FREE INVOLUTIONS ON COMPLEX PROJECTIVE SPACES

Richard H. Sady

0. INTRODUCTION

Let $h\mathbb{C}P^N$ denote a compact manifold of the same homotopy type as complex projective space $\mathbb{C}P^N$. An easy consequence of the Lefschetz fixed point theorem is: the only group which can act freely on $h\mathbb{C}P^N$ is the cyclic group $\mathbb{Z}/2\mathbb{Z}$. Moreover, in this case it also follows that N must be odd. The purpose of this paper is to classify all free (PL and TOP) $\mathbb{Z}/2\mathbb{Z}$-actions on $h\mathbb{C}P^{2n+1}$. (The existence of a free smooth involution on any $h\mathbb{C}P^3$ was proven by Petrie [2]. His results are implied by Corollary B, below, and the fact that $\pi_i(PL/O) = 0$ for $i < 7$.) We will give invariants which detect the existence of a free involution on $h\mathbb{C}P^{2n+1}$ and determine the structure of the set of equivalence classes of free $\mathbb{Z}/2\mathbb{Z}$-actions on $h\mathbb{C}P^{2n+1}$. In particular, we show that there exist exactly 2^n distinct free PL involutions on $\mathbb{C}P^{2n+1}$. We assume familiarity with the surgery exact sequence [7]:

$$L^s_{n+1}(\pi, w) \xrightarrow{\omega} \mathscr{S}^s_H(M) \xrightarrow{\theta} [M, G/H] \xrightarrow{\sigma} L^s_n(\pi, w),$$

where H = PL or TOP, M is a PL or TOP n-manifold, $\pi = \pi_1(M)$, and $w = w_1(M)$. $\mathscr{S}^s_H(M)$ denotes the set of simple homotopy structures on M.

1. STATEMENT OF RESULTS

In this section, we explain our main results in the topological category. The minor modifications necessary for extending these results to the PL case are given in Section 8.

We write Free Inv($h\mathbb{C}P^{2n+1}$) for the set of conjugacy classes of free involutions on $h\mathbb{C}P^{2n+1}$. Note, for example, that Free Inv($\mathbb{C}P^{2n+1}$) $\neq \emptyset$. In terms of homogeneous coordinates, we can easily describe an element of this set:

$$T[z_0 : z_1 : \cdots : z_{2n+1}] = [-\bar{z}_1 : \bar{z}_0 : -\bar{z}_3 : \bar{z}_2 : \cdots].$$

(In fact, this is the free involution on $\mathbb{C}P^{2n+1}$ induced by the antipodal map in the fibres of the fibration $S^2 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$, where $\mathbb{H}P^n$ denotes quaternionic projective space.) Henceforth, T will denote the above involution and $X = X^{4n+2}$ will denote the orbit space $\mathbb{C}P^{2n+1}/\langle T \rangle$. Let π: $\mathbb{C}P^{2n+1} \to X$ be the natural projection.

Now suppose there exists a free (TOP) involution S on $h\mathbb{C}P^{2n+1}$. Let η denote the $\mathbb{Z}/2\mathbb{Z}$-bundle ρ: $h\mathbb{C}P^{2n+1} \to Y$, where $Y = Y^{4n+2}$ is the orbit space $h\mathbb{C}P^{2n+1}/\langle S \rangle$ and ρ is the natural projection. Define $\mathscr{S}_{TOP}(Y)^P$ to be the set of

Received July 12, 1976. Revision received April 21, 1977.

This work was completed at the University of Notre Dame as part of the author's doctoral thesis. The author wishes to express his great appreciation to Frank Connolly, whose encouragement and assistance have been invaluable in the preparation of this paper.

Michigan Math. J. 24 (1977).

those elements $[M, f] \in \mathscr{S}_{TOP}(Y)$ satisfying $f^*P(Y) = P(M)$, where $P(\ )$ is the function which assigns to any compact manifold its total Pontryagin class in $H^*(\ ;\mathbb{Q})$. (Any such f will be called a *Pontryagin equivalence.*) Letting $j\colon G/TOP \to BTOP$ denote the inclusion, we define $[Y, G/TOP]^P$ to be the set of homotopy classes of maps $\phi\colon Y \to G/TOP$ satisfying $(j\phi)^*(P) = 1$, where $P \in H^*(BTOP;\mathbb{Q})$ is the universal total Pontryagin class. (Such a map will be called a *Pontryagin normal map.*) Our main results follow.

THEOREM I. *There exists a bijection of sets:*

$$\text{Free Inv}(h\mathbb{C}P^{2n+1}) \cong \mathscr{S}_{TOP}(Y)^P .$$

THEOREM II. *$\mathscr{S}_{TOP}(Y)$ admits an abelian group structure such that $\mathscr{S}_{TOP}(Y)^P$ is a subgroup and*

$$0 \to \mathscr{S}_{TOP}(Y)^P \xrightarrow{\theta} [Y, G/TOP]^P \xrightarrow{\sigma} L_{4n+2}(\mathbb{Z}/2\mathbb{Z}, -) \to 0$$

is an exact sequence of abelian groups. Moreover, as abelian groups,

$$[Y, G/TOP]^P \cong \prod_{i=1}^{n+1} (\mathbb{Z}/2\mathbb{Z})_i .$$

An easy consequence of Theorems I and II is:

COROLLARY A. *If it is nonempty,* Free Inv$(h\mathbb{C}P^{2n+1})$ *is an abelian group. Moreover, in this case,* Free Inv$(h\mathbb{C}P^{2n+1}) \cong [n]\mathbb{Z}/2\mathbb{Z}$, *a direct sum of* n *copies of* $\mathbb{Z}/2\mathbb{Z}$.

Finally, we prove:

THEOREM III. *There exists an exact sequence of abelian groups:*

$$0 \longrightarrow \mathscr{S}_{TOP}(Y)^P \longrightarrow \mathscr{S}_{TOP}(Y) \xrightarrow{\rho\#} \mathscr{S}_{TOP}(h\mathbb{C}P^{2n+1}) \xrightarrow{\kappa} \prod_{i=1}^{n} (\mathbb{Z}/2\mathbb{Z})_i \longrightarrow 0 .$$

(*Here $\mathscr{S}_{TOP}(h\mathbb{C}P^{2n+1})$ is given an abelian group structure as described by Sullivan* [5], *and κ is defined in Section* 7.)

Letting $f\colon M \to \mathbb{C}P^{2n+1}$ be a homotopy equivalence, we also have:

COROLLARY B. *There exists a free involution on* M *if and only if* $\kappa[M, f] = 0$.

Remark. The above results are all true in the piecewise linear category, with the one exception that in Theorem III, "$i = 1$" must be replaced by "$i = 2$".

Note. After this work was completed, I discovered that a version of this corollary was proven independently by F. Hegenbarth [1].

2. THE GROUP $[X, G/TOP]^P$

In this section, we determine the structure of $[X, G/TOP]^P$ (Theorem 2.6). The following results are needed.

