearly natural in X, for fixed n. The following lemma shows that the map is "independent of n" in a reasonable sense.

LEMMA 5.8. *For* n *odd and* $X \in \mathscr{CM}$ *there is a commutative diagram*

$$\begin{array}{ccc} \mathscr{E}_1(X) & \xrightarrow{\mu_n^X/} & K^{-2}(F(X, S^n)) \\ \downarrow{\scriptstyle \mu_{n-2}^X/} & & \downarrow{\scriptstyle \rho_n^*} \\ K^{-2}(F(X, S^{n-2})) & \xrightarrow{a\wedge} & K^{-2}(F(X, S^{n-2}) \wedge S^2). \end{array}$$

*Proof.* Let $\tau \in \mathscr{E}_1(X)$. Then by (5.6) and by [18, (5.8)],

$$\rho_n^*(\mu_n^X/\tau) = ((1 \wedge T)^*(\rho_n \wedge 1)^* \mu_n^X)/\tau = (a \wedge \mu_{n-2}^X)/\tau = a \wedge (\mu_n^X/\tau),$$

as required.

Now suppose that $X \in \mathscr{FCM}$. Choose n to be odd and large, so that X embeds in $S^{n+1}$. Then (4.5) implies that

$$\hat{e}_n : \Sigma^{-n} \underline{S}(S^{n+1} \setminus X) \to F(X, \underline{S})$$

is a CW-substitute. Hence by periodicity and by Steenrod duality,

$$K^{-2}(S^{n+1}\setminus X) \cong K^{n-1}(S^{n+1}\setminus X) \cong {}^{s}K_1(X),$$

and so

$$K^{-2}(S^{n+1}\setminus X) \cong {}^{s}K_1(X). \tag{5.9}$$

Define $\Gamma^X_{1,n}$ (for n odd) to be the composite

$$\begin{array}{ccccc} \mathscr{E}_1(X) & \xrightarrow{\mu^X_n/} & K^{-2}(F(X, S^n)) & \xrightarrow{\hat{e}^*_n} & K^{-2}(S^{n+1}\setminus X) \\ & & & & \big\downarrow \cong (5.9) \\ & & & & {}^{s}K_1(X) \end{array} .$$

LEMMA 5.10. *The collection* $\{\Gamma^X_{1,n}\}$ *defines a natural transformation of functors* $\Gamma_1\colon \mathscr{E}_1 \to {}^{s}K_1$ *on* $\mathscr{F}\mathscr{C}\mathscr{M}$.

*Proof.* (*Cf.* [39] for an analogous argument in greater detail.) Suppose $X \subset S^n \subset S^m$ with n, m odd. Then a routine check involving (5.8) shows $\Gamma^X_{1,n} = \Gamma^X_{1,m}$. Thus $\Gamma^X_1$ is well-defined. Naturality will follow if $\hat{e}^*_n$ is natural for $n \gg \dim X$. Suppose $f\colon X \to Y$, with $n \gg 2\dim X,\ 2\dim Y$. Then $\hat{e}_n(W)$ is a $2(n - \dim W)$ equivalence ($W = X$ or $Y$), so there is a unique homotopy class of maps $g\colon S^{n+1}\setminus Y \to S^{n+1}\setminus X$, making the diagram

$$\begin{array}{ccc} S^{n+1}\setminus Y & \xrightarrow{\hat{e}_n(Y)} & F(Y, S^n) \\ \big\downarrow g & & \big\downarrow F(f) \\ S^{n+1}\setminus X & \xrightarrow{\hat{e}_n(X)} & F(X, S^n) \end{array}$$

homotopy-commute. This implies that $\hat{e}^*_n$ is natural.

