ON PRIME DIVISORS OF I^n, n LARGE

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1. INTRODUCTION

The terminology in this article is, in general, the same as that in [2] (so all rings are commutative with nonzero identity element). For convenience of discussion, throughout this section, A is a Noetherian domain and $I \subseteq p$ are ideals in A such that $I \neq (0)$ and p is prime.

Consider the following statement

(PD_0): If p is a prime divisor of I^k, for some $k \geq 1$, then p is a prime divisor of I^n, for all large n.

This is known to hold when I is principal [2, (12.6)], and the original purpose of this paper was to prove that (PD_0) holds for all ideals in all Noetherian domains. Although it is shown below that many ideals do satisfy (PD_0), we have been unable to show that all ideals do. However, it is hoped that the results which have been obtained are of sufficient interest and importance to merit consideration by others.

A brief discussion of the results in this paper will now be given.

In Section 2, we prove that if p is a prime divisor of the integral closure $(I^k)_a$ of I^k, for some $k \geq 1$, then p is a prime divisor of $(I^n)_a$, for all large n (2.5). From this, it is shown that p is also a prime divisor of H^n, for all ideals H such that $H_a = I_a$ and for all large n (2.6.1). After proving some further corollaries, it is shown in (2.11) that, for all large m, I^m satisfies (PD_0). In (2.14) and (2.15), we show that (PD_0) holds for certain ideals in a number of other important cases. Then Section 2 is closed by proving three equivalences of the conjecture that (PD_0) holds for all ideals in all Noetherian domains (2.18).

In proving the results in Section 2, considerable use is made of the Rees ring $\mathscr{R} = \mathscr{R}(A, I)$ of A with respect to I (2.3). The failure to prove (PD_0) is due to the fact that $\mathscr{P} = (pA[t, u] \cap \mathscr{R}, u)\mathscr{R}$ may be an irrelevant prime divisor of $u\mathscr{R}$ (see (3.1)). (If $u\mathscr{R}$ has no irrelevant prime divisor, then I satisfies (PD_0).) In Section 3, (3.3) determines when $\mathscr{P}$ is contained in an (irrelevant) prime divisor of $u\mathscr{R}$. Then three equivalences to "$u\mathscr{R}$ has no irrelevant prime divisor" are given in (3.6.2), one of which is "$I^{n+1} : I = I^n$, for all $n \geq 1$".

In Section 4, we consider $I^{n+1} : I$ and show that $I^{n+1} : I = I^n$, for all large n (4.1). After proving some corollaries of this, we show that if I is normal (that is, $(I^n)_a = I^n$ for all $n \geq 1$), then $I^n : I^m = I^{n-m}$ for all $n \geq m \geq 1$; hence, $u\mathscr{R}$ has no irrelevant prime divisor and I satisfies (PD_0), when I is normal ((4.7) and (4.8)).

Section 5 considers $K + I^n$, where K is another ideal in A. (Such ideals arise in (2.18) mentioned above.) The prime divisors of $K + I^n$ (for $n \geq 1$) are

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characterized in (5.2), and then in (5.5) it is shown that $\bigcap (K + I^n)$ is the intersection of all p_i-primary components of K contained in a prime divisor of $K + I^k$ for some $k \geq 1$. Therefore, if A is local, then each prime divisor of K is contained in a prime divisor of $K + I^k$, for some $k \geq 1$ (5.8). The paper is closed by showing in (5.10) that frequently, in a semi-local ring, $p^n \neq p^{(n)}$, for all large n.

2. PRIME DIVISORS OF I^n, n LARGE

In this section, we consider a condition on ideals in a Noetherian ring A (condition (PD), which is more general than (PD_0)), which we conjecture holds for all ideals in A. We show that certain ideals do satisfy (PD), and this section is then closed by giving three equivalences of the conjecture.

For each ideal I in a Noetherian ring A, consider the following condition

(PD): For each prime ideal p in A such that IA_p is not nilpotent and p is a prime divisor of I^k, for some $k \geq 1$, it holds that p is a prime divisor of I^n, for all large n.

It should be noted that if I is such that IA_p is nilpotent for every prime ideal p which contains I, then I vacuously satisfies (PD). Also, (2.17) below shows that the nilpotent condition in (PD) is necessary. Finally, it will be shown below that only finitely many elements of Spec A are prime divisors of I^k for some $k \geq 1$.

(2.1) CONJECTURE. *Every ideal in each Noetherian ring satisfies* (PD).

The author has been unable to settle this conjecture. However, a number of positive results have been obtained and will be given in this section. To prove our first main result, we need the following two definitions and some facts concerning them.

(2.2) DEFINITION. Let I be an ideal in a ring A. Then the *integral closure* I_a *of* I *in* A is the set of elements $x \in A$ which satisfy an equation of the form $x^k + b_1 x^{k-1} + \cdots + b_k = 0$, where $b_i \in I^i$.

It is clear that $I \subseteq I_a \subseteq \text{Rad } I$, and it is known that I_a is an ideal in A [5, Section 6]. Further, it is readily seen that if S is a multiplicatively closed set in A, then $I_a A_S = (IA_S)_a$. These facts will be used below without further mention.

(2.3) DEFINITION. Let $I = (b_1, \cdots, b_k)A$ be an ideal in a Noetherian ring A, let t be an indeterminate, and let $u = 1/t$. Then the *Rees ring* $\mathscr{R} = \mathscr{R}(A, I)$ *of* A *with respect to* I is the subring $\mathscr{R} = A[tb_1, \cdots, tb_k, u]$ of $A[t, u]$.

The following remark lists most of the properties of Rees rings which are needed in what follows.

