EXTRINSIC SPHERES IN KÄHLER MANIFOLDS, II

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1. INTRODUCTION

An n-dimensional submanifold M^n of an arbitrary Riemannian manifold is called an *extrinsic sphere* if it is umbilical and has parallel mean curvature vector $H \neq 0$. In the first part of this series, we have proved that a complete, simply connected, even-dimensional extrinsic sphere in any Kähler manifold is isometric to an ordinary sphere if its normal connection is flat. Moreover, we have proved that there exist no complete orientable extrinsic spheres of codimension 2 in any positively (or negatively) curved Kähler manifold.

On the other hand, it is known that there exist complete extrinsic spheres of codimension $p > m$ in 2m-dimensional complex projective space $P^{2m}(\mathbb{C})$ and complex sphere Q^{2m}. Since $P^{2m}(\mathbb{C})$ and Q^{2m} are Hermitian symmetric spaces of rank 1 and 2, and Hermitian symmetric spaces are the most important class of Kähler manifolds, it seems to be interesting to determine the codimensions of extrinsic spheres in all irreducible Hermitian symmetric spaces. In this paper, we shall study such codimensions.

We shall use the same notations as in the first part of this series [8], unless mentioned otherwise.

2. HERMITIAN SYMMETRIC SPACES

Let G/K be an irreducible Hermitian symmetric space with an involution τ, and let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Then the eigenvalues of τ as a linear transformation of $\mathfrak{g}$ are 1 and -1, and $\mathfrak{k}$ is the eigenspace for 1. Let $\mathfrak{m}$ be the eigenspace for -1. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, and $\mathfrak{m}$ can be regarded as the tangent space of G/K at the origin. This decomposition of $\mathfrak{g}$ is called the *canonical decomposition* of $\mathfrak{g}$. On the Lie algebra $\mathfrak{g}$, the *Killing-Cartan form* ϕ is given by

$$\phi(X, Y) = \operatorname{tr}(\operatorname{ad} X \cdot \operatorname{ad} Y), \qquad \text{where } X, Y \in \mathfrak{g}.$$

The restriction of ϕ to $\mathfrak{m}$ defines a G-invariant Hermitian metric on G/K. It is well known that every G-invariant Hermitian metric is Kählerian and it is a constant multiple of the Killing-Cartan form.

The irreducible Hermitian symmetric spaces have been classified (up to constant multiples for the metric) by É. Cartan. Throughout this paper we shall assume that the maximal (respectively, minimal) holomorphic sectional curvatures are 1 (respectively, -1) for the Hermitian symmetric spaces of compact (respectively, noncompact) type. The dimensions, Ricci tensors $\tilde{S}$, metric tensors $\tilde{g}$, and their holomorphic sectional curvatures $\tilde{H}$ are given in Tables I and II [2, 7, 10]. We shall follow the notations of Helgason [9] for Lie groups and Hermitian symmetric spaces.

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TABLE I. Irreducible Hermitian Symmetric Spaces of Compact Type

$\tilde{M}$		dimension	rank	$\tilde{S}$	$\tilde{H}$
AIII	$SU(q+q')/S(U_q \times U_{q'})$, $q \geq q' \geq 1$	$2qq'$	q'	$\frac{1}{2}(q+q')\tilde{g}$	$1 \geq \tilde{H} \geq 1/q'$
BDI	$SO(2+q)/SO(2) \times SO(q)$, $q \geq 3$	$2q$	2	$\frac{q}{2}\tilde{g}$	$1 \geq \tilde{H} \geq 1/2$
DIII	$SO(2q)/U(q)$, $q \geq 5$	$q(q-1)$	$[q/2]$	$(q-1)\tilde{g}$	$1 \geq \tilde{H} \geq 1/[q/2]$
CI	$Sp(q)/U(q)$, $q \geq 3$	$q(q+1)$	q	$\frac{1}{2}(q+1)\tilde{g}$	$1 \geq \tilde{H} \geq 1/q$
EIII	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbb{R})$	32	2	$3\tilde{g}$	$1 \geq \tilde{H} \geq 1/2$
EVII	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbb{R})$	54	3	$\frac{9}{2}\tilde{g}$	$1 \geq \tilde{H} \geq 1/3$