LEMMA 2.1 (Sullivan [4]). *There are primitive cohomology classes*

$$K = K_{4*-2} = \sum_{i \geq 1} k_{4i-2} \in H^{4*-2}(G/TOP; \mathbb{Z}/2\mathbb{Z})$$

and

$$K_{4*} = \sum_{i \geq 1} k_{4i} \in H^{4*}(G/TOP; \mathbb{Z}/2\mathbb{Z})$$

such that K_{4*} *is the* mod 2 *reduction of an integral class*

$$\hat{L} = \hat{L}_{4*} = \sum_{i \geq 1} \hat{L}_{4i} \in H^{4*}(G/TOP; \mathbb{Z}),$$

and such that the map $K \times \hat{L}$: $G/TOP \to \prod_{n \geq 1} K(\pi_n(G/TOP), n)$ *induces an isomorphism in* $\mathbb{Z}/2\mathbb{Z}$*-cohomology.*

In particular, the localization of G/TOP at the prime 2 has the homotopy type of a product of Eilenberg-MacLane spaces. Hence, it is immediate that

$$[X, (G/TOP)_{(2)}] \cong \prod_{i \geq 1} H^{4i}(X; \mathbb{Z}_{(2)}) \times \prod_{i \geq 1} H^{4i-2}(X; \mathbb{Z}/2\mathbb{Z}),$$

the correspondence being $[\phi] \leftrightarrow \phi^*\mathcal{L} \times \phi^*K$, where

$$\mathcal{L} = \sum_{i \geq 1} \ell_{4i} \in H^{4*}(G/TOP; \mathbb{Z}_{(2)})$$

satisfies $(j_0)_*\hat{L} = \mathcal{L}$, j_0 denoting the inclusion $\mathbb{Z} \to \mathbb{Z}_{(2)}$. Letting

$$i: G/TOP \to (G/TOP)_{(2)}$$

denote inclusion, we have:

PROPOSITION 2.2. *The induced map* $i_\#$: $[X, G/TOP]^p \to [X, (G/TOP)_{(2)}]^p$ *is a bijection.*

Note. Here $[X, (G/TOP)_{(2)}]^p$ is the set of homotopy classes of maps $\psi: X \to (G/TOP)_{(2)}$ satisfying $(j_{(2)}\psi)^*(P_{(2)}) = 1$. Before proving this proposition, we first determine the structure of $H^*(X; \mathbb{Z})$ as an algebra.

LEMMA 2.3. *Let* A *denote the ring* $\mathbb{Z}$ *or* $\mathbb{Z}/m\mathbb{Z}$. *Then, as algebras,* $H^*(X; A) \cong H^*(\mathbb{H}P^n; A) \otimes_A H^*(\mathbb{R}P^2; A)$.

Proof. The base of the fibration $\mathbb{R}P^2 \to X \to \mathbb{H}P^n$ is 3-connected so $H^*(X; A) \to H^*(\mathbb{R}P^2; A)$ is a split epimorphism. By the Leray-Hirsch theorem, the isomorphism follows, as a map of $H^*(\mathbb{H}P^n; A)$-modules. But if $z \in H^2(\mathbb{R}P^2; A)$ is any element, then $2z = 0$. Thus $2z^2 = 0$. Now if $A = \mathbb{Z}$, then clearly $z^2 = 0$, since $H^4(X; \mathbb{Z}) \cong \mathbb{Z}$. If $A = \mathbb{Z}/m\mathbb{Z}$, then $z = \rho(a)$ for some $a \in H^2(X; \mathbb{Z})$ (ρ denoting reduction mod m). Thus $a^2 = 0$ and so $z^2 = 0$. Hence, the above isomorphism is a map of algebras. (The reader may try to convince himself that Lemma 2.3 is also true for any commutative ring A.)

COROLLARY 2.4. $H^{4i+3}(X; M) = 0$ *for any abelian group* M.

Proof. If M is cyclic, the result follows from Lemma 2.3. Hence, if M is finitely generated, the result is also true. Now, for M arbitrary, take direct limits.

Proof of Proposition 2.2. First we show that $i_{\#}: [X, G/TOP] \to [X, (G/TOP_{(2)}]$ is injective. Since G/TOP is an H-space, $[X, G/TOP]$ and $[X, (G/TOP)_{(2)}]$ are groups and $i_{\#}$ a homomorphism, so it suffices to show kernel $i_{\#} = (0)$. Let $\phi: X \to G/TOP$ be such that $i_{\#}(\phi) = i \circ \phi$ is null-homotopic. We can assume $i: G/TOP \to (G/TOP)_{(2)}$ is a fibration with homotopy-theoretic fibre F. We have:

$$\begin{array}{ccc} F & \longrightarrow & G/TOP \\ & \overset{\phi}{\nearrow} & \downarrow i \\ X & \xrightarrow[i \circ \phi]{} & (G/TOP)_{(2)} . \end{array}$$

Now ϕ is null-homotopic if and only if a sequence of obstructions in $H^i(X; \pi_i(F))$ vanish, $i \geq 1$. The homotopy sequence of the above fibration yields:

$$\pi_i(F) \cong \begin{cases} \mathbb{Z}_{(2)}/\mathbb{Z} & \text{if } i \equiv 3 \pmod 4 \\ 0 & \text{if } i \not\equiv 3 \pmod 4 . \end{cases}$$

But for $i \equiv 3 \pmod 4$, $H^i(X; \mathbb{Z}_{(2)}/\mathbb{Z}) = 0$, by Corollary 2.4. This proves injectivity. Restricting $i_{\#}$ to Pontryagin normal maps, we show that $i_{\#}$ is surjective. Let $\phi: X \to (G/TOP)_{(2)}$ be a Pontryagin normal map. Recall that G/TOP is, up to homotopy type, the fibre product of i_1 and i_2 in the diagram:

$$\begin{array}{ccccc} & \overset{i}{\nearrow} & (G/TOP)_{(2)} & \overset{i_1}{\searrow} & \\ G/TOP & & & & (G/TOP)_{(\mathbb{Q})} . \\ & \underset{i'}{\searrow} & (G/TOP)_{(odd)} & \underset{i_2}{\nearrow} & \end{array}$$

Now, by Sullivan [4], $[X, (G/TOP)_{(\mathbb{Q})}] \cong [X, (BO)_{(\mathbb{Q})}] \underset{P_*}{\cong} H^{4*}(X; \mathbb{Q})$, where P is the Pontryagin character. Thus, $[X, (G/TOP)_{(\mathbb{Q})}]^P = 0$, and therefore $i_1 \phi$ is null-homotopic. Thus, assuming i_2 is a fibration, we can lift $i_1 \phi$ to a map

$$\tilde{\phi}: X \to (G/TOP)_{(odd)};$$

hence, $i_2 \tilde{\phi} = i_1 \phi$. We therefore have a unique map $\psi: X \to G/TOP$ such that $i' \psi = \tilde{\phi}$ and $i\psi = \phi$. Clearly, $\psi \in [X, G/TOP]^P$, so

$$i_{\#}: [X, G/TOP]^P \to [X, (G/TOP)_{(2)}]^P$$

is onto.

Note that Proposition 2.2 greatly simplifies the calculation of $[X, G/TOP]^P$ because of Sullivan's determination of the homotopy type of $(G/TOP)_{(2)}$. The following proposition leads to the main result of this section.

PROPOSITION 2.5. $[X, (G/TOP)_{(2)}]^P \cong \prod_{i \geq 1} H^{4i-2}(X; \mathbb{Z}/2\mathbb{Z})$.