(2.4) REMARK. *Let* A, I, *and* $\mathscr{R}$ *be as in* (2.3). *Then:*

(2.4.1) *The elements in* $\mathscr{R}$ *are finite sums* $\sum_{-m}^{n} c_i t^i$, *where* $c_i \in I^i$ (*with the convention that* $I^i = A$, *if* $i \leq 0$). *Therefore* $\mathscr{R}$ *is a graded Noetherian ring.*

(2.4.2) u *is a regular element in* $\mathscr{R}$ *and* $u^i \mathscr{R} \cap A = I^i$, *for all* $i \geq 1$.

(2.4.3) $\mathscr{R}/u\mathscr{R}$ *is* (*isomorphic to*) *the form ring of* A *with respect to* I [11, Theorem 2.1].

(2.4.4) *For an ideal* B *in* A, *let* $B^* = BA[t, u] \cap \mathscr{R}$. *Then* $\mathscr{R}/B^* \cong \mathscr{R}(A/B, (I + B)/B)$ [11, Lemma 1.1].

(2.4.5) *If* p *is a prime ideal in* A *and* q *is* p*-primary, then* p^* *is prime in* $\mathscr{R}$ *and* q^* *is* p^**-primary* [11, Lemma 1.4]. *Further, if* $B = \bigcap_1^g q_i$ *is a normal primary decomposition of an ideal* B *in* A, *then* $B^* = \bigcap_1^g q_i^*$ *is a normal primary decomposition of* B^* *in* $\mathscr{R}$ [11, Theorem 1.5].

(2.4.6) *If* H *is a homogeneous ideal in* $\mathscr{R}$, *then let* $H_{[i]} = \{b \in A: bt^i \in H\}$, *for* $-\infty < i < \infty$. *Then* $I^i \supseteq H_{[i]} \supseteq H_{[i+1]} \supseteq IH_{[i]}$ [12, p. 11]. *If* H' *is another homogeneous ideal in* $\mathscr{R}$, *then* $(H + H')_{[i]} = H_{[i]} + H'_{[i]}$.

We can now prove our first main result. This theorem gives information concerning prime divisors of $(I^n)_a$ rather than of I^n. However, from this information we will derive some results concerning prime divisors of I^n.

(2.5) THEOREM. *Let* I *be an ideal in a Noetherian ring* A *such that* height $I \geq 1$, *and let* p *be a prime divisor of* $(I^k)_a$, *for some* $k \geq 1$. *Then* p *is a prime divisor of* $(I^n)_a$, *for all large* n.

Proof. Let $\mathscr{R} = \mathscr{R}(A, I)$, and let $\mathscr{R}'$ be the integral closure of $\mathscr{R}$ in its total quotient ring $\mathscr{T}$. Then, for each $k \geq 1$, $u^k\mathscr{R}' \cap A = (I^k)_a$, as is seen by considering equations of integral dependence (for example, see the last part of the proof of [6, Theorem 3.7(4)]). Also, $u^k\mathscr{R}' \cap \mathscr{R} = (u^k\mathscr{R})_a$ [7, Lemma 2.4(1)], so $(u^k\mathscr{R})_a \cap A = (I^k)_a$, for all $k \geq 1$. Therefore, there exists a prime divisor P of $(u^k\mathscr{R})_a$ such that $P \cap A = p$, and there exists a prime divisor P' of $u^k\mathscr{R}'$ such that $P' \cap \mathscr{R} = P$. Then P' has height 1 and, with $z'' = \text{Rad}\,\mathscr{R}'_{P'}$, $V^* = \mathscr{R}'_{P'}/z''$ is a discrete valuation ring [7, Corollary 2.11 and Proposition 2.7(1)]. Let $z' = z'' \cap \mathscr{R}'$ and $z = z' \cap A$. Then $z' = z\mathscr{T} \cap \mathscr{R}'$, so z is a minimal prime ideal in A; hence, $I \not\subseteq z$ (since height $I \geq 1$). Also, $A/z \subseteq \mathscr{R}/(z\mathscr{T} \cap \mathscr{R}) \subseteq \mathscr{R}'/z' \subseteq \mathscr{T}/z\mathscr{T}$, $\mathscr{R}'/z'$ is integral over $\mathscr{R}/(z\mathscr{T} \cap \mathscr{R})$, and $\mathscr{R}/(z\mathscr{T} \cap \mathscr{R}) \cong \mathscr{R}(A/z, (I + z)/z)$, by (2.4.4). Therefore, by the proof of [6, Lemma 3.2(1)] (from the third sentence on),

$$t^* = \text{trd}\,(\mathscr{R}'/P')/(A/p) \geq 1.$$

Thus, since $t^* = \text{trd}\,(\mathscr{R}/P)/(A/p)$ (by integral dependence) and $u \in P$, $tI \not\subseteq P$. Hence, there exists a homogeneous element $ct \in tI$, such that $ct \notin P$.

Also, since P is homogeneous (since $(u^k\mathscr{R})_a$ is), there exists a homogeneous element $h = bt^m$ ($b \in I^m$) in $\mathscr{R}$ such that $(u^k\mathscr{R})_a : h\mathscr{R} = P$ (so $m > -k$, since $h \notin u^k\mathscr{R}$). Therefore, for all $i \geq 0$, $P = (u^k\mathscr{R})_a : bt^m c^i t^i \mathscr{R}$. So

$$p = P \cap A = ((u^k\mathscr{R})_a : bt^m c^i t^i \mathscr{R}) \cap A = (I^{m+i+k})_a : bc^i A,$$

since $r \in (I^{m+i+k})_a : bc^i A$ if and only if $rbc^i \in (I^{m+i+k})_a = u^{m+i+k}\mathscr{R}' \cap A$ if and only if (since bt^m and $c^i t^i$ are in $\mathscr{R}$) $rbt^m c^i t^i \in u^k\mathscr{R}' \cap \mathscr{R}$ if and only if $r \in ((u^k\mathscr{R})_a : bt^m c^i t^i \mathscr{R}) \cap A$. Therefore, there exist $b \in A$, $c \in I$, and $m > -k$ such that, for all $i \geq 0$, $p = (I^{m+i+k})_a : bc^i A$; so p is a prime divisor of $(I^n)_a$, for all large n. *q. e. d.*

In [10, Theorem 6.7], D. Rees proved that if A is a Noetherian domain in (2.5), then p is a prime divisor of I^n, for infinitely many n. (2.6.1) sharpens this result.