TABLE II. Irreducible Hermitian Symmetric Spaces of Noncompact Type

$\tilde{M}$		dimension	rank	$\tilde{S}$	$\tilde{H}$
AIII	$SU(q, q')/S(U_q \times U_{q'})$, $q \geq q' \geq 1$	$2qq'$	q'	$-\frac{1}{2}(q+q')\tilde{g}$	$-1 \leq \tilde{H} \leq -1/q'$
BDI	$SO_0(2, q)/SO(2) \times SO(q)$, $q \geq 3$	$2q$	2	$-\frac{q}{2}\tilde{g}$	$-1 \leq \tilde{H} \leq -1/2$
DIII	$SO^*(2q)/U(q)$, $q \geq 5$	$q(q-1)$	$[q/2]$	$(1-q)\tilde{g}$	$-1 \leq \tilde{H} \leq -1/[q/2]$
CI	$Sp(q, \mathbb{R})/U(q)$, $q \geq 3$	$q(q+1)$	q	$-\frac{1}{2}(q+1)\tilde{g}$	$-1 \leq \tilde{H} \leq -1/q$
EIII	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + \mathbb{R})$	32	2	$-3\tilde{g}$	$-1 \leq \tilde{H} \leq -1/2$
EVII	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 + \mathbb{R})$	54	3	$-\frac{9}{2}\tilde{g}$	$-1 \leq \tilde{H} \leq -1/3$

3. POSITIVELY OR NEGATIVELY CURVED KÄHLER MANIFOLDS

Let M^n be an n-dimensional extrinsic sphere in a 2m-dimensional Kähler manifold $\tilde{M}^{2m}$. Then by definition, the second fundamental form σ and the mean curvature vector H satisfy

$$\sigma(X, Y) = g(X, Y)H, \quad D_X H = 0, \quad \text{and} \quad H \neq 0, \tag{3.1}$$

for all vectors X, Y tangent to M^n. Thus the covariant derivative of σ satisfies

$$(3.2) \qquad (\overline{\nabla}_X\sigma)(Y, Z) \equiv D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = 0.$$

Substituting this into the equation of Codazzi, we see that the curvature tensor $\tilde{R}$ of $\tilde{M}^{2m}$ satisfies

$$(3.3) \qquad \tilde{R}(X, Y; Z, \xi) = 0$$

for all vectors X, Y, Z tangent to M^n and ξ normal to M^n.

On the other hand, since H is parallel and the second fundamental tensors commute, the equation of Ricci gives

$$(3.4) \qquad \tilde{R}(X, Y; H, \xi) = 0.$$

Combining (3.3) and (3.4), we have

$$(3.5) \qquad \tilde{R}(X, JX; H, JH) = 0$$

for all $X \in TM^n \cap J(TM^n)$, where TM^n is the tangent bundle of M^n and J is the complex structure of $\tilde{M}^{2m}$. Let $\tilde{K}(X, H)$ denote the sectional curvature of $\tilde{M}^{2m}$ on the plane section spanned by X and H. Then (3.5) and the first Bianchi identity imply the following.

LEMMA 1. *Let* M^n *be an extrinsic sphere in a Kähler manifold* $\tilde{M}^{2m}$. *Then*

$$(3.6) \qquad \tilde{K}(X, H) + \tilde{K}(JX, H) = 0$$

for any vector $X \in TM^n \cap J(TM^n)$.

From Lemma 1, we get immediately the following.

THEOREM 1. *Let* $\tilde{M}^{2m}$ *be a positively (or negatively) curved* 2m-*dimensional Kähler manifold. Then* $\tilde{M}^{2m}$ *admits no extrinsic spheres of codimension* $p < m$.

This theorem generalizes the corollary of [8].

For later use, we mention the following.

LEMMA 2. *Let* N_1 *and* N_2 *be orthonormal vectors in a Kähler manifold* $\tilde{M}^{2m}$ *such that* $g(N_1, JN_2) = 0$. *Then*

$$\tilde{K}(N_1, N_2) + \tilde{K}(N_1, JN_2) = \frac{1}{4}[\tilde{H}(N_1 + N_2) + \tilde{H}(N_1 - N_2) + \tilde{H}(N_1 + JN_2) + \tilde{H}(N_1 - JN_2) - \tilde{H}(N_1) - \tilde{H}(N_2)].$$

This lemma can be found in [6].

4. EXTRINSIC SPHERES IN HERMITIAN SYMMETRIC SPACES

Let M^n be an n-dimensional extrinsic sphere in a 2m-dimensional irreducible Hermitian symmetric space $\tilde{M}^{2m}$. Then it is well known that the sectional curvatures of $\tilde{M}^{2m}$ are nonnegative (respectively, nonpositive) if $\tilde{M}^{2m}$ is of compact type (respectively, noncompact type). (See, for instance, [9, p. 205].) Let $x \in M^n$ and $V = T_x^\perp \oplus JT_x^\perp$, where $T_x^\perp$ denotes the normal space of M^n in $\tilde{M}^{2m}$ at x. Then V is a complex vector space whose dimension is less than or equal to $2p \equiv 4m - 2n$. We denote by $2s$ the dimension of V. Let $N_1, \cdots, N_s, JN_1, \cdots, JN_s$ be $2s$ orthonormal vectors in V such that $N_1 = H/|H|$. Then by Lemma 1, the Ricci tensor of $\tilde{M}^{2m}$ satisfies