Proof. Let $\psi: X \to (G/TOP)_{(2)}$ be a Pontryagin normal map. By Lemma 2.1, it suffices to show that $\psi^* \ell_{4i} = 0$ for $i \geq 1$. Recall the relationship between Sullivan's $\mathscr{L}$-class and the universal Hirzebruch class $L = \sum_{i \geq 1} L_{4i} \in H^{4*}(BTOP; \mathbb{Q})$ is simply $j^*L = 8\mathscr{L}$ [5]. Hence, $8\psi^*(\ell_{4i}) = (j\psi)^*(L_{4i}) = 0$, since each L_{4i} is a rational linear combination of the universal Pontryagin classes $P_1, P_2, \cdots, P_i$ and, by hypothesis, $(j\psi)^*(P_i) = 0$ for $i \geq 1$.

Finally, we have:

THEOREM 2.6. $[X^{4n+2}, G/TOP]^p \cong \prod_{i=1}^{n+1} (\mathbb{Z}/2\mathbb{Z})_i$.

Proof. By Propositions 2.2 and 2.5 and Lemma 2.3,

$$[X, G/TOP]^p \cong [X, (G/TOP)_{(2)}]^p \cong \prod_{i=1}^{n+1} H^{4i-2}(X; \mathbb{Z}/2\mathbb{Z}) \cong \prod_{i=1}^{n+1} (\mathbb{Z}/2\mathbb{Z})_i .$$

3. UNIQUENESS OF THE ORBIT SPACE X

Let t be any free involution on $h\mathbb{C}P^{2n+1}$. The following theorem enables us to generalize the results of Section 2 to any homotopy $\mathbb{C}P^{2n+1}$.

THEOREM 3.1. *The orbit space* $h\mathbb{C}P^{2n+1}/\langle t \rangle$ *has the same homotopy type as* X^{4n+2}.

Proof. Let Y^{4n+2} denote $h\mathbb{C}P^{2n+1}/\langle t \rangle$ and $\rho: h\mathbb{C}P^{2n+1} \to Y$ be the natural projection. The 2-skeleton of X is just $X^{(2)} = \mathbb{C}P^1/\langle T \rangle = \mathbb{R}P^2$. Since $\pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$, there exists a map $g: \mathbb{R}P^2 \to Y$ which induces an isomorphism of fundamental groups. Now, the image of $g_\#: \pi_2(\mathbb{R}P^2) \to \pi_2(Y) \cong \mathbb{Z}$ is $d \cdot \pi_2(Y)$ for some integer d. We claim that, for any integer m, the map g may be altered to obtain a π_1-isomorphism $f: \mathbb{R}P^2 \to Y$ such that the image of $f_\#: \pi_2(\mathbb{R}P^2) \to \pi_2(Y)$ is $(d+2m) \cdot \pi_2(Y)$. The map f is defined to be $\alpha \circ (g \vee (\rho \circ h)) \circ i$, where $i: \mathbb{R}P^2 \to \mathbb{R}P^2 \vee S^2$ is the pinch map, $\alpha: Y \vee Y \to Y$ is the folding map, and $h: S^2 \to h\mathbb{C}P^{2n+1}$ has the property:

$$\text{image}(h_\#: \pi_2(S^2) \to \pi_2(h\mathbb{C}P^{2n+1})) = m \cdot \pi_2(h\mathbb{C}P^{2n+1}).$$

The reader may convince himself that f is as required. We now appeal to the following:

LEMMA 3.2. *If* $f: \mathbb{R}P^2 \to Y^{4n+2}$ *is any* π_1*-isomorphism, then*

$$f_\#: \pi_2(\mathbb{R}P^2) \to \pi_2(Y)$$

has image $d \cdot \pi_2(Y)$, *where* d *is an* odd *integer.*

Proof. Suppose not. Then d is an even integer, and by the argument preceding the lemma, we can assume $d = 0$; that is, $f_\#: \pi_2(\mathbb{R}P^2) \to \pi_2(Y)$ is zero. Then f extends to a map $f_1: \mathbb{R}P^3 \to Y$. But

$$H^i(\mathbb{R}P^{4n+3}, \mathbb{R}P^3; \pi_{i-1}(Y)) = 0 \quad \text{if } i \leq 3,$$

and

$$\pi_{i-1}(Y) \cong \pi_{i-1}(\mathbb{C}P^{2n+1}) = 0 \quad \text{if } 3 < i \leq 4n+3.$$

Thus, the groups $H^i(\mathbb{R}P^{4n+3}, \mathbb{R}P^3; \pi_{i-1}(Y)) = 0$ for all i. Hence, f_1 extends to a π_1-isomorphism $f_2: \mathbb{R}P^{4n+3} \to Y$. This implies that

$$f_2^*: H^1(Y, \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^{4n+3}; \mathbb{Z}/2\mathbb{Z})$$

is at least an epimorphism. Hence, if $0 \neq \beta \in H^1(\mathbb{R}P^{4n+3}; \mathbb{Z}/2\mathbb{Z})$, there exists an $\alpha \in H^1(Y; \mathbb{Z}/2\mathbb{Z})$ such that $f_2^*(\alpha) = \beta$, and $\alpha \neq 0$. But $0 \neq \beta^{4n+3} = f_2^*(\alpha^{4n+3})$, so $\alpha^{4n+3} \neq 0$. This is absurd since $\alpha^{4n+3} \in H^{4n+3}(Y^{4n+2}; \mathbb{Z}/2\mathbb{Z}) = 0$.

By Lemma 3.2 and the remarks preceding it, we can construct a map $f: \mathbb{R}P^2 \to Y$ which induces an isomorphism on the first two homotopy groups. This map extends to X, since the obstruction groups $H^i(X, X^{(2)}; \pi_{i-1}(Y))$ are all zero for $i \geq 3$ (for $i = 3$, examine the long exact sequence of $(X, X^{(2)})$ and use Corollary 2.4; for $3 < i \leq 4n+3$, $\pi_{i-1}(Y) = 0$). Hence, we have a map $f: X \to Y$ satisfying $f_\#: \pi_i(X) \cong \pi_i(Y)$ for $i = 1, 2$. Thus $\tilde{f}_\#: \pi_2(\tilde{X}) \to \pi_2(\tilde{Y})$ is an isomorphism, where $\tilde{f}$ lifts f. (Here $\tilde{X}$, $\tilde{Y}$ denote the universal covers of X, Y, respectively.) Thus $\tilde{f}: \tilde{X} \to \tilde{Y}$ is $(4n+2)$-connected, since

$$\pi_i(\tilde{X}) = \pi_i(\tilde{Y}) = \pi_i(\mathbb{C}P^{2n+1}) = 0 \quad \text{if } 3 \leq i \leq 4n+2 .$$

The Hurewicz theorem implies that $\tilde{f}_*: H_i(\tilde{X}; \mathbb{Z}) \to H_i(\tilde{Y}; \mathbb{Z})$ is an isomorphism for $i < 4n+2$ and an epimorphism for $i = 4n+2$. Since

$$H_{4n+2}(\tilde{X}; \mathbb{Z}) \cong \mathbb{Z} \cong H_{4n+2}(\tilde{Y}; \mathbb{Z}),$$

$\tilde{f}_*$ is also an isomorphism for $i = 4n+2$. Hence, $\tilde{f}: \tilde{X} \to \tilde{Y}$ is a homology equivalence $(H_i(\tilde{X}; \mathbb{Z}) = H_i(\tilde{Y}; \mathbb{Z}) = 0$ if $i > 4n+2)$. Whitehead's theorem now applies, so $f: X \to Y$ is a homotopy equivalence.