(2.6) COROLLARY. *Let* A, I, *and* p *be as in* (2.5). *Then:*

(2.6.1) *For each ideal* H *such that* $H_a = I_a$, p *is a prime divisor of* H^n, *for all large* n.

(2.6.2) *For each large* m *and each ideal* K *such that* $K_a = (I^m)_a$, p *is a prime divisor of* K^n, *for all large* n.

Proof. (2.6.1) It will first be shown that p is a prime divisor of I^n, for all large n. For this, let $\mathscr{R}$, $\mathscr{R}'$, P, and P' be as in the first paragraph of the proof of (2.5), so that P' is a prime divisor of $u^k\mathscr{R}'$, $P = P' \cap \mathscr{R}$ is a prime divisor of $(u^k\mathscr{R})_a$, $p = P \cap A$, and $tI \not\subseteq P$. Also, since P' is a prime divisor of $u^k\mathscr{R}'$, P is a prime divisor of $u\mathscr{R}$ [7, Theorem 2.15]. Therefore, $u\mathscr{R}:h'\mathscr{R} = P$ for some homogeneous element $h' = dt^m \in \mathscr{R}$. Thus, with ct as at the end of the first paragraph of the proof of (2.5), $u\mathscr{R}:dt^m c^i t^i \mathscr{R} = P$, for all $i \geq 0$. Therefore

$$p = P \cap A = (u\mathscr{R}:dt^m c^i t^i \mathscr{R}) \cap A = I^{m+i+1}:dc^i A, \quad \text{for all } i \geq 0.$$

Hence, p is a prime divisor of I^n, for all large n. Since $H_a = I_a$, it follows from what has already been proved that p is a prime divisor of H^n, for all large n.

(2.6.2) Let m be large, so p is a prime divisor of $(I^m)_a$ by (2.5). Since $K_a = (I^m)_a$, it follows from (2.6.1) that p is a prime divisor of K^n, for all large n. *q. e. d.*

The following remark will be used below.

(2.7) REMARK. *The proof of* (2.6.1) *shows that if* P *is a prime divisor of* $u\mathscr{R}(A, I)$ *such that* $tI \not\subseteq P$, *then* $p = P \cap A$ *is a prime divisor of* I^n, *for all large* n.

Therefore, if I is such that tI is not contained in any prime divisor of $u\mathscr{R}(A, I)$, then I satisfies (PD). This will be examined further in Section 3.

The next corollary to (2.5) relates the prime divisors of bB in the monadic transformation $B = A[I/b]$ to prime divisors of I^n.

(2.8) COROLLARY. *Let* I *be an ideal in a Noetherian ring* A *such that* $(0):I = (0)$. *Let* b *be a regular element in* I, *and let* $B = A[I/b]$. *Then, for each prime divisor* Q *of* bB, $Q \cap A$ *is a prime divisor of* I^n, *for all large* n.

Proof. If bB = B, then there is nothing to prove, so assume that bB is proper and let Q be a prime divisor of bB. Then, with $\mathscr{R} = \mathscr{R}(A, I)$,

$$\mathscr{R}[1/bt] = B[bt, 1/bt] = (\text{say})\ \mathscr{S},$$

so $Q\mathscr{S}$ is a prime divisor of $b\mathscr{S} = u\mathscr{S}$; hence, $P = Q\mathscr{S} \cap \mathscr{R}$ is a prime divisor of $u\mathscr{R}$, and $bt \notin P$. Therefore, since $Q \cap A = P \cap A$, the conclusion follows from (2.7).

The final corollary to (2.5) shows an interesting fact concerning symbolic powers of certain prime ideals. For this corollary (and also for (4.5) and (5.10)), recall that if p is a prime ideal in a ring A and q is p-primary, then the n^{th} *symbolic power* $q^{(n)}$ of q is the p-primary component of q^n; that is, $q^{(n)} = q^n A_p \cap A$.

(2.9) COROLLARY. *Let* p *be a prime ideal in a regular domain* A. *If* $(p^k)_a \neq p^{(k)}$, *for some* $k \geq 1$, *then* $p^n \neq p^{(n)}$, *for all large* n.

Proof. Since A_p is a regular local ring, for all $n \geq 1$, $p^n A_p = (p^n A_p)_a$, so $p^n A_p = (p^n)_a A_p$; hence, $p^n \subseteq (p^n)_a \subseteq p^{(n)}$. Therefore, if $(p^k)_a \neq p^{(k)}$, for some $k \geq 1$, then $(p^k)_a$ has an imbedded prime divisor, say Q; hence, Q is an imbedded prime divisor of $(p^n)_a$, for all large n by (2.5). Therefore, $p^n \subseteq (p^n)_a \subset p^{(n)}$, for all large n, so $p^n \neq p^{(n)}$, for all large n. *q. e. d.*

The following remark gives a known result which is closely related to (2.5). Recall that an ideal I in a ring A is said to be of the *principal class* if I is generated by h elements, where h is the height of I. Also, I is said to be *height unmixed* if all prime divisors p of I have height h.

(2.10) REMARK [8, Theorem 2.12]. *Let* A *be a Noetherian domain which satisfies the altitude formula. Then, for each ideal* I *of the principal class in* A, $(I^n)_a$ *is height unmixed, for all* $n \geq 1$.

Therefore, if in (2.10) p is a prime divisor of $(I^k)_a$, for some $k \geq 1$, then p is a prime divisor of I^n and of $(I^n)_a$, for all $n \geq 1$, since p is a minimal prime divisor of I.

The second main result in this section shows that if I contains a regular element, then all large powers of I satisfy (PD).