Let $a\wedge\colon {}^{s}K_r(X) \to {}^{s}K_{r+2}(X)$ be the composite

$${}^{s}K_r(X) = K^{-r}(\underline{F}(X)) \xrightarrow{a\wedge} K^{-r-2}(\underline{F}(X)) = {}^{s}K_{r+2}(X),$$

and let $\sigma\colon {}^{s}K_r(X) \to {}^{s}K_{r+1}(SX)$ be suspension. Define $\Gamma_*$ as follows. The map $\Gamma_{2k}$ is the composite

$$\begin{array}{ccc} \mathscr{E}_{2k}(X) & \overset{\Gamma^X_{2k}}{\dashrightarrow} & {}^{s}K_{2k}(X) \\ \| & & \big\uparrow (a\wedge)^k \\ \mathscr{E}_0(X) & & {}^{s}K_0(X) \\ \| & & \big\uparrow \sigma^{-1} \\ \mathscr{E}_1(SX) & \xrightarrow{\Gamma^{SX}_1} & {}^{s}K_1(SX) \end{array} ,$$

and $\Gamma_{2k+1}$ is the composite

$$\begin{array}{ccc} \mathscr{E}_{2k+1}(X) & \xrightarrow{\Gamma^X_{2k+1}} & {}^sK_{2k+1}(X) \\ \| & & \uparrow (a\wedge)^k \\ \mathscr{E}_1(X) & \xrightarrow{\Gamma^X_1} & {}^sK_1(X) \end{array}$$

It remains to relate $\Gamma_*$ to periodicity and suspension. Topological periodicity and BDF periodicity may both be expressed as cap products. Let $p\colon X^+ \to S^0$ be the canonical map, and define $a_X = p^* a \in K^{-2}(X^+)$. Then the BDF periodicity is

$$a_X \cap\colon \mathscr{E}_1(S^2X) = \mathscr{E}_{-1}(X) \to \mathscr{E}_1(X)$$

and the topological periodicity is

$$a_X \cap\colon {}^sK_1(S^2X) \to {}^sK_1(X),$$

where this latter map is defined to be the composite

$${}^sK_1(S^2X) \xrightarrow{\sigma^{-2}} {}^sK_{-1}(X) \xrightarrow{a\wedge} {}^sK_1(X)$$

(which agrees with the usual cap product on finite complexes).

PROPOSITION 5.11. *The diagram*

$$\begin{array}{ccc} \mathscr{E}_1(S^2X) & \xrightarrow{\Gamma^{S^2X}_1} & {}^sK_1(S^2X) \\ \downarrow a_X\cap & & \downarrow a_X\cap \\ \mathscr{E}_1(X) & \xrightarrow{\Gamma^X_1} & {}^sK_1(X) \end{array}$$

*commutes.*

*Proof.* After unraveling a large diagram, the problem reduces to showing that

(5.12)
$$\begin{array}{ccc} \mathscr{E}_1(S^2X) & \xrightarrow{\mu^{S^2X}_{n+2}/} & K^{-2}(F(S^2X, S^{n+2})) \\ \downarrow a_X\cap & & \downarrow \psi^* \\ \mathscr{E}_1(X) & \xrightarrow{\mu^X_n/} & K^{-2}(F(X, S^n)) \end{array}$$

commutes, where $\psi\colon F(X, S^n) \to F(S^2X, S^{n+2})$ is the double suspension map: $\psi(f)(x \wedge t) = f(x) \wedge t$. It is immediate that $e^{S^2X}_{n+2} \circ (\psi \wedge 1) = e^X_n \wedge 1$. Hence

$$(\psi \wedge 1)^* \mu^{S^2X}_{n+2} = (\psi \wedge 1)^*(e^{S^2X}_{n+2})^*(d_n \wedge a) = (e^X_n \wedge 1)^*(d_n \wedge a) = \mu^X_n \wedge a .$$

Thus

$$\psi^*(\mu_{n+2}^{S^2X}/\tau) = ((\psi \wedge 1)^*(\mu_{n+2}^{S^2X}))/\tau = (\mu_n^X \wedge a)/\tau$$
$$= (\mu_n^X \cup (1 \wedge a_X))/\tau = \mu_n^X/(a_X \cap \tau).$$

proving the proposition.

Finally, the section concludes with its main result.

THEOREM 5.13. $\Gamma_*$: $\mathscr{E}_* \to {}^sK_*$ *is a natural transformation of Steenrod homology theories on* $\mathscr{F}\mathscr{C}\mathscr{M}$.