4. DETERMINATION OF $\mathscr{S}_{TOP}(Y)^P$

Recall that $Y^{4n+2} = h\mathbb{C}P^{2n+1}/\langle S \rangle$ and η denotes the bundle $\rho: h\mathbb{C}P^{2n+1} \to Y$. For any space Z, let $\tilde{Z}$ denote its universal cover. The theory of nonsimply connected surgery gives us the following commutative diagram, with the rows exact:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{S}_{TOP}(h\mathbb{C}P^{2n+1}) & \xrightarrow{\theta} & [h\mathbb{C}P^{2n+1}, G/TOP] & \xrightarrow{\sigma} & L_{4n+2}(1) \\
& \uparrow \rho_1^\# & & \uparrow \rho_2^\# & & \uparrow i_0^\# \\
0 \longrightarrow & \mathscr{S}_{TOP}(Y) & \xrightarrow{\theta} & [Y, G/TOP] & \xrightarrow{\sigma} & L_{4n+2}(\mathbb{Z}/2\mathbb{Z}, -) .
\end{array}$$

The zeroes on the left follow from: $L_{4n+3}(1) = 0$, $L_{4n+3}(\mathbb{Z}/2\mathbb{Z}, -) = 0$ [6, page 162]. Here $Y = Y^{4n+2}$ and $\rho_1 = \rho_2 = \rho$. The map $\rho_1^\#$ is defined by $\rho_1^\#[M, f] = [\tilde{M}, \tilde{f}]$, where $\tilde{f}$ lifts f. The map $\rho_2^\#$ is defined by composition and $i_0^\#$ is the "transfer" induced by the inclusion $i_0: \{1\} \to \mathbb{Z}/2\mathbb{Z}$. The goal of this section (Theorem 4.4) is to show that $\mathscr{S}_{TOP}(Y)^P = \text{kernel } \rho_1^\#$. We first show the following.

THEOREM 4.1. *If* $\pi^\#: [X^{4n+2}, G/TOP] \to [\mathbb{C}P^{2n+1}, G/TOP]$ *is defined by composition, then* kernel $\pi^\# = [X, G/TOP]^P$.

Note. $\pi^\#$ is in fact a homomorphism of groups since G/TOP is an H-space using the operation of Whitney sum.

Proof. Suppose ϕ: X $\to$ G/TOP and $\pi^{\#}(\phi) = \phi\pi$ represents zero in $[\mathbb{C}P^{2n+1}, G/TOP]$. Then, if $\hat{L}$ is as given in Lemma 2.1, we have:

$$0 = 8(\phi\pi)^*\hat{L}_{4i} = (\phi\pi)^*(j^*L_{4i}) = \pi^*(j\phi)^*L_{4i},$$

which implies that $\pi^*(j\phi)^*L_{4i} = 0$, since $H^{4i}(X; \mathbb{Z}) \cong \mathbb{Z}$, $i \geq 1$ (Lemma 2.3). But $\pi^*: H^{4i}(X; \mathbb{Z}) \to H^{4i}(\mathbb{C}P^{2n+1}; \mathbb{Z})$ is an isomorphism for all i (from Lemma 2.3). Hence, $(j\phi)^*L_{4i} = 0$, $i \geq 1$. Thus $(j\phi)^*P_i = 0$ for $i \geq 1$, and so $(j\phi)^*P = 1$ ($P = 1 + P_1 + P_2 + \cdots$ is the universal total Pontryagin class); that is,

$$\phi \in [X, G/TOP]^P.$$

We have shown: kernel $\pi^{\#} \subset [X, G/TOP]^P$. To prove the reverse inclusion, recall [7] that there exists a bijection:

$$s_{4*} \times s_{4*+2}: [\mathbb{C}P^{2n+1}, G/TOP] \to \prod_{i=0}^{n} (\mathbb{Z})_{4i} \times \prod_{i=0}^{n} (\mathbb{Z}/2\mathbb{Z})_{4i+2},$$

where $s_{2i}(f) = \sigma(f|_{\mathbb{C}P^i})$, $1 \leq i \leq 2n+1$, σ denoting surgery obstruction. Moreover, it is not difficult to show that $\sigma(f|_{\mathbb{C}P^{2i}}) = \langle L(\mathbb{C}P^{2i}) \cdot (f^*j^*L - 1), [\mathbb{C}P^{2i}] \rangle$. Now suppose ψ: X $\to$ G/TOP is a Pontryagin normal map. Then $(j\psi)^*L = 1$. Clearly then, the homotopy class $[\psi\pi] \in [\mathbb{C}P^{2n+1}, G/TOP]$ is zero if and only if $s_{2i}(\psi\pi) = 0$ for $1 \leq i \leq 2n+1$. But

$$\begin{aligned} s_{4i}(\psi\pi) &= \langle L(\mathbb{C}P^{2i}) \cdot ((\psi\pi)^*j^*L - 1), [\mathbb{C}P^{2i}] \rangle \\ &= \langle L(\mathbb{C}P^{2i}) \cdot (\pi^*(j\psi)^*L - 1), [\mathbb{C}P^{2i}] \rangle = \langle 0, [\mathbb{C}P^{2i}] \rangle = 0. \end{aligned}$$

$1 \leq i \leq 2n$. Also, $s_{4i+2}(\psi\pi) = \sigma(\psi\pi|_{\mathbb{C}P^{2i+1}}) = i_0^{\#}\sigma(\psi|_{X^{4i+2}})$.

We now use the following:

LEMMA 4.2. $i_0^{\#}: L_{4i+2}(\mathbb{Z}/2\mathbb{Z}, -) \to L_{4i+2}(1)$ *is zero.*

Proof. By periodicity of the L-groups, it suffices to prove this for $i = 0$. Let Λ denote the group ring $\mathbb{Z}\pi$, where $\pi = \{1, T\}$, $T^2 = 1$. Let

$$\alpha = [H, \lambda, \mu] \in L_2(\mathbb{Z}/2\mathbb{Z}, -),$$

where H is a free Λ-module with basis e, f such that $\lambda(e, e) = \lambda(f, f) = 0$, $\lambda(e, f) = 1$, and $\mu(e) = \mu(f) = 1$. Thus, $i_0^{\#}\alpha = [i_0^{\#}H, i_0^{\#}\lambda, i_0^{\#}\mu] \in L_2(1)$ and $i_0^{\#}H$ has a basis $\{e, f, eT, fT\}$. It is easily checked that the mod 2 reductions of e, f, eT and fT (denoted by the same symbols) give a symplectic basis for $i_0^{\#}H \otimes_\Lambda \mathbb{Z}/2\mathbb{Z}$ (with respect to $i_0^{\#}\lambda$, reduced mod 2). Recall [6] that there exists an isomorphism c: $L_2(1) \to \mathbb{Z}/2\mathbb{Z}$, given by the Arf-invariant. Now, by definition,

$$c(i_0^{\#}\alpha) = i_0^{\#}\mu(e)\, i_0^{\#}\mu(f) + i_0^{\#}\mu(eT)\, i_0^{\#}\mu(fT) = 1 \cdot 1 + (-1) \cdot (-1) \cong 0 \pmod 2.$$

Hence, $i_0^{\#}\alpha = 0$ in $L_2(1)$.