(2.11) THEOREM. *Let* I *be an ideal in a Noetherian ring* A *such that* $(0):I = (0)$. *Then, for each large* m, *the ideal* $B = I^m$ *satisfies* (PD).

Proof. Let W be the set of all p in Spec A which are prime divisors of I^n, for some $n \geq 1$. Then W is a finite set, since $u^n \mathscr{R}(A, I) \cap A = I^n$ and the prime divisors of $u^n \mathscr{R}$ are the prime divisors of $u\mathscr{R}$, for all $n \geq 1$. Thus, to prove the theorem, it may be assumed that the maximal ideals in A are the ideals which are maximal in W (since, for each $n \geq 1$, a prime ideal q in A is a prime divisor of I^n if and only if qA_S is a prime divisor of $I^n A_S$, where $S = A - \bigcup \{p: p \in W\}$). Therefore, assume that A is a semi-local ring with maximal ideals $M_1, \cdots, M_h$ and $I \subseteq J = \bigcap_1^h M_i$. Then, since a prime ideal q in A is a prime divisor of an ideal H in A if and only if qA(X) is a prime divisor of HA(X), it may be assumed that each A/M_i is an infinite field.

Let $z_1, \cdots, z_g$ be the prime divisors of (0) in A, so $I \not\subseteq \bigcup_1^g z_j$ (by hypothesis). Then, since $I \subseteq J$, the proof of [2, (22.3)] (with $\{z_1, \cdots, z_g\} = \{B_1, \cdots, B_t\}$ of [2]) shows that some superficial element b (of degree one) of I is a regular element. (The assumption that Rad I = J in [2] is not essential.) By definition, there exists $c \geq 0$ such that $(I^{n+1}:bA) \cap I^c = I^n$, for all $n \geq c$. Hence, by [2, (3.12)] (since $(0):I = (0)$), $I^{n+1}:bA = I^n$, for all large n (say $n \geq n^*$). Fix $m \geq n^*$, let $B = I^m$, and let $\mathscr{R} = \mathscr{R}(A, B)$. Since $B^{n+1}:b^m A = B^n$, for all $n \geq 1$, [15, (3.4) and (3.6)] say that for each prime divisor P of $u\mathscr{R}$, $tB \not\subseteq P$. Therefore, let p be a prime divisor of B^k, for some $k \geq 1$. Then there exists a prime divisor P of $u\mathscr{R}$ such that $P \cap A = p$. By (2.7), p is a prime divisor of B^n, for all large n. Therefore, B satisfies (PD). *q. e. d.*

(4.3) reconsiders I as in (2.11) and gives a sharper conclusion.

The conclusion of (2.11) continues to hold if the condition "$(0):I = (0)$" is replaced by "if a prime divisor z of (0) contains I, then z is a maximal prime divisor of I", for each such z is a prime divisor of I^n, for all large n [9, (3.16.1)]; so on localizing to A_S, with S the complement in A of the union of the elements of Spec A which are prime divisors of I^k (for some $k \geq 1$) but are not prime divisors of (0), one finds that $(0)A_S : IA_S = (0)A_S$. From this and (2.11), the desired conclusion readily follows.

(2.12) COROLLARY. *Let* I *be a nonzero ideal in a Noetherian domain* A. *Then, for each large* m, I^m *satisfies* (PD).

Proof. Clear by (2.11).

(2.13) REMARK. *Let* A *and* I *be as in* (2.11), *and fix a large positive integer* m. *Let* S *be the set of* p *in* Spec A *which are prime divisors of* I^{mk}, *for some*

$k \geq 1$; *and, for each* $n \geq 1$, *let* S_n *be the set of* q *in* Spec A *which are prime divisors of* I^{mn}. *Then:*

(2.13.1) $S_n = S$, *for all large* n.

(2.13.2) *For all large* n, $S_{nk} = S_{nh}$, *for all* k, $h \geq 1$; *that is, if a prime ideal* p *is a prime divisor of* I^{mnk}, *for some* $k \geq 1$, *then* p *is a prime divisor of* I^{mnh}, *for all* $h \geq 1$.

Proof. (2.13.1) is clear by (2.11), and (2.13.2) follows immediately from (2.13.1).

(2.13.2) is a generalization of the result: if b is a regular element in a Noetherian ring A and a prime ideal p is a prime divisor of $b^k A$, for some $k \geq 1$, then p is a prime divisor of $b^h A$, for all $h \geq 1$ [2, (12.6)].

There are a number of more or less unrelated but important special cases when a prime ideal is a prime divisor of I^n, for all large n. These are listed in the following two remarks.

(2.14) REMARK. *Let* I *be an ideal in a Noetherian ring* A *and assume that there exists* $x \in A$ *such that* $I^n : xA = I^n$, *for all* $n \geq 1$. *Then, for each prime ideal* p *in* A:

(2.14.1) *If* p *is a prime divisor of* $((I, x)^k A)_a$, *for some* $k \geq 1$, *then* p *is a prime divisor of* $((I, x)^n A)_a$, *for all large* n.

(2.14.2) (I, x)A *satisfies* (PD).

Proof. It follows easily from the hypothesis that height $(I, x)A \geq 1$. Thus, (2.14.1) follows from (2.5).

(2.14.2) $(I, x)^n A : xA = (I^n, x(I, x)^{n-1})A : xA = (I^n : xA, (I, x)^{n-1})A = (I, x)^{n-1} A$, for all $n \geq 1$. Therefore, for each prime divisor P of $u\mathscr{R}(A, (I, x)A)$, $t(I, x) \not\subseteq P$ [15, (3.4) and (3.6)], so (2.14.2) follows as in the last three sentences of the proof of (2.11).

(2.15) REMARK. *Let* $I \subseteq p$ *be ideals in a Noetherian ring* A *such that* p *is prime,* IA_p *is not nilpotent, and* p *is a prime divisor of* I^k, *for some* $k \geq 1$. *Then in each of the following cases,* p *is a prime divisor of* I^n, *for all large* n:

(2.15.1) p *is a minimal prime divisor of* I.