$$(4.1)\qquad \tilde{S}(N_1, N_1) = \sum_{r=2}^{s} [\tilde{K}(N_1, N_r) + \tilde{K}(N_1, JN_r)] + \tilde{H}(N_1),$$

where $\tilde{H}(N_1)$ denotes the holomorphic sectional curvature of $\tilde{M}^{2m}$ at N_1. Since $g(N_1, JN_r) = g(N_1, N_r) = 0$, (4.1) and Lemma 2 imply

$$(4.2)\qquad \tilde{S}(N_1, N_1) = \frac{5-s}{4}\tilde{H}(N_1) + \frac{1}{4}\sum_{r=2}^{s} [\tilde{H}(N_1 + N_r) + \tilde{H}(N_1 - N_r) + \tilde{H}(N_1 + JN_r) + \tilde{H}(N_1 - JN_r) - \tilde{H}(N_r)].$$

LEMMA 3. *Let* M^n *be an extrinsic sphere in a compact* (*respectively, noncompact*) *irreducible Hermitian symmetric space of rank* 2. *Then*

$$(4.3)\qquad \tilde{H}(X) = \tilde{H}(N_1) = 1 \quad \text{(respectively, } -1),$$

where X *is any unit vector in* $TM^n \cap J(TM^n)$ *and* $N_1 = H/|H|$.

Proof. Let M^n be an extrinsic sphere in an irreducible Hermitian symmetric space $\tilde{M}^{2m}$ of rank 2. Then Lemma 1 implies

$$(4.4)\qquad \tilde{K}(H, X) + \tilde{K}(H, JX) = 0,$$

for any unit vector $X \in TM^n \cap J(TM^n)$. Thus Lemma 2 gives

$$(4.5)\qquad \tilde{H}(N_1 + X) + \tilde{H}(N_1 - X) + \tilde{H}(N_1 + JX) + \tilde{H}(N_1 - JX) = \tilde{H}(N_1) + \tilde{H}(X).$$

On the other hand, Tables I and II tell us that the holomorphic sectional curvature $\tilde{H}$ of $\tilde{M}^{2m}$ is (1/2)-pinching. Thus, (4.5) implies (4.3).

LEMMA 4. *Let* $\tilde{M}^{2m}$ *be a compact* (*respectively, noncompact*) *irreducible Hermitian symmetric space of rank* 2. *Then*

$$\dim \{X \in T_x\tilde{M}^{2m} : |X| = 1, \tilde{H}(X) = 1\} = m - 1$$

(*respectively,* $\dim \{X \in T_x\tilde{M}^{2m} : |X| = 1, \tilde{H}(X) = -1\} = m - 1$) *for* $x \in \tilde{M}^{2m}$.

Proof. Let $G = I_0(\tilde{M}^{2m})$ be the identity component of the group of isometry of $\tilde{M}^{2m}$ and K its isotropy. Then $\tilde{M}^{2m} = G/K$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$. Then $\mathfrak{m}$ can be regarded as the tangent space of $\tilde{M}^{2m}$ at the origin 0. Let $\mathfrak{U}$ be the maximal abelian subspace of $\mathfrak{m}$. Then $\dim \mathfrak{U} = \operatorname{rank} \tilde{M}^{2m} = 2$. Moreover, every vector X of $\tilde{M}^{2m}$ at the origin can be mapped into a vector in $\mathfrak{U}$ by the isotropy. The set of unit vectors X in $\mathfrak{U}$ with $\tilde{H}(X) = 1$ (respectively, -1) is discrete if $\tilde{M}^{2m}$ is of compact type (respectively, noncompact type). Thus

$$\dim \{X \in T_0\tilde{M}^{2m} : |X| = 1, \tilde{H}(X) = 1\} = m - 1$$

(respectively, $\dim\{X \in T_0\tilde{M}^{2m}: |X| = 1, \tilde{H}(X) = -1\} = m - 1$). Since $\tilde{M}^{2m}$ is symmetric, a similar result holds at every other point $x \in \tilde{M}^{2m}$.

THEOREM 2. *The Hermitian symmetric spaces* $(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbb{R})$ *and* $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + \mathbb{R})$ *admit no extrinsic spheres of codimension* $p < 8$.

Proof. Since both $(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbb{R})$ and $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + \mathbb{R})$ are of rank 2, Lemmas 3 and 4 imply that $\dim(TM^n \cap J(TM^n)) \leq 16$ for every extrinsic sphere M^n. This shows that $n \leq 28$ and $p \geq 8$.