Therefore, $s_{4i+2}(\psi\pi) = 0$. We conclude that $[\psi\pi] = 0$, so

$$[X, G/TOP]^p \subset \text{kernel } \pi^{\#}$$

and the proof of Theorem 4.1 is complete.

Since Y is of the same homotopy type as X (Theorem 3.1), we have:

COROLLARY 4.3. kernel $\rho_2^{\#} = [Y, G/TOP]^p$.

The main result of this section is:

THEOREM 4.4. kernel $\rho_1^{\#} = \mathcal{S}_{TOP}(Y)^p$.

Proof. By kernel $\rho_1^{\#}$ we mean those elements $[M, f] \in \mathcal{S}_{TOP}(Y)$ satisfying $\rho_1^{\#}[M, f] = [h\mathbb{C}P^{2n+1}, id]$. Note that θ restricts to a map (also called θ): $\mathcal{S}_{TOP}(Y)^p \to [Y, G/TOP]^p$. Suppose $[M, f] \in \ker \rho_1^{\#}$. Then

$$0 = \theta\rho_1^{\#}[M, f] = \rho_2^{\#}\theta[M, f].$$

Thus $\theta[M, f] \in \ker \rho_2^{\#} = [Y, G/TOP]^p$ (Corollary 4.3). Thus f: M → Y is a Pontryagin equivalence. Hence, $\ker \rho_1^{\#} \subset \mathcal{S}_{TOP}(Y)^p$. If $[M, f] \in \mathcal{S}_{TOP}(Y)^p$, then $\theta[M, f] \in [Y, G/TOP]^p = \ker \rho_2^{\#}$. Thus, $0 = \rho_2^{\#}\theta[M, f] = \theta\rho_1^{\#}[M, f]$, which implies $\rho_1^{\#}[M, f] = [h\mathbb{C}P^{2n+1}, id]$, since θ is injective. This proves the reverse inclusion and therefore the theorem.

5. PROOF OF THEOREM I

In this section, we prove Theorem I of Section 1. Let $c: \mathbb{C}P^{2n+1} \to \mathbb{C}P^{2n+1}$ be the diffeomorphism defined by conjugation of complex coordinates. Then c is equivariant with respect to T and so induces a diffeomorphism c: X → X.

LEMMA 5.1. *If* $c^{\#}: [X, G/TOP] \to [X, G/TOP]$ *is defined by composition, then* $c^{\#} = \text{identity}$.

Proof. We have a commutative diagram:

$$\begin{array}{ccccc} 0 \longrightarrow & [X, G/TOP] & \xrightarrow{i_{\#}} & [X, (G/TOP)_{(2)}] \\ & \downarrow c^{\#} & & \downarrow c^{\#} \\ 0 \longrightarrow & [X, G/TOP] & \xrightarrow{i_{\#}} & [X, (G/TOP)_{(2)}] \end{array}$$

with the rows exact (see the proof of Proposition 2.2), so it suffices to prove that $c^{\#}: [X, (G/TOP)_{(2)}] \to [X, (G/TOP)_{(2)}]$ is the identity. There exists a commutative diagram (by Lemma 2.1):

$$\begin{array}{ccc} [X, (G/TOP)_{(2)}] & \xrightarrow[\cong]{} & \prod_{i \geq 1} H^{4i}(X; \mathbb{Z}_{(2)}) \times \prod_{i \geq 1} H^{4i-2}(X; \mathbb{Z}/2\mathbb{Z}) \\ \downarrow c^{\#} & & \downarrow c_1^* \times c_2^* \\ [X, (G/TOP)_{(2)}] & \xrightarrow[\cong]{} & \prod_{i \geq 1} H^{4i}(X, \mathbb{Z}_{(2)}) \times \prod_{i \geq 1} H^{4i-2}(X; \mathbb{Z}/2\mathbb{Z}), \end{array}$$

where $c_1 = c_2 = c$. Since c_2^* is an isomorphism, it must be the identity because each $H^{4i-2}(X; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Also, $c^*: H^{4i}(\mathbb{C}P^{2n+1}; \mathbb{Z}) \to H^{4i}(\mathbb{C}P^{2n+1}; \mathbb{Z})$ is the identity. (For $i = 1$, $H^4(\mathbb{C}P^{2n+1}; \mathbb{Z})$ is generated by α^2, where $\alpha \in H^2(\mathbb{C}P^{2n+1}; \mathbb{Z})$ is the generator. But $c^*(\alpha) = -\alpha$, so $c^*(\alpha^2) = \alpha^2$; similarly for $2 \leq i \leq n$.) Since $\pi^*: H^{4i}(X; \mathbb{Z}) \to H^{4i}(\mathbb{C}P^{2n+1}; \mathbb{Z})$ is an isomorphism for all i (Lemma 2.3), it follows that $c_1^* =$ identity. Thus $c^\#$ is the identity.

COROLLARY 5.2. $c_*: \mathscr{S}_{TOP}(X) \to \mathscr{S}_{TOP}(X)$ *is the identity, where* $c_*[M, f] = [M, c \circ f]$.

Proof. We have a commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathscr{S}_{TOP}(X) & \xrightarrow{\theta} & [X, G/TOP] \\ & & \downarrow c_* & & \downarrow c^\# \\ 0 & \longrightarrow & \mathscr{S}_{TOP}(X) & \xrightarrow{\theta} & [X, G/TOP], \end{array}$$

where the rows are part of the surgery exact sequence. Now use Lemma 5.1.

PROPOSITION 5.3. *Let* M *have the same homotopy type as* X *and let* h: M → M *be a homotopy equivalence. Then* h *is homotopic to a homeomorphism.*

Proof. Using Corollary 5.2, one can easily show that there exists a homeomorphism $\gamma: M \to M$ such that $\gamma_* = -\mathrm{id}: \pi_2(M) \to \pi_2(M)$. We show that either $h \simeq \mathrm{id}$ or $h \simeq \gamma$. Let g denote either id or γ. The first obstruction $O_1(h, g)$ for making h homotopic to g lies in $H^1(M; \pi_1(M)) \cong \mathrm{Hom}(H_1(M), \pi_1(M)) \cong \mathrm{Hom}(\pi_1(M), \pi_1(M))$, and it is well known [8] that $O_1(h, g)$ corresponds to $h_* - g_* \in \mathrm{Hom}(\pi_1(M), \pi_1(M))$ under the above isomorphisms. But clearly $h_* - g_* = 0$, so $O_1(h, g) = 0$. The second obstruction $O_2(h, g)$ lies in $H^2(M; \pi_2(M)^t)$, where t denotes the action of $\pi_1(M)$ on $\pi_2(M)$. Let Λ denote the integral group ring $\mathbb{Z}\pi_1(M)$. Then if j is given as in the coefficient sequence: $0 \to \mathbb{Z}^t \xrightarrow{j} \Lambda \to \mathbb{Z} \to 0$, we have a monomorphism