(2.15.2) p *is a prime divisor of zero.*

(2.15.3) $(0) : I = (0)$ *and* $(I^n)_a = I^n$, *for all* $n \geq 1$.

(2.15.4) *There exists* $b \in I$ *such that* $I^{n+1} : bA = I^n$, *for all* $n \geq k$.

(2.15.5) A *is a flat* B*-algebra* (*with* B *Noetherian*) *and* I = qA, *for some primary ideal* q *in* B.

(2.15.6) A = B[X] (X *is transcendental over* B) *and* I = (H, f)A, *for some ideal* H *in* B *and* $f \in A$ *whose content is* B (*that is, the ideal generated in* B *by the coefficients of* f *is* B).

Proof. (2.15.1) is clear, and (2.15.2) is given in [9, (3.16.1)].

For (2.15.3), let $\mathscr{R} = \mathscr{R}(A, I)$. Then there exists a prime divisor P of $u\mathscr{R}$ such that $P \cap A = p$ (as in the second sentence of the proof of (2.11)). Further, by hypothesis and (4.8), $tI \not\subseteq P$. Therefore, the conclusion follows from (2.7).

(2.15.4) Let $c \in A$ such that $I^k : cA = p$. Then $I^{k+i} : b^i cA = I^k : cA = p$, so p is a prime divisor of I^{k+i}, for all $i \geq 0$.

(2.15.5) Let q be p'-primary. Then, since p' is a prime divisor of q^n, for all $n \geq 1$, [2, (18.11)] says that p is a prime divisor of $q^n A = I^n$, for all $n \geq 1$.

(2.15.6) follows immediately from (2.14.2).

(2.16) REMARK. *With the notation of* (2.15), *the following statements hold:*

(2.16.1) *If* A *is an analytically unramified local ring and* height $I > 0$, *then, for all large* m, $(I^m)_a$ *satisfies the conditions on* I *in* (2.15.3).

(2.16.2) *In* (2.15.4), p *is a prime divisor of* I^n, *for all* $n \geq k$; *in* (2.15.5) p *is a prime divisor of* I^n, *for all* $n \geq 1$; *and in* (2.15.6) I *satisfies* (PD).

Proof. (2.16.1) follows from [13, Theorem 1], and (2.16.2) was proved in the proofs of (2.15.4), (2.15.5), and (2.15.6).

The following example shows the necessity of the condition in (PD) that IA_p not be nilpotent. (Of course, if IA_p is nilpotent and p is a prime divisor of (0), then p is a prime divisor of I^n, for all large n.)

(2.17) EXAMPLE. Let (R, M) be a regular local ring, and let

$$M = (b_1, \cdots, b_k)R$$

with height $M = k \geq 2$. Let $b = b_1$, let $A = R/b^2 R$, and let $I = bM/b^2 R$. Then $p = M/b^2 R$ is a prime divisor of I^i if and only if $i = 1$. To see this, note that M is a prime divisor of bM, since $bM : bR = M$, so p is a prime divisor of I. Also, $I^2 = (0)$, and p is not a prime divisor of (0), since (0) in A is $bR/b^2 R$-primary.

On the other hand, it may happen that a prime ideal p in A is a prime divisor of I^k, for some $k > 1$, and is not a prime divisor of I^j, for some $j < k$. For example, this holds if I is prime and I^k is not I-primary.

This section will be closed by proving three equivalences of Conjecture 2.1.

(2.18) PROPOSITION. *The following statements are equivalent:*

(2.18.1) *Every ideal in each Noetherian ring satisfies* (PD).

(2.18.2) *Every ideal in each local ring* (R, M) *such that* R/M *is infinite satisfies* (PD).

(2.18.3) *For each local ring* (R, M) *the following condition holds: if* K *and* I *are ideals in* R *such that* M *is a prime divisor of* $K + I^k$, *for some* $k \geq 1$, *and, for each* $j \geq 1$, $I^j \not\subseteq K$, *then* M *is a prime divisor of* $K + I^n$, *for all large* n.

(2.18.4) *For each unramified complete regular local ring* (R, M), *the condition in* (2.18.3) *holds.*

Proof. It is clear that (2.18.1) $\Rightarrow$ (2.18.2) and (2.18.3) $\Rightarrow$ (2.18.4).

Assume that (2.18.2) holds, let (R, M) be a local ring, and let the hypotheses in the condition in (2.18.3) hold. Let (S, N) = (R(X), MR(X)), let $I^* = IR(X)$, and let $K^* = KR(X)$. Then it is readily seen that the hypotheses on (R, M), I, and K are satisfied by (S, N), I^*, and K^*. Also, for each ideal H in R, N is a prime divisor of HR(X) if and only if M is a prime divisor of H. Therefore, it may be assumed that R/M is an infinite field. Then, in R/K, H = (I + K)/K is not nilpotent and M/K is a prime divisor of H^k, for some $k \geq 1$. Therefore, by (2.18.2), M/K is a prime divisor of H^n, for all large n, so M is a prime divisor of $K + I^n$, for all large n. Therefore (2.18.2) ⇒ (2.18.3).

