LUMER'S HARDY SPACES

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In the present paper, the term *pluriharmonic* will always refer to real-valued functions. A pluriharmonic function is thus one whose domain is an open subset Ω of $\mathbb{C}^n$ and which is locally the real part of a holomorphic function.

We define $(LH)^p(\Omega)$ to be the class of all holomorphic functions $f: \Omega \to \mathbb{C}$ such that $|f|^p \le u$ for some pluriharmonic u. (Here $0 < p < \infty$.) This is Lumer's definition of H^p-spaces [1]. When $n = 1$, pluriharmonic is the same as harmonic, so that this definition coincides with the old one ([2], [3]) which involves harmonic majorants of $|f|^p$. But when $n > 1$, then $(LH)^p(\Omega)$ is a proper subclass of what is usually called $H^p(\Omega)$. (See, for example, [6].)

The use of pluriharmonic majorants leads to some appealing properties of $(LH)^p(\Omega)$. For example, holomorphic invariance is a triviality: if Φ is a holomorphic map of Ω_1 into Ω_2 and if $f \in (LH)^p(\Omega_2)$, then obviously $f \circ \Phi \in (LH)^p(\Omega_1)$.

To see another example, let Ω be simply connected. If $f \in (LH)^p(\Omega)$ for some $p \in (0, \infty)$, then $\log |f| \le \Re g$ for some holomorphic g in Ω. Setting $h = f \cdot \exp(-g)$, it follows that $|h| \le 1$. *Thus every* $f \in (LH)^p(\Omega)$ *has the same zeros as some* $h \in H^\infty(\Omega)$. This is in strong contrast to what is known [4] about zero sets of the usual H^p-functions in the unit ball or the unit polydisc of $\mathbb{C}^n$.

However, from the standpoint of functional analysis, the $(LH)^p$-spaces have unexpectedly pathological properties. The purpose of the present paper is to describe some of these for the case $\Omega = B$, the open unit ball of $\mathbb{C}^n$; from now on, $n > 1$.

When $1 \le p < \infty$, $(LH)^p(B)$ can be normed by defining

$$\|f\|_p = \inf u(0)^{1/p}, \tag{1}$$

the infimum being taken over all pluriharmonic majorants u of $|f|^p$ in B. As pointed out in [1], this norm turns $(LH)^p(B)$ into a Banach space.

For $0 \le r < 1$, we use the notation f_r to denote the function defined for $z \in B$ by $f_r(z) = f(rz)$.

We let $\mathcal{U}$ denote the (compact topological) group of all unitary transformations of $\mathbb{C}^n$. Clearly, every $U \in \mathcal{U}$ maps B onto B.

As usual ℓ^∞ is the Banach space of all bounded complex sequences, and c_0 is the subspace of ℓ^∞ consisting of those sequences that converge to 0.

Here is our main result:

THEOREM. *Fix* p, $1 \le p < \infty$, *and fix* $\varepsilon > 0$.

(i) *There exists a linear map of* ℓ^∞ *into* $(LH)^p(B)$ *which assigns to each* $\gamma \in \ell^\infty$ *a function* f_γ *that satisfies* $\|\gamma\|_\infty \le \|f_\gamma\|_p \le \|f_\gamma\|_\infty \le (1+\varepsilon)\|\gamma\|_\infty$.

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(ii) *If* γ *is not in* c_0, *then* $U \to f_\gamma \circ U$ *is a discontinuous map of* $\mathcal{U}$ *into* $(LH)^p(B)$.

(iii) *If* γ *is not in* c_0, *then* $(f_\gamma)_r$ *does not converge to* f_γ *in the norm topology of* $(LH)^p(B)$, *as* $r \to 1$.

(Originally, in answer to a question raised by Stout, I constructed an $f \in (LH)^2(B)$ such that f_r did not converge to f. Joel Shapiro pointed out to me that very small modifications of my construction would yield the theorem as stated. Similar gap series constructions occur in [4] and [5].)

Recall that every holomorphic $f: B \to \mathbb{C}$ has an expansion of the form

$$f(z) = \sum_{k=0}^{\infty} f_k(z) \tag{2}$$

in which each f_k is a homogeneous polynomial of degree k.

We let S denote the sphere that is the boundary of B.

LEMMA. *If* $1 \leq p < \infty$ *and* $f \in (LH)^p(B)$, *then*

$$\|f\|_p \geq |f_m(\zeta)| \tag{3}$$

for every $\zeta \in S$ *and for every* m.

Proof. Fix $\zeta \in S$, fix m, and let u be a pluriharmonic majorant of $|f|^p$ in B. By (1), we have to show that

$$|f_m(\zeta)|^p \leq u(0). \tag{4}$$

Since u is pluriharmonic in B, the function $\lambda \to u(\lambda\zeta)$ is harmonic in the unit disc, so that

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}\zeta)\, d\theta \tag{5}$$

for every $r \in (0, 1)$. In the unit disc, the coefficients of a power series are dominated by its H^p-norm. Apply this to the series

$$f(\lambda\zeta) = \sum_{k=0}^{\infty} f_k(\zeta)\lambda^k \qquad (|\lambda| < 1) \tag{6}$$

to obtain

$$|f_m(\zeta)| \leq \sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}\zeta)|^p\, d\theta \right\}^{1/p}. \tag{7}$$

Since $|f|^p \leq u$, (7) and (5) give (4). This completes the proof of the lemma.