*Proof.* By (5.10), it suffices to show that $\Gamma_*$ commutes with suspension. After unraveling again, the relevant commutative diagram is (5.12).

## 6. PROOF OF THEOREM C

THEOREM C. *There is a natural equivalence of Steenrod homology theories* $\Gamma_*$: $\mathscr{E}_* \to {}^sK_*$ *on* $\mathscr{F}\mathscr{C}\mathscr{M}$.

*Proof.* Theorem (5.13) defines a natural transformation of Steenrod homology theories $\Gamma_*$ on $\mathscr{F}\mathscr{C}\mathscr{M}$. Such a natural transformation induces a morphism of spectral sequences [28] which on the $E^2$ level is the coefficient homomorphism ${}^sH_p(X; \mathscr{E}_q(S^0)) \to {}^sH_p(X; {}^sK_q(S^0))$ and which on the $E^\infty$ level is an associated graded map to $\Gamma_*$. To prove Theorem C, then, it suffices to show that $\Gamma_*^{S^0}$ is an isomorphism. This is equivalent to checking $\Gamma_1^{S^1}$ (by periodicity), which corresponds to showing that the composite

$$\mathscr{E}_1(S^1) \xrightarrow{\mu_n/} K^0(F(S^1, S^n)) \xrightarrow{g_n^*} K^0(S^{n+1} \setminus S^1)$$

is an isomorphism, where $g_n = \hat{e}_{D_n}$: $S^{n+1} \setminus X \to F(S^1, S^n)$ as in (4.4), and n is odd. Proving (6.1), then, is the remaining step.

LEMMA 6.1. $g_n^* \circ (\mu_n/)$: $\mathscr{E}_1(S^1) \to K^0(S^{n+1} \setminus S^1)$ *is an isomorphism, for* n *odd.*

*Proof.* Let $\mathscr{H}$ be the Hardy space $H^2(S^1)$ (those $L^2$ functions on $S^1$ with analytic extensions to the whole disk) and $\mathscr{L} = \mathscr{L}(H)$. Let $\mathscr{E}$ be the $C^*$-algebra generated by $\mathscr{K}$, I, and the Toeplitz operator $T_{z^{-1}}$. Then

$$C(S^1) \cong \mathscr{E}/\mathscr{K} \xrightarrow{\tau} \mathfrak{A}$$

represents a generator of $\mathscr{E}_1(S^1) = Z$, and $\tau(z) = \pi(T_{z^{-1}})$. Note that

$$\gamma[\tau](1_{S^1}) = \text{index}(T_{z^{-1}}) = 1.$$

Since $K^0(S^{n+1} \setminus S^1) \cong Z$ also, it suffices to verify that $g_n^*(\mu_n/[\tau])$ generates $K^0(S^{n+1} \setminus S^1)$. Explicitly writing down the element is an easy task. Choose an inclusion $S^{n-1} \to S^{n+1} \setminus S^1$ which generates $\pi_{n-1}(S^{n+1} \setminus S^1)$. Then our problem reduces to showing that the composite $S^{n-1} \xrightarrow{\phi} \mathscr{U}(N)^{S^1} \xrightarrow{\tau \otimes 1_N} \mathfrak{A}^r$ generates $\pi_{n-1}(\mathfrak{A}^r) = Z$, where $\phi$ is the composite

$$\begin{array}{ccccccc} S^{n-1} & \longrightarrow & S^{n+1}\backslash S^1 & \xrightarrow{\hat{e}_D} & F(S^1, S^n) & \xrightarrow{\hat{\mu}} & \mathscr{U}^{S^1} \\ & \searrow^{\phi} & & & & & \uparrow \\ & & & & & & \mathscr{U}(N)^{S^1} \end{array} .$$

But the adjoint $\phi\#: S^{n-1} \wedge S^1 \to \mathscr{U}(N)$ is homotopic to $d_n$, the Bott generator of $\pi_n(\mathscr{U}(N)) = Z$. Atiyah's proof [9] of periodicity implies that, for N large, $(\tau \otimes 1_N)_*: [S^{n-1}, \Omega\mathscr{U}(N)] \to [S^{n-1}, \mathfrak{A}^*]$ induces the periodicity isomorphism (for our particular choice of $\tau$!). Hence $(\tau \otimes 1_N) \circ \phi$ generates $\pi_{n-1}(\mathfrak{A}^r)$, and the proof is complete.