Finally, assume that (2.18.4) holds, let A be a Noetherian ring, and let $I \subseteq p$ be ideals in A such that IA_p is not nilpotent and p is a prime divisor of I^k, for some $k \geq 1$. Then it suffices to prove that p is a prime divisor of I^n, for all large n. For this, it is sufficient to prove that pA_p is a prime divisor of $I^n A_p$, for all large n. Therefore, assume that A is local and p is the maximal ideal in A. Let (A^*, p^*) be the completion of A. Then, by [2, (18.11)], p is a prime divisor of I^n, for all large n, if and only if p^* is a prime divisor of $I^n A^*$, for all large n. Therefore, assume that A is a complete local ring. Then there exists an unramified complete regular local ring (R, M) and an ideal K in R such that $A = R/K$ [1, Corollary 2, p. 89]. Let H be the pre-image of I in R. Then M is a prime divisor of $K + H^k$ and, for each $j \geq 1$, $H^j \not\subseteq K$ (since I is not nilpotent). Therefore, by (2.18.4), M is a prime divisor of $K + H^n$, for all large n, so p is a prime divisor of I^n, for all large n. *q. e. d.*

By [9, (3.15)], if (R, M) is a local ring and M is a prime divisor of an ideal K in R, then for each ideal I in R, M is a prime divisor of $K + I^n$, for all large n. Thus, the conclusion in the condition in (2.18.3) holds when M is a prime divisor of K.

Prime divisors of $K + I^n$ will be considered in Section 5.

3. IRRELEVANT PRIME DIVISORS OF $u\mathcal{R}$

In (2.7) and in the comment following it, it was noted that if tI is not contained in any prime divisor of $u\mathcal{R}(A, I)$, then I satisfies (PD). In this section we investigate when this condition on the prime divisors of $u\mathcal{R}$ holds. We begin with the following definition.

(3.1) DEFINITION. Let I be an ideal in a Noetherian ring A, and let $\mathcal{R} = \mathcal{R}(A, I)$. Then a homogeneous ideal H in $\mathcal{R}$ is said to be *irrelevant* if H contains all homogeneous elements of sufficiently large degree; otherwise, H is said to be *relevant*.

(3.2) REMARK. *With the notation of* (3.1), *let*

$$U = \{p \in \text{Spec } A : I \subseteq p\} \quad \textit{and} \quad U' = \{P \in \text{Spec } \mathcal{R} : (u, tI)\mathcal{R} \subseteq P\}.$$

Then U' *is the set of possible irrelevant prime divisors of* $u\mathcal{R}$, *and there exists a one-to-one correspondence between* U *and* U' *given by:* p *and* P *correspond if and only if* $p = P \cap A$ *and* $P = (p^*, u)\mathcal{R}$, *where* $p^* = pA[t, u] \cap \mathcal{R}$.

Proof. It is clear that U' is the set of possible irrelevant prime divisors of $u\mathcal{R}$. Also, if $p \in U$, then $tI \subseteq p^*$ and $\mathcal{P} = (p^*, u)\mathcal{R}$ is a prime ideal in $\mathcal{R}$ (since $\mathcal{R}/\mathcal{P} = A/p$, by (2.4.6)), so $\mathcal{P} \in U'$ and $\mathcal{P} \cap A = p$. On the other hand, if $P \in U'$, then $p = P \cap A \in \text{Spec } A$ and $I = u\mathcal{R} \cap A \subseteq p$. Therefore

$$\mathcal{P} = (p^*, u)\mathcal{R} = (u, p, tI)\mathcal{R} \subseteq P, \quad \text{and} \quad \mathcal{R}/\mathcal{P} = A/p = \mathcal{R}/P,$$

so $\mathcal{P} = P$. *q. e. d.*

The following theorem determines when an ideal $p \in U$ is such that $\mathcal{P} = (p^*, u)\mathcal{R}$ is contained in an (irrelevant) prime divisor of $u\mathcal{R}$.

(3.3) THEOREM. *Let* $I \subseteq p$ *be ideals in a Noetherian ring* A *such that* p *is prime, let* $\mathscr{R} = \mathscr{R}(A, I)$, *and let* $\mathscr{P} = (u, p, tI)\mathscr{R}$ ($= (p^*, u)\mathscr{R}$). *Then* $\mathscr{P}$ *is contained in a prime divisor of* $u\mathscr{R}$ *if and only if* $I^{n+1} \subset I^n \cap (I^{n+1}: p) \cap (I^{n+2}: I)$ *for some* $n \geq 0$.

Proof. Assume first that $\mathscr{P}$ is contained in a prime divisor Q of $u\mathscr{R}$, and let bt^n $(n \geq 0)$ be in $\mathscr{R}$ with $u\mathscr{R}: bt^n\mathscr{R} = Q$. Then $b \in I^n$, $b \notin I^{n+1}$ (since $bt^n \notin u\mathscr{R}$), and $(p, tI)bt^n \subseteq u\mathscr{R}$. Therefore

$$bp \subseteq u^{n+1}\mathscr{R} \cap A = I^{n+1} \quad \text{and} \quad bI \subseteq u^{n+2}\mathscr{R} \cap A = I^{n+2},$$

so $b \in I^n \cap (I^{n+1}: p) \cap (I^{n+2}: I)$, but $b \notin I^{n+1}$.

Conversely, let $b \in I^n \cap (I^{n+1}: p) \cap (I^{n+2}: I)$, with $b \notin I^{n+1}$. Then $bt^n \in \mathscr{R}$, $bt^n \notin u\mathscr{R}$, $bt^n p \subseteq t^n I^{n+1} \subseteq u\mathscr{R}$, and $bt^n(tI) \subseteq t^{n+1}I^{n+2} \subseteq u\mathscr{R}$. Therefore, $u\mathscr{R}: bt^n\mathscr{R} \supseteq (u, p, tI)\mathscr{R} = \mathscr{P}$ and is a proper ideal; hence, $\mathscr{P}$ is contained in a prime divisor of $u\mathscr{R}$.

(3.4) COROLLARY. *With the notation of* (3.3), *if* $I \subset (I: p) \cap (I^2: I)$, *then* $\mathscr{P}$ *is contained in a prime divisor of* $u\mathscr{R}$.

Proof. This is merely the case $n = 0$ in (3.3).

(3.5) COROLLARY. *With the notation of* (3.3), *the following statements hold:*

(3.5.1) *If, for each* $n \geq 1$, p *is not contained in a prime divisor of* I^n, *then* p^* *is not contained in an (irrelevant) prime divisor of* $u\mathscr{R}$.