Proof of the theorem. Choose points $\zeta_k \in S$, $k = 1, 2, 3, \cdots$, so that no circle

$$\Gamma_k = \{e^{i\theta}\zeta_k : -\pi \leq \theta \leq \pi\} \tag{8}$$

contains a limit point of the union of the other Γ_i. Then there are disjoint open sets V_k in $\mathbb{C}^n$ such that $\Gamma_k \subset V_k$.

Choose unitary transformations $U_k \in \mathscr{U}$ so that U_k converges to the identity element of $\mathscr{U}$ as $k \to \infty$, and so that $|\langle U_k \zeta_k, \zeta_k \rangle| < 1$. (Here $\langle z, w \rangle = \sum z_i \overline{w}_i$ is the usual inner product in $\mathbb{C}^n$.)

We can then find an increasing sequence of natural numbers n_k such that

(9) $$|\langle z, \zeta_k \rangle|^{n_k} < \varepsilon/2^k \quad \text{if } z \in B - V_k$$

and

(10) $$|\langle U_k \zeta_k, \zeta_k \rangle|^{n_k} < 1/2 .$$

The linear map mentioned in (i) is the one that assigns to every $\gamma = \{c_k\} \in \ell^\infty$ the function

(11) $$f_\gamma(z) = \sum_{k=1}^{\infty} c_k \langle z, \zeta_k \rangle^{n_k} \quad (z \in B).$$

Since no two of the sets V_k intersect, the inequality (9) fails (for any given $z \in B$) for at most one term in the series (11). Thus

$$|f_\gamma(z)| \leq \|\gamma\|_\infty + \varepsilon \sum_{k=1}^{\infty} |c_k| 2^{-k} \leq (1+\varepsilon)\|\gamma\|_\infty ,$$

so that $\|f_\gamma\|_\infty \leq (1+\varepsilon)\|\gamma\|_\infty$. That $\|f_\gamma\|_p \leq \|f_\gamma\|_\infty$ is trivial, and $\|\gamma\|_\infty \leq \|f_\gamma\|_p$ follows from an application of the lemma to (11). This proves (i).

Next,

$$(f_\gamma - f_\gamma \circ U_i)(z) = \sum_{k=1}^{\infty} c_k [\langle z, \zeta_k \rangle^{n_k} - \langle U_i z, \zeta_k \rangle^{n_k}] .$$

When $z = \zeta_i$, the absolute value of the i^{th} term of this series is

$$|c_i| \; |1 - \langle U_i \zeta_i, \zeta_i \rangle^{n_i}| \geq \frac{1}{2} |c_i| ,$$

by (10). Another application of the lemma shows therefore that

$$\limsup_{i \to \infty} \|f_\gamma - f_\gamma \circ U_i\|_p \geq \frac{1}{2} \limsup_{i \to \infty} |c_i| .$$

Hence, $f_\gamma \circ U_i$ does not converge to f_γ in $(LH)^p(B)$ if $\{c_i\}$ fails to converge to 0. This proves (ii).

The proof of (iii) is quite similar: choose r_i so that $(r_i)^{n_i} = 1/2$. Then $r_i \to 1$ as $i \to \infty$, and

$$f_\gamma(z) - f_\gamma(r_i z) = \sum_{k=1}^{\infty} c_k [1 - (r_i)^{n_k}] \langle z, \zeta_k \rangle^{n_k} .$$

With $z = \zeta_i$, it follows from the lemma that

$$\|f_\gamma - (f_\gamma)_{r_i}\|_p \geq \frac{1}{2} |c_i|$$

for $i = 1, 2, 3, \cdots$. Thus

$$\limsup_{r \to 1} \|f_\gamma - (f_\gamma)_r\|_p \geq \frac{1}{2} \limsup_{i \to \infty} |c_i| ,$$

which proves (iii).

We now list some consequences of the theorem. Recall that two Banach spaces are said to be *isomorphic* if there is a linear homeomorphism of one onto the other.

COROLLARY. (a) $(LH)^p(B)$ *contains a closed subspace that is isomorphic to* ℓ^∞ *and lies in* $H^\infty(B)$.

(b) $(LH)^p(B)$ *is not separable.*

(c) *The ball algebra* $A(B)$ *is not dense in* $(LH)^p(B)$.

(d) $(LH)^2(B)$ *is not isomorphic to a Hilbert space.*

Proof. (a) follows immediately from (i), and obviously implies (b). Since $A(B)$ is separable in the sup-norm topology, it is *a fortiori* separable in the norm topology of $(LH)^p(B)$; thus (b) implies (c). Finally, (d) follows from (a) since every closed subspace of a Hilbert space is a Hilbert space, but ℓ^∞ (not being reflexive) is not isomorphic to any Hilbert space.

Here are some open questions.

In view of (c), is $H^\infty(B)$ dense in $(LH)^p(B)$? (This was asked by Stout.)

If $1 \leq p < q < \infty$, is $(LH)^q(B)$ dense in $(LH)^p(B)$?

If f is holomorphic in B and if there is a $C < \infty$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}\zeta)|^p \, d\theta \leq C$$

for all $\zeta \in S$ and for all $r \in (0, 1)$, does it follow that $f \in (LH)^p(B)$? Even the case $p = 2$ is open. (This question is suggested by Theorem 2 of [1].)

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