*Remark* 6.2. Suppose $\overline{\Gamma}_*: \mathscr{E}_* \to {}^sK_*$ were another natural equivalence of Steenrod homology theories, and $\Gamma_*|\mathscr{W} = \overline{\Gamma}_*|\mathscr{W}$. Then $\psi_* = \Gamma_* \circ \overline{\Gamma}_*^{-1}: {}^sK_* \to {}^sK_*$ is a natural equivalence, and $\psi_* = 1$ on $\mathscr{W}$. Then $(\psi - 1): BU \times Z \to BU \times Z$ is a map of spaces which vanishes when restricted to skeleta. Since $BU \times Z$ has no phantom self-maps, we conclude $\psi - 1 = 0$; hence $\Gamma_* = \overline{\Gamma}_*$ on $\mathscr{F}\mathscr{C}\mathscr{M}$. Thus $\Gamma_*$ is canonical in a very strong sense.

## 7. APPLICATIONS AND REMARKS

Theorem C yields short proofs of two results obtained first by L. G. Brown. His methods involve algebraic K-theory and are quite interesting. Our proofs (7.1) and (7.5) are completely topological in nature but more limited—they hold only in $\mathscr{F}\mathscr{C}\mathscr{M}$. The section closes with assorted remarks.

THEOREM 7.1 (L. G. Brown). *There is a natural Universal Coefficient sequence*

$$0 \to \mathrm{Ext}(K^0(X), Z) \to \mathscr{E}xt(X) \to \mathrm{hom}(K^1(X), Z) \to 0$$

*on* $\mathscr{F}\mathscr{C}\mathscr{M}$. *There is a similar sequence for* $\mathscr{E}_0(X)$. *Both sequences split* (*unnaturally*).

*Proof.* Let $\underline{W} = \Sigma^{-n}\, \underline{S}(S^{n+1}\backslash X)$. Then there is a natural Universal Coefficient sequence [44, 4]

$$0 \to \mathrm{Ext}(K_0(\underline{W}), Z) \to K^{-1}(\underline{W}) \to \mathrm{hom}(K_{-1}(\underline{W}), Z) \to 0 . \tag{7.2}$$

Then by definition and by Theorem C, $K^{-1}(\underline{W}) = {}^sK_1(X) \cong \mathscr{E}xt(X)$, and

$$K_j(\underline{W}) = K_j\left(\Sigma^{-n}\, \underline{S}(S^{n+1}\backslash X)\right) = K_{j-n}(S^{n+1}\backslash X) = K^{-j}(X),$$

by Alexander duality [41]. These isomorphisms are natural. Thus (7.2) yields (7.1) by translating. The sequence (7.1) splits (unnaturally), since Ext(A, Z) is algebraically compact and hom(B, Z) is torsion-free for any abelian groups A, B.

*Remark* 7.3. Let G be a finitely generated abelian group and let M be a Moore space of type (G, 2). Then coefficients may be introduced in the theory $\mathscr{E}_*$ by defining $\mathscr{E}_n(X; G) = \mathscr{E}_{n+2}(X \wedge M)$ and this new theory satisfies the Universal Coefficient sequence

(7.4) $$0 \to \mathrm{Ext}(K^{n+1}(X), G) \to \mathscr{E}_n(X; G) \to \hom(K^n(X), G) \to 0$$

for all $X \in \mathscr{F}\mathscr{C}\mathscr{M}$.

THEOREM 7.5 (L. G. Brown). ("Strong homotopy invariance") *Let* $X \in \mathscr{F}\mathscr{C}\mathscr{M}$ *and let* $\tau_s$: $C(X) \to \mathfrak{A}$ *be a continuous path of extensions*, $0 \leq s \leq 1$. *Then* $[\tau_0] = [\tau_1]$ *in* $\mathscr{E}\mathrm{xt}(X)$.