(3.5.2) *If* $I^{n+1}: I = I^n$, *for all* $n \geq 1$, *then* $u\mathscr{R}$ *has no irrelevant prime divisor. Therefore* I *satisfies* (PD) *and there exist* k *and* $b \in I^k$ *such that* $I^{n+k}: bA = I^n$, *for all* $n \geq 1$.

Proof. (3.5.1) is clear by (3.3).

The first statement in (3.5.2) is clear by (3.3) and the fact that U' in (3.2) is the set of possible irrelevant prime divisors of $u\mathscr{R}$. Therefore I satisfies (PD) (by (2.7)), and [15, Theorem 3.4] gives the existence of such k and b.

(3.6) REMARK. *With the notation of* (3.5):

(3.6.1) *The converse of* (3.5.1) *is false.*

(3.6.2) *The following statements are equivalent:*

(i) $I^{n+1}: I = I^n$, *for all* $n \geq 1$.

(ii) $u\mathscr{R}$ *has no irrelevant prime divisor.*

(iii) *There exist* k *and* $b \in I^k$ *such that* $I^{n+k}: bA = I^n$, *for all* $n \geq 1$.

(iv) $I^{n+1} = I^n \cap (I^{n+1}: p) \cap (I^{n+2}: I)$, *for all* $n \geq 0$ *and all* $p \in \operatorname{Spec} A$ *such that* $I \subseteq p$.

Proof. (3.6.1) Let (R, M) be a regular local ring, let $I = M$, and let $\mathscr{R} = \mathscr{R}(R, M)$. Then M is a prime divisor of I^n, for all $n \geq 1$, $(M, tM)\mathscr{R} = M^* \not\subseteq u\mathscr{R}$, and $u\mathscr{R}$ is prime (since the form ring $\mathscr{F}$ of R with respect to M is a domain and by (2.4.3) $\mathscr{F} \cong \mathscr{R}/u\mathscr{R}$).

(3.6.2) (i) $\Rightarrow$ (ii), by (3.5.2); (ii) $\Leftrightarrow$ (iii), by [15, Theorem 3.4]; and (ii) $\Leftrightarrow$ (iv), by (3.3). Finally, if (iii) holds, then $I^{n+k}: I^k = I^n$ (since $b \in I^k$). Therefore

$$I^n \subseteq I^{n+1}: I \subseteq I^{n+2}: I^2 \subseteq \cdots \subseteq I^{n+k}: I^k = I^n, \quad \text{for all } n \geq 1,$$

so (i) holds.

Each of the equivalent conditions in (3.6.2) implies that I satisfies (PD), by (2.7).

The condition in (3.3) becomes simpler when I = p, as is shown in the following corollary.

(3.7) COROLLARY. *Let* p *be a prime ideal in a Noetherian ring* A, *and let* $\mathscr{R} = \mathscr{R}(A, p)$. *Then* $u\mathscr{R}$ *has an irrelevant prime divisor if and only if there exists* $n \geq 0$ *such that* $p^{n+1} \subset p^n \cap (p^{n+2} : p)$.

Proof. Assume that Q is an irrelevant prime divisor of $u\mathscr{R}$, so $(u, tp)\mathscr{R} \subseteq Q$. Therefore $p \subseteq Q \cap A$, so $(p^*, u)\mathscr{R} = (u, p, tp)\mathscr{R} \subseteq Q$. Therefore, by (3.3), there exists $n \geq 0$ such that $p^{n+1} \subset p^n \cap (p^{n+1} : p) \cap (p^{n+2} : p) = p^n \cap (p^{n+2} : p)$.

The converse follows easily from (3.3).

(3.8) REMARK. *With* A *and* p *as in* (3.7), *the following statements are equivalent:*

(i) $p^{n+1} : p = p^n$, *for all* $n \geq 1$.

(ii) $u\mathscr{R}(A, p)$ *has no irrelevant prime divisor.*

(iii) *There exist* k *and* $b \in p^k$ *such that* $p^{n+k} : bA = p^n$, *for all* $n \geq 1$.

(iv) $p^{n+1} = p^n \cap (p^{n+2} : p)$, *for all* $n \geq 0$.

Proof. Clear by (3.6.2) and (3.7).

This section will be closed by using (3.3) to determine when $(p^*, u)\mathscr{R}$ *is* an irrelevant prime divisor of $u\mathscr{R}$.

(3.9) COROLLARY. *Let* A, $I \subseteq p$, *and* $\mathscr{R}$ *be as in* (3.3). *Then* $(p^*, u)\mathscr{R}$ *is an irrelevant prime divisor of* $u\mathscr{R}$ *if and only if*

$$I^{n+1} A_p \subset I^n A_p \cap (I^{n+1} A_p : pA_p) \cap (I^{n+2} A_p : IA_p).$$

Proof. Let S = A - p. Then

$$\mathscr{R}_S \cong \mathscr{R}(A_p, IA_p) \quad \text{and} \quad (p^*, u)\mathscr{R}_S = (pA_p[t, u] \cap \mathscr{R}_S, u)\mathscr{R}_S = ((pA_p)^*, u)\mathscr{R}_S.$$

Therefore $(p^*, u)\mathscr{R}$ is an irrelevant prime divisor of $u\mathscr{R}$ if and only if $(p^*, u)\mathscr{R}_S$ is an irrelevant prime divisor of $u\mathscr{R}_S$; hence, the conclusion follows from (3.3) applied to IA_p (since $(p^*, u)\mathscr{R}_S$ is a maximal ideal).

4. NOTES ON $I^{n+1} : I = I^n$

The condition in (3.3) for $u\mathscr{R}$ to have no irrelevant prime divisor involved $I^{n+1} : I$. In this section, we investigate ideals of this form. (4.1) shows that if I contains a regular element, then $I^{n+1} : I = I^n$, for all large n; so there are only finitely many values of n in (3.3) that have to be checked.