$$j_*: H^2(M; \pi_2(M)^t) \to H^2(M; \Lambda) = H^2(\tilde{M}; \pi_2(M)).$$

But $H^2(\tilde{M}; \pi_2(M)) \cong H^2(\tilde{M}; \pi_2(\tilde{M})) \cong \mathrm{Hom}(\pi_2(\tilde{M}), \pi_2(\tilde{M}))$ and, as before, $O_2(\tilde{h}, \tilde{g})$ corresponds to $\tilde{h}_* - \tilde{g}_* \in \mathrm{Hom}(\pi_2(\tilde{M}), \pi_2(\tilde{M}))$ under the above isomorphisms, where $\tilde{h}$ and $\tilde{g}$ are liftings of h, g, respectively, to $\tilde{M}$. But $j^*O_2(h, g) = O_2(\tilde{h}, \tilde{g})$. Now if $\tilde{h}$ is orientation preserving, then $\tilde{h}_* - \tilde{id}_* = 0$, so $O_2(\tilde{h}, \tilde{g}) = 0$. Thus $O_2(h, g) = 0$. Similarly, if $\tilde{h}$ is orientation reversing, then $\tilde{h}_* - \tilde{\gamma}_* = 0$, so $O_2(h, g) = 0$. Since the groups $H^i(M, \pi_i(M))$ are zero for $i \geq 3$ ($\pi_i(M) = 0$ if $3 \leq i \leq 4n + 2$; $H^i(M; A) = 0$ if $i > 4n + 2$, where A is any abelian group), there are no further obstructions. Hence, either $h \simeq \mathrm{id}$ or $h \simeq \gamma$ and the proof of Proposition 5.3 is complete.

Define a map $\psi: \mathscr{S}_{TOP}(Y^{4n+2})^p \to \mathrm{Free\ Inv}(h\mathbb{C}P^{2n+1})$ as follows: let $[M, f] \in \mathscr{S}_{TOP}(Y)^p$. By Theorem 4.4, $[\tilde{M}, \tilde{f}] = [h\mathbb{C}P^{2n+1}, \mathrm{id}]$, so $\tilde{f}$ is homotopic to a homeomorphism $g: \tilde{M} \to h\mathbb{C}P^{2n+1}$. Let $\psi[M, f]$ be the conjugacy class of gtg^{-1}, where $t \in \pi_1(M) \cong \mathbb{Z}/2\mathbb{Z}$ is the generator, considered as a covering transformation: $\tilde{M} \to \tilde{M}$. Clearly, $\psi[M, f]$ is independent of the choice of g. We first show that ψ is surjective. Let $S' \in \mathrm{Free\ Inv}(h\mathbb{C}P^{2n+1})$ and let M denote the orbit space $h\mathbb{C}P^{2n+1}/\langle S' \rangle$. By Theorem 3.1, there exists a homotopy equivalence $f: M \to Y$. Hence, $[M, f] \in \mathscr{S}_{TOP}(Y)$. Since any homotopy equivalence: $h\mathbb{C}P^{2n+1} \to h\mathbb{C}P^{2n+1}$ is homotopic to a homeomorphism [5], $[h\mathbb{C}P^{2n+1}, \mathrm{id}] = [\tilde{M}, \tilde{f}] = \rho_2^\#[M, f]$, so

$[M, f] \in$ kernel $\rho_2^{\#} = \mathscr{S}_{TOP}(Y)^p$. Clearly $\psi[M, f] \sim S'$. We now show that ψ is injective. Let $[M_i, f_i] \in \mathscr{S}_{TOP}(Y)^p$, $i = 1, 2$. Suppose $\psi[M_1, f_1] = \psi[M_2, f_2]$. Let $g_i\colon \tilde{M}_i \to h\mathbb{C}P^{2n+1}$, $i = 1, 2$, be homeomorphisms such that $g_i \simeq \tilde{f}_i$, where $\tilde{f}_i$ is a lifting of f_i to $\tilde{M}_i$. Let $t_i \in \pi_1(M_i) \cong \mathbb{Z}/2\mathbb{Z}$ be generators, $i = 1, 2$. By hypothesis, $g_1 t_1 g_1^{-1} \sim g_2 t_2 g_2^{-1}$; that is, there exists a homeomorphism

$$H\colon h\mathbb{C}P^{2n+1} \to h\mathbb{C}P^{2n+1}, \qquad \text{such that } Hg_1 t_1 g_1^{-1} = g_2 t_2 g_2^{-1} H.$$

If we let $h = g_2^{-1} H g_1$, then $h\colon \tilde{M}_1 \to \tilde{M}_2$ is a homeomorphism which is equivariant. Thus h induces a homeomorphism (also called h) $M_1 \to M_2$. If $\bar{f}_2$ is a homotopy inverse of f_2, then $h^{-1}\bar{f}_2 f_1\colon M_1 \to M_1$ is a homotopy equivalence. By Proposition 5.3, there exists a homeomorphism $g\colon M_1 \to M_1$ such that $h^{-1}\bar{f}_2 f_1 \simeq g$. Set $G = hg$. Then $G\colon M_1 \to M_2$ is a homeomorphism, and

$$f_2 G = f_2 hg \simeq f_2 hh^{-1}\bar{f}_2 f_1 = f_2\bar{f}_2 f_1 \simeq f_1 .$$

Hence, $[M_1, f_1]$ and $[M_2, f_2]$ are equal in $\mathscr{S}_{TOP}(Y)^p$. This completes the proof of Theorem I.

6. PROOF OF THEOREM II

According to Rourke and Sullivan [3], the surgery obstruction

$$\sigma\colon [Y, G/TOP] \to L_{4n+2}(\mathbb{Z}/2\mathbb{Z}, -)$$

is given by the formula: $\sigma(f) = \langle V^2 \cdot f^*K, [Y]\rangle$, where V is the total Wu class of Y and where K is as in Lemma 2.1. Primitivity of the class K implies that σ is a homomorphism. Recall (Theorems 2.6 and 3.1) that there exists a bijection

$$[Y^{4n+2}, G/TOP]^p \underset{\cong}{\longleftrightarrow} \prod_{i=1}^{n+1} H^{4i-2}(Y, \mathbb{Z}/2\mathbb{Z}) \cong \prod_{i=1}^{n+1} (\mathbb{Z}/2\mathbb{Z})_i ,$$

the correspondence being $\phi \longleftrightarrow \phi^*K$. Hence, there exists a Pontryagin normal map $f\colon Y \to G/TOP$ satisfying: $f^*k_{4i-2} = 0$ for $1 \le i \le n$ and $\langle f^*k_{4n+2}, [Y]\rangle \ne 0$. In this case, Sullivan's formula yields: $\sigma(f) = \langle f^*k_{4n+2}, [Y]\rangle \ne 0$. Hence, σ is surjective (since $L_{4n+2}(\mathbb{Z}/2\mathbb{Z}, -) \cong \mathbb{Z}/2\mathbb{Z}$ [7]). The surgery exact sequence $0 \to \mathscr{S}_{TOP}(Y) \xrightarrow{\theta} [Y, G/TOP] \xrightarrow{\sigma} L_{4n+2}(\mathbb{Z}/2\mathbb{Z}, -)$ implies that kernel $\sigma \cong \mathscr{S}_{TOP}(Y)$. Thus, $\mathscr{S}_{TOP}(Y)$ has the structure of an abelian group. Moreover, Theorem 4.4 implies that $\mathscr{S}_{TOP}(Y)^p$ is a subgroup. Hence, restricting σ to Pontryagin normal maps, we have an exact sequence of abelian groups:

$$0 \to \mathscr{S}_{TOP}(Y)^p \xrightarrow{\theta} [Y, G/TOP]^p \xrightarrow{\sigma} L_{4n+2}(\mathbb{Z}/2\mathbb{Z}, -) \to 0 .$$

Combining Theorem I and the above comments, this exact sequence becomes:

$$0 \to \text{Free Inv}(h\mathbb{C}P^{2n+1}) \to \prod_{i=1}^{n+1} (\mathbb{Z}/2\mathbb{Z})_i \to \mathbb{Z}/2\mathbb{Z} \to 0 .$$

Corollary A (Section 1) follows easily from this.