*Proof.* Let $\tilde{\tau}$: $C(X) \times I \to \mathfrak{A}$ by $\tilde{\tau}(f, s) = \tau_s(f)$. Then $\mu_n^X/\tilde{\tau}$ gives a homotopy, which implies $\mu_n^X/\tau_0 \simeq \mu_n^X/\tau_1$, and this yields $\Gamma_*([\tau_0]) = \Gamma_*([\tau_1])$. Since $\Gamma_*$ is an isomorphism, $[\tau_0] = [\tau_1]$.

*Remark* 7.6. Theorem 7.5 implies that $\mathscr{E}\mathrm{xt}(X) = [C(X), \mathfrak{A}]$, where the right-hand side denotes homotopy classes of extensions (in the category of C*-algebras with identity). There is then the following amusing fact: ${}^sK_1(X) \cong [C(X), \mathfrak{A}]$ and $K^1(X) \cong [SX, \Omega\mathfrak{A}^r]$. BDF first noted that Theorem 7.5 implies that

$$\mathscr{E}\mathrm{xt}(X) \cong \pi_0(\mathrm{E11}(X)),$$

answering Atiyah's question [8] as to the suitable equivalence relation to be put on E11(X). This shows that the homology theory of Kasparov [30] coincides with $\mathscr{E}_*$.

*Remark* 7.7. Suppose $h_*$ is a Steenrod homology theory with $h_*(S^0)$ of finite type. Then the $\lim^1$ sequence splits, and so $h_n(X)$ is unnaturally isomorphic to the direct sum $\varprojlim h_n(X_j) \oplus \varprojlim^1 h_{n+1}(X_j)$, where $X = \varprojlim X_j$. Thus the *groups* $h_*(X)$ are uniquely specified by the behavior of $h_*$ on $\mathscr{W}$. A stronger uniqueness theorem seems to require some additional axiom specifying the relation of $h_*$ to $h^*$. For example, suppose one assumes the existence of a slant product

$$h^n(X \wedge \underline{Y}) \otimes h_k(X) \xrightarrow{/} h^{n-k}(\underline{Y})$$

and a duality class $\mu \in h^n(X \wedge \underline{F}(X))$ with obvious properties. Take $\underline{Y} = \underline{F}(X)$, and one obtains a natural map of theories $\mu/$: $h_k(X) \to h^{n-k}(\underline{F}(X))$ which is easily seen to be a natural equivalence. (This is essentially the argument in Sections 5-6.) This would imply that the Steenrod homology theory ${}^sh_*$ produced by Theorem A is naturally isomorphic to $h_*$ on $\mathscr{F}\mathscr{C}\mathscr{M}$. A strong uniqueness theorem would result. Suppose $h_*$ and $k_*$ are Steenrod homology theories "with slant products", and suppose $\Gamma_{\mathscr{W}}$: $h_*|\mathscr{W} \to k_*|\mathscr{W}$ is a natural equivalence. Then $\Gamma_{\mathscr{W}}$ extends to a natural equivalence $\Gamma$: $h_* \to k_*$ on $\mathscr{F}\mathscr{C}\mathscr{M}$ (or even on $\mathscr{C}\mathscr{M}$ if the duality class existed). Even in such circumstances $\Gamma$ would not be unique; there might be phantom natural equivalences on $h_*$ or $k_*$ which were the identity on $\mathscr{W}$ but not on $\mathscr{F}\mathscr{C}\mathscr{M}$. This fortunately does not occur for $K_*$.

*Remark* 7.8. Here is an analytic description of $\Gamma_1$ due to Atiyah and Segal. First recall the Clifford algebra description [9] of the generator $d_{2n-1} \in \pi_{2n-1}(\mathscr{U})$. Let $H_1, \cdots, H_{2n}$ be complex $N \times N$ matrices ($N = 2^{n-1}$) with $H_iH_j = -H_jH_i$ ($i \neq j$) and $H_i^2 = -I$. Define $\delta_{2n-1}$: $\mathbb{R}^{2n} - \{0\} \to GL(N, \mathbb{C})$ by $\delta_{2n-1}(x) = \sum_{j=1}^{2n} x_jH_j$, where $x = (x_1, \cdots, x_{2n})$. The restriction of this map to $S^{2n-1}$ is in the homotopy class of $d_{2n-1}$.