THEOREM 3. *The Hermitian symmetric spaces* $SO(2 + m)/SO(2) \times SO(m)$ *and* $SO_0(2, m)/SO(2) \times SO(m)$ *admit no extrinsic spheres of codimension* $p < 4m/5 - 3$.

Proof. If M^n is an extrinsic sphere of codimension p in

$$SO(2 + m)/SO(2) \times SO(m),$$

then from Lemma 3 and (4.2) we get

$$\tilde{S}(N_1, N_1) = \frac{5 - s}{4} + \frac{1}{4}\sum_{r=2}^{s} [\tilde{H}(N_1 + N_r) + \tilde{H}(N_1 - N_r) + \tilde{H}(N_1 + JN_r) + \tilde{H}(N_1 - JN_r) - \tilde{H}(N_r)]. \tag{4.6}$$

Since $s \leq p$, Table I and (4.6) imply

$$\frac{m}{2} = \tilde{S}(N_1, N_1) \leq \frac{5s + 3}{8} \leq \frac{5p + 3}{8},$$

from which we get $p \geq 4m/5 - 3$. A similar argument applies to extrinsic spheres in $SO_0(2, m)/SO(2) \times SO(m)$.

THEOREM 4. *The Hermitian symmetric spaces* $SO(2q)/U(q)$ *and* $SO^*(2q)/U(q)$ *admit no extrinsic spheres of codimension* $p \leq q$ (*respectively,* $p < q$) *for* $q > 6$ (*respectively,* $q = 5, 6$).

Proof. Let M^n be an extrinsic sphere of codimension p in $SO(2q)/U(q)$, $q \geq 5$. Then (4.2) holds for some $s \leq p$.

Case 1. If $s \geq 5$, Table I and (4.2) give

$$4[q/2](q - 1) \leq (4[q/2] - 1)s - 4[q/2] + 6.$$

Combining this with $p \geq s$, we get $p \geq q + (q - 6)/(4[q/2] - 1)$.

Case 2. If $s = p < 5$, then Table I and (4.2) imply $p \geq 29/5$, which is a contradiction.

Case 3. If $s \leq p - 1$ and $s < 5$, then Table I and (4.2) imply

$$p \geq (4q[q/2] - 2[q/2] - 2)/(3[q/2] - 1) > q.$$

Consequently, we get $p > q$ for $q > 6$ and $p \geq q$ for $q = 5, 6$. A similar argument applies to extrinsic spheres in $SO^*(2q)/U(q)$.

Remark 1. By arguments similar to the proof of Theorem 4, we may also prove the following.

(a) The Hermitian symmetric spaces $(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbb{R})$ and $(\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 + \mathbb{R})$ admit no extrinsic spheres of codimension $p < 6$;

(b) The Hermitian symmetric spaces $Sp(q)/U(q)$ and $Sp(q, \mathbb{R})/U(q)$ admit no extrinsic spheres of codimension $p \leq q/2 + 1$; and

(c) The Hermitian symmetric spaces

$$SU(q + q')/S(U_q \times U_{q'}) \quad \text{and} \quad SU(q, q')/S(U_q \times U_{q'}), \qquad q \geq q' \geq 2,$$

admit no extrinsic spheres of codimension $p \leq q'(q + q')/(2q' - 1)$.

Remark 2. If $\tilde{M}^{2m}$ is an irreducible Hermitian symmetric space of rank ℓ, then $\tilde{M}^{2m}$ admits extrinsic spheres of dimensions less than or equal to $\ell - 1$ with flat normal connection; namely, extrinsic spheres of maximal flat totally geodesic submanifolds of $\tilde{M}^{2m}$.

Remark 3. Since an m-sphere can be isometrically imbedded in the complex sphere $Q^{2m} = SO(2 + m)/SO(2) \times SO(m)$ as a totally geodesic submanifold, Q^{2m} admits extrinsic spheres of dimension $n < m$.

REFERENCES

6. R. L. Bishop and S. I. Goldberg, *Some implications of the generalized Gauss-Bonnet theorem.* Trans. Amer. Math. Soc. 112 (1964), 508-535.
7. E. Calabi and E. Vesentini, *On compact, locally symmetric Kähler manifolds.* Ann. of Math. (2) 71 (1960), 472-507.
8. B.-Y. Chen, *Extrinsic spheres in Kähler manifolds.* Michigan Math. J. 23 (1976), 327-330.
9. S. Helgason, *Differential geometry and symmetric spaces.* Academic Press, New York, 1962.
10. A. Korányi, *Analytic invariants of bounded symmetric domains.* Proc. Amer. Math. Soc. 19 (1968), 279-284.

(For references [1-5] see [8].)

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