7. PROOF OF THEOREM III

Let $\kappa = \prod_{i=1}^{n+1} k_{4i-2}$, where k_{4i-2}: $[\mathbb{C}P^{2n+1}, G/TOP] \to \mathbb{Z}/2\mathbb{Z}$ denotes the function $k_{4i-2}(f) = \langle f^*K, x_{4i-2} \rangle$; x_{4i-2} is Poincaré dual to the generator $\zeta_{2i-1} \in H^0(\mathbb{C}P^{2i-1}; \mathbb{Z}/2\mathbb{Z})$; and $K = \sum_{i\geq 1} k_{4i-2}$ is given in Lemma 2.1.

LEMMA 7.1. $[X^{4n+2}, G/TOP] \xrightarrow{\pi^\#} [\mathbb{C}P^{2n+1}, G/TOP] \xrightarrow{\kappa} \prod_{i=1}^{n+1} (\mathbb{Z}/2\mathbb{Z})_{4i-2}$ *is an exact sequence of abelian groups.*

Proof. It has been noted (Section 4) that $\pi^\#$ is a homomorphism. Also, the primitivity of K again implies that κ is a homomorphism. We first show: $\kappa\pi^\# = 0$. If ϕ: $X \to G/TOP$, then $k_{4i-2}(\pi^\#\phi) = k_{4i-2}(\phi\pi) = \langle (\phi\pi)^*K, x_{4i-2} \rangle = 0$ since π^*: $H^{4i-2}(X; \mathbb{Z}/2\mathbb{Z}) \to H^{4i-2}(\mathbb{C}P^{2n+1}; \mathbb{Z}/2\mathbb{Z})$ is identically zero (from the Gysin sequence of the 0-sphere bundle $\mathbb{C}P^{2n+1} \to X$, with $\mathbb{Z}/2\mathbb{Z}$-coefficients). Hence, image $\pi^* \subset$ kernel κ. Now suppose ψ: $\mathbb{C}P^{2n+1} \to G/TOP$ satisfies $\kappa(\psi) = 0$. Let K_0 denote the product $\prod_{i=1}^{n+1} K(\mathbb{Z}/2\mathbb{Z}, 4i-2)$. The obvious map j: $G/TOP \to K_0$ will be assumed to be a fibration, with fibre F. Note that the homotopy sequence of this fibration yields:

$$\pi_i(F) \cong \begin{cases} \mathbb{Z} & i \equiv 0 \pmod 4 \\ 0 & i \not\equiv 0 \pmod 4. \end{cases}$$

If $\iota_{4i-2} \in H^{4i-2}(K(\mathbb{Z}/2\mathbb{Z}, 4i-2), \mathbb{Z}/2\mathbb{Z}) \cong [K(\mathbb{Z}/2\mathbb{Z}, 4i-2), K(\mathbb{Z}/2\mathbb{Z}, 4i-2)]$ corresponds to the identity map under the above isomorphism, then

$$\left\langle (j\psi)^*\left(\sum_{i=1}^{n+1} \iota_{4i-2}\right), x_{4i-2}\right\rangle = \langle \psi^*K, x_{4i-2} \rangle = \kappa(\psi) = 0.$$

Thus the composite $\mathbb{C}P^{2n+1} \xrightarrow{\psi} G/TOP \xrightarrow{j} K_0$ is null-homotopic, so ψ factors through F: there exists a map ψ_0: $\mathbb{C}P^{2n+1} \to F$ such that $i\psi_0 = \psi$, where i: $F \to G/TOP$ is the inclusion map. The obstructions to extending ψ_0 to X^{4n+2} lie in the groups $H^i(X, \mathbb{C}P^{2n+1}; \pi_{i-1}(F)) = 0$ unless $i - 1 \equiv 0 \pmod 4$. But, examining the long exact sequence of $(X, \mathbb{C}P^{2n+1})$, we see that

$$H^{4i+1}(X, \mathbb{C}P^{2n+1}; \pi_{4i}(F)) = 0.$$

(Recall from Corollary 2.4 that π^*: $H^{4i}(X, \mathbb{Z}) \to H^{4i}(\mathbb{C}P^{2n+1}; \mathbb{Z})$ is an isomorphism and $H^{4i+1}(X; \mathbb{Z}) = 0$.) Thus, there exists a map ψ': $X \to F$ such that $\psi'\pi = \psi_0$. Clearly, $i\psi'$: $X \to G/TOP$ satisfies $\pi^\#(i\psi') = \psi$. Hence, kernel $\kappa \subset$ image $\pi^\#$.

LEMMA 7.2. *If* $\kappa = \prod_{i=1}^{n} k_{4i-2}$, *then*

$$\mathcal{S}_{TOP}(X) \xrightarrow{\pi^\#} \mathcal{S}_{TOP}(\mathbb{C}P^{2n+1}) \xrightarrow{\kappa} \prod_{i=1}^{n} (\mathbb{Z}/2\mathbb{Z})_{4i-2} \longrightarrow 0$$

is an exact sequence of abelian groups.

Proof. Consider the following commutative diagram, the rows being exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{i=1}^{n} (\mathbb{Z}/2\mathbb{Z})_{4i-2} & \longrightarrow & \prod_{i=1}^{n+1} (\mathbb{Z}/2\mathbb{Z})_{4i-2} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
 & & \uparrow \kappa & & \uparrow \kappa & & & & \\
0 & \longrightarrow & \mathscr{S}_{TOP}(\mathbb{C}P^{2n+1}) & \xrightarrow{\theta} & [\mathbb{C}P^{2n+1}, G/TOP] & \xrightarrow{\sigma} & L_{4n+2}(1) & & \\
 & & \uparrow \pi_1^{\#} & & \uparrow \pi_2^{\#} & & \uparrow i_0^{\#} & & \\
0 & \longrightarrow & \mathscr{S}_{TOP}(X) & \xrightarrow{\theta} & [X, G/TOP] & \xrightarrow{\sigma} & L_{4n+2}(\mathbb{Z}/2\mathbb{Z}, -1), & &
\end{array}$$

where $\pi_1 = \pi_2 = \pi$, $X = X^{4n+2}$. Now $\kappa\pi_1^{\#} = 0$, since $\kappa\pi_2^{\#} = 0$ by Lemma 7.1. We need only show: kernel $\kappa \subset$ image $\pi_1^{\#}$. Suppose $x \in \mathscr{S}_{TOP}(\mathbb{C}P^{2n+1})$ and $\kappa(x) = 0$. Let $\theta(x) = y \in [\mathbb{C}P^{2n+1}, G/TOP]$. Then $\kappa(y) = 0$. By exactness (Lemma 7.1), there exists $z \in [X, G/TOP]$ such that $\pi_2^{\#} z = y$. Choose $f \in [X, G/TOP]^p$ as in Section 6. Then $\sigma(f)$ is a generator of $L_{4n+2}(\mathbb{Z}/2\mathbb{Z}, -)$. We have:

$$\pi_2^{\#}(f + z) = \pi_2^{\#}(f) + \pi_2^{\#}(z) = \pi_2^{\#}(z) \qquad \text{(Theorem 4.1)}.$$

If $\sigma(z) = 0$, replace z by $f + z$. Hence, we can assume that $\sigma(z) = 0$, so there exists $x_0 \in \mathscr{S}_{TOP}(X)$ with $\theta(x_0) = z$. But $\theta\pi_1^{\#}(x_0) = \pi_2^{\#}\,\theta(x_0) = y$. Since $\theta(x) = y$ and θ is injective, we conclude that $\pi_1^{\#}(x_0) = x$, and so kernel $\kappa \subset$ image $\pi_1^{\#}$. Finally, κ is onto, by calculations of Sullivan [5].