Now suppose $X \subset \mathbb{C}^n = \mathbb{R}^{2n}$ and $\tau \in \mathscr{E}\mathrm{xt}(X)$. Then $\Gamma_1(\tau)$ is the homotopy class of the composite

$$\mathbb{C}^n \backslash X \xrightarrow{\ \phi\ } F(X, \mathbb{C}^n \backslash \{0\}) \xrightarrow{\delta_{2n-1}^{\#}} F(X, GL(N, \mathbb{C})) \xrightarrow{\tau \otimes I_N} \mathfrak{A} \otimes M_N,$$

where $\phi(\lambda)(x) = x - \lambda$, and an easy check shows that $\Gamma_1(\tau)$ is the map

$$\lambda \mapsto \sum_{j=1}^{2n} (\tau(p_j) - \lambda_j) \otimes H_j ,$$

where $p_j: X \to \mathbb{R}$ are the coordinate functions. A detailed description of the Atiyah-Segal map may be found in [10].

## REFERENCES


1. J. F. Adams, *A variant of E. H. Brown's representability theorem.* Topology 10 (1971), 185-198.

2. ———, *Stable Homotopy and Generalized Homology.* University of Chicago Press, 1974.

3. D. W. Anderson, *Universal coefficient theorems for* K-*theory*, to appear.

4. S. Araki and Z. Yosimura, *A spectral sequence associated with a cohomology theory of infinite* CW-*complexes.* Osaka J. Math. 9 (1972), 351-365.

5. M. F. Atiyah, *Algebraic topology and elliptic operators.* Comm. Pure Appl. Math. 20 (1967), 237-249.

6. ———, *Bott periodicity and the index of elliptic operators.* Quart. J. Math. Oxford Ser. (2) 19 (1968), 113-140.

7. ———, K-*Theory. Lecture Notes by D. W. Anderson.* W. A. Benjamin, Inc., New York-Amsterdam, 1967.

8. ———, *Global theory of elliptic operators.* Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969), pp. 21-30. Univ. of Tokyo Press, Tokyo, 1970.

9. ———, *Algebraic topology and operators in Hilbert space.* Lectures in Modern Analysis and Applications I, Lecture Notes in Mathematics, Vol. 103, pp. 101-121. Springer-Verlag, N.Y., 1969.

10. ———, *A survey of* K-*theory.* K-theory and Operator Algebras, Lecture Notes in Mathematics, Vol. 575, pp. 1-9. Springer-Verlag, N.Y., 1977.

11. M. F. Atiyah and I. M. Singer, *The index of elliptic operators.* I. III, Ann. of Math. (2) 87 (1968), 484-530, 546-604.

12. ———, *Index theory for skew-adjoint Fredholm operators.* Inst. Hautes Études Sci. Publ. Math. No. 37 (1969), 5-26.

13. E. H. Brown, *Cohomology theories.* Ann. of Math. (2) 75 (1962), 467-484.

14. L. G. Brown, *Operator algebras and algebraic* K-*theory.* Bull. Amer. Math. Soc. 81 (1975), 1119-1121.

15. ———, *Characterizing Ext.* K-theory and Operator Algebras, Lecture Notes in Mathematics, Vol. 575, pp. 10-18. Springer-Verlag, N.Y., 1977.

16. L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Extensions of* C*-*algebras, operators with compact self-commutators, and* K-*homology.* Bull. Amer. Math. Soc. 79 (1973), 973-978.

17. L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of* $C^*$*-algebras*. Proc. Conf. on Operator Theory. Lecture Notes in Mathematics, Vol. 345, pp. 58-128. Springer-Verlag, N.Y., 1973.

18. ———, *Extensions of* $C^*$*-algebras and* K*-homology*. Ann. of Math. (2), 105 (1977), 265-324.