(4.1) THEOREM. *Let* I *be an ideal in a Noetherian ring* A *such that* $(0) : I = (0)$. *Then* $I^{n+1} : I = I^n$, *for all large* n.

Proof. Let W be as in the proof of (2.11). Then, with

$$S = A - \bigcup \{p : p \in W\},$$

$(0)A_S : IA_S = (0)A_S$ and $I^n A_S \cap A = I^n$, for all $n \geq 1$, so, by [16, Proof, p. 220], it may be assumed that A is semi-local and $I \subseteq J$, the Jacobson radical of A. Likewise (considering A(X)), it may be assumed that A/M is an infinite field, for all maximal ideals M in A. Then, by the second paragraph of the proof of (2.11), there exists $b \in A$ such that $I^{n+1} : bA = I^n$, for all large n. Therefore

$$I^n \subseteq I^{n+1} : I \subseteq I^{n+1} : bA = I^n;$$

so $I^{n+1} : I = I^n$, for all large n. *q. e. d.*

The condition $(0) : I = (0)$ in (4.1) is necessary, for if A is local and $(0) : I \neq (0)$, then $\bigcap (I^{n+1} : I) = \left(\bigcap I^{n+1}\right) : I = (0) : I \neq (0) = \bigcap I^n$, so $I^{n+1} : I \neq I^n$, for infinitely many n.

(4.2) COROLLARY. *If* I *is a nonzero ideal in a Noetherian domain* A, *then* $I^{n+1} : I = I^n$, *for all large* n.

Proof. Clear by (4.1).

(4.3) COROLLARY. *Let* I *be an ideal in a Noetherian ring* A *such that* $(0) : I = (0)$. *Then, for all large* m, $u\mathscr{R}(A, I^m)$ *has no irrelevant prime divisor,* I^m *satisfies* (PD), *and there exist* k *and* $b \in I^{mk}$ *such that* $(I^m)^{n+k} : bA = (I^m)^n$, *for all* $n \geq 1$.

Proof. Let $B = I^m$. Then, since $(H : I) : J = H : IJ$, it follows from (4.1) that $B^{n+1} : B = B^n$, for all $n \geq 1$. Therefore, the conclusions follow from (3.5.2).

For the next corollary to (4.1), recall that if I is an ideal in a ring A and if S is a multiplicatively closed set in A, then the S-*component* I_S *of* I is the ideal $I_S = \{x \in A : xs \in I, \text{ for some } s \in S\}$ of A; that is, $I_S = IA_S \cap A$.

(4.4) COROLLARY. *Let* I *be an ideal in a Noetherian ring* A, *and let* S *be a multiplicatively closed set in* A *such that* $(0)_S : I_S = (0)_S$. *Then* ${I^{n+k}}_S : {I^k}_S = {I^n}_S$, *for all large* n *and all* $k \geq 1$.

Proof. Let $B = S^{-1}A$. Then, in B, $(0) : IB = (0)$ (by hypothesis), so $I^{n+1}B : IB = I^nB$, for all large n by (4.1). Thus, $I^{n+k}B : I^kB = I^nB$, for all large n and all $k \geq 1$. Therefore, the conclusion follows by contracting to A.

(4.4) is a little awkward, but it becomes much more manageable when I is primary. Specifically, we have the following corollary to (4.1) which was proved by P. Samuel in [14].

(4.5) COROLLARY. *Let* q *be a* p-*primary ideal in a Noetherian ring* A *such that* p *is not a prime divisor of* (0). *Then* $q^{(n+k)} : q^{(k)} = q^{(n+k)} : q^k = q^{(n)}$, *for all large* n *and all* $k \geq 1$.

Proof. Fix a large n and $k \geq 1$ and let $H = q^{(n+k)} : q^{(k)}$. Then $H = q^{(n)}$, by (4.4). Also, $H \subseteq q^{(n+k)} : q^k$, which is p-primary, and

$$HA_p = q^{n+k}A_p : q^kA_p = (q^{(n+k)} : q^k)A_p,$$

so that $q^{(n+k)} : q^k = q^{(n)}$. *q. e. d.*

The following remark shows that some prime ideals do satisfy the hypothesis of (3.5.2).

(4.6) REMARK. *Let* (R, M) *be a regular local ring, and let* p *be a nonzero prime ideal in* R *such that* $p^n = p^{(n)}$, *for all* $n \geq 1$. *Then there exists* $b \in p$ *such* *that* $p^{n+1}:bR = p^n$, *for all* $n \geq 1$. *Therefore*, $p^{n+1}:p = p^n$, *for all* $n \geq 1$.

Proof. There exists $b \in p$ such that $p^{n+1}R_p:bR_p = p^nR_p$, for all $n \geq 1$ [17, Lemma 6, p. 402]. Therefore, by hypothesis and with $B = bR_p \cap R$,

$$p^n = p^{n+1}:B \subseteq p^{n+1}:bR, \quad \text{for all } n \geq 1$$

(see [16, Proof, p. 220]). Fix n and let $H = p^{n+1}:bR$. Then H is p-primary and $HR_p = p^{n+1}R_p:bR_p = p^nR_p$, so $H = p^{(n)} = p^n$. Therefore, since $b \in p$, for all $n \geq 1$, $p^{n+1}:p = p^n$. *q. e. d.*

The next result is related to (4.1). Recall that an ideal I in a ring A is said to be *normal* if $I^n = (I^n)_a$, for all $n \geq 1$.

(4.7) PROPOSITION. *Let* I *be an ideal in a Noetherian ring* A *such that* $(0):I = (0)$, *and assume that* I *is normal. Then* $I^n:I^m = I^{n-m}$, *for all* $n \geq m \geq 1$.

Proof. Fix $n \geq m \geq 1$ and let $H = I^m(I^n:I^m)$. Then $H = I^n$, since $H \