It is clear that the proofs of Lemmas 7.1 and 7.2 also work if $\mathbb{C}P^{2n+1}$ is replaced by $h\mathbb{C}P^{2n+1}$ and X is replaced by Y. Theorem III now follows from Lemma 7.2 and Theorem 4.4. Corollary B is an immediate consequence of Theorem III.

8. EXTENSION TO THE PL CATEGORY

We now briefly indicate how these results may be extended to the piecewise linear category. Most of the details are left to the reader.

Theorem I remains unchanged if PL replaces TOP. This is true since Theorem 4.1 is still valid with PL replacing TOP. However, the proof of Theorem 4.1 must be slightly altered. Here one needs to show that

$$[\mathbb{C}P^N, G/PL] \to [\mathbb{C}P^N, G/TOP]$$

is a monomorphism. Theorem 4.4 is then true with PL replacing TOP, and the proof of Theorem I proceeds as before.

In Proposition 8.3, we show that $[Y, G/PL]^p \cong [Y, G/TOP]^p$, so Theorem III is still valid if PL replaces TOP. Hence, Corollary A is also true in the PL category.

Lemma 7.1 is true word for word if PL replaces TOP. But Lemma 7.2 must be altered to read:

LEMMA 7.2'. *If* $\kappa = \prod_{i=2}^{n} k_{4i-2}$, *then*

$$\mathscr{S}_{PL}(X) \xrightarrow{\pi^{\#}} \mathscr{S}_{PL}(\mathbb{C}P^{2n+1}) \xrightarrow{\kappa} \prod_{i=2}^{n} (\mathbb{Z}/2\mathbb{Z})_{4i-2} \longrightarrow 0$$

is an exact sequence of abelian groups.

It will become evident why k_2 does not appear here after one reads the proof of Proposition 8.3 below. (Obviously, the diagram given in the proof of Lemma 7.2 must be similarly changed in order to prove Lemma 7.2'.) Recall the following result, due to Sullivan:

LEMMA 8.1 [4]. *There exists a homotopy commutative diagram (localized at* 2):

$$\begin{array}{ccc} G/PL & \longrightarrow & E \times \prod_{n \geq 6} K(\pi_n(G/PL), n) \\ \downarrow & & \downarrow p \times id \\ G/TOP & \xrightarrow{K \times \hat{L}} & \prod_{n \geq 2} K(\pi_n(G/TOP), n) \end{array} ,$$

where E *is determined by the fibration* $K(\mathbb{Z}, 4) \xrightarrow{i} E \xrightarrow{p} K(\mathbb{Z}/2\mathbb{Z}, 2)$ *with* K-*invariant* $\delta Sq^2(\iota) \in H^5(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$ *and* $p = k_2' \times \ell_4'$, *with*

$$k_2' \in H^2(E; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \quad \textit{and} \quad \ell_4' \in H^4(E; \mathbb{Z}) \cong \mathbb{Z}$$

being generators.

Remark. E is an H-space since $E = \Omega E_3$, where E_3 is the 2-stage Postnikov system obtained as the fibre in the fibration $E_3 \to K(\mathbb{Z}/2\mathbb{Z}, 3) \to K(\mathbb{Z}, 6)$ with K-invariant $\delta Sq^2(\iota_3)$.

Hence, localized at 2, we have:

$$[Y, G/PL] \cong [Y, E] \times \prod_{i \geq 2} H^{4i}(Y; \mathbb{Z}) \times \prod_{i \geq 2} H^{4i-2}(Y, \mathbb{Z}/2\mathbb{Z}).$$

LEMMA 8.2. *There exists an isomorphism of groups (localized at* 2):

$$[Y, E] \cong A = \{(a, b) \in H^2(Y; \mathbb{Z}/2\mathbb{Z}) \times H^4(Y; \mathbb{Z}): a^2 = \rho(b)\},$$

ρ *denoting reduction* mod 2.

Proof. There exists a map $g: E \to K(\mathbb{Z}, 4)$ satisfying $(gi)^*(\iota_4) = 2\iota_4$, where $\iota_4 \in H^4(K(\mathbb{Z}, 4); \mathbb{Z})$ is the canonical generator (this map is determined by ℓ_4'). Let $\iota_2 \in H^2(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{Z}/2\mathbb{Z})$ be the generator. Define a map $\psi: [Y, E] \to A$ as follows: $\psi(f) = ((pf)^*(\iota_2), (gf)^*(\iota_4))$. One easily checks that ψ is an epimorphism. Suppose $\psi(f) = (0, 0)$. By the covering homotopy property we can assume pf is constant, so f factors through $K(\mathbb{Z}, 4)$: there exists a map $f_1: Y \to K(\mathbb{Z}, 4)$ such that $if_1 = f$. But $0 = (gf)^*(\iota_4) = (gif_1)^*(\iota_4) = f_1^*(gi)^*(\iota_4) = 2f_1^*(\iota_4) \in H^4(Y; \mathbb{Z}) \cong \mathbb{Z}$, so $f_1^*(\iota_4) = 0$. This implies that f is null-homotopic, and therefore ψ is also a monomorphism.

Finally, we have:

PROPOSITION 8.3. $[Y, G/PL]^p \cong [Y, G/TOP]^p$.

Proof. As before (Proposition 2.2), one can prove that

$$[Y, G/PL]^p \cong [Y, (G/PL)_{(2)}]^p \cong [Y, E_{(2)}]^p \times \prod_{i \geq 2} H^{4i-2}(Y, \mathbb{Z}/2\mathbb{Z}).$$

(Here $[Y, E]^p$ is defined in the obvious way.) Thus, we need only show that $[Y, E_{(2)}]^p \cong H^2(Y, \mathbb{Z}/2\mathbb{Z})$. Consider the composite (localized at 2):

$$E \xrightarrow{i} G/PL \xrightarrow{j} BPL.$$

Now, if $f: Y \to E$ satisfies $(jif)^*(P) = 1$, then, in particular, $(jif)^*(L_4) = 0$. We have $0 = (jif)^*L_4 = f^*(ji)^*L_4 = f^*\ell_4' = (gf)^*(\iota_4)$, where g is given in the proof of Lemma 8.2. Hence, restricting the map ψ (see the proof of Lemma 8.2) to $[Y, E]^p$, we have an isomorphism $\psi: [Y, E]^p \to H^2(Y; \mathbb{Z}/2\mathbb{Z})$.

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Department of Mathematics
University of Notre Dame
Notre Dame, Indiana 46556

Current mailing address:
Department of Mathematics
Pembroke State University
Pembroke, North Carolina 28372