19. L. G. Brown and C. Schochet, $K_1$ *of the compact operators is zero*. Proc. Amer. Math. Soc. 59 (1976), 119-122.

20. R. G. Douglas, *Extensions of* $C^*$*-algebras and* K*-homology*. K-theory and Operator Algebras, Lecture Notes in Mathematics, Vol, 575, pp. 44-52. Springer-Verlag, N.Y., 1977.

21. ———, *The relation of Ext to* K*-theory*. Instituto Nazionale di alta matematica, Symposia Mathematica, Vol. XX (1976), pp. 513-529.

22. E. Dyer, *The functors of algebraic topology*. Studies in Modern Topology, pp. 134-164. Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.), 1968.

23. D. A. Edwards and H. M. Hastings, *Čech and Steenrod Homotopy Theories with Applications to Geometric Topology*. Lecture Notes in Mathematics, Vol. 542, Springer-Verlag, N.Y., 1976.

24. R. H. Fox, *On fibre spaces I, II.* Bull. Amer. Math. Soc. 49 (1943), 555-557, 733-735.

25. J. B. Giever, *On the equivalence of two singular homology theories*. Ann. of Math. (2) 51 (1950), 178-191.

26. J. Kaminker and C. Schochet, Ext(X) *from a homological point of view*. Proc. Conf. on Operator Theory, Lecture Notes in Mathematics Vol. 345, pp. 129-140. Springer-Verlag, N.Y., 1973.

27. ———, *Steenrod homology and operator algebras*. Bull. Amer. Math. Soc. 81 (1975), 431-434.

28. ———, K*-theory and Steenrod homology; Applications to the Brown-Douglas-Fillmore theory of operator algebras*. Trans. Amer. Math. Soc. 227 (1977), 63-107.

29. ———, *Topological obstructions to perturbations of pairs of operators*. K-theory and Operator Algebras, Lecture Notes in Mathematics, Vol. 575, pp. 70-77. Springer-Verlag, N.Y., 1977.

30. G. G. Kasparov, *The generalized index of elliptic operators*. Functional Anal. Appl. 7 (1973), 82-83.

31. P. Kohn, *A homotopy theory of* $C^*$*-algebras*, thesis, University of Penn., 1972.

32. S. Mardešić, *On inverse limits of compact spaces*. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II 13 (1958), 249-255.

33. J. Milnor, *On spaces having the homotopy type of* CW*-complex*. Trans. Amer. Math. Soc. 90 (1959), 272-280.

34. ———, *On the Steenrod homology theory*. Berkeley, 1961, mimeo.

35. J. C. Moore, *On a theorem of Borsuk*. Fund. Math. 43 (1956), 195-201.

36. I. M. Singer, *Operator theory and periodicity*. Indiana Univ. Math. J. 20 (1971), 949-951.

37. E. H. Spanier, *Infinite symmetric products, function spaces, and duality.* Ann. of Math. (2) 69 (1959), 142-198.

38. ———, *Function spaces and duality.* Ann. of Math. (2) 70 (1959), 338-378.

39. E. H. Spanier and J. H. C. Whitehead, *Duality in homotopy theory.* Mathematika 2 (1955), 56-80.

40. N. E. Steenrod, *Regular cycles of compact metric spaces.* Ann. of Math. (2) 41 (1940), 833-851.

41. R. E. Stong, *Notes on Cobordism Theory.* Mathematical Notes, Princeton University Press, Princeton, N.J., 1968.

42. R. M. Switzer, *Algebraic Topology—Homotopy and Homology.* Springer-Verlag, N.Y., 1975.

43. G. W. Whitehead, *Generalized homology theories.* Trans. Amer. Math. Soc. 102 (1962), 227-283.

44. Z. Yosimura, *Projective dimension of complex bordism modules of* CW-*spectra* I. Osaka J. Math. 10 (1973), 545-564.

Northwestern University
Evanston, Ill. 60201
and
Indiana University—Purdue University at Indianapolis
Indianapolis, In. 46202
and
Wayne State University
Detroit, Mi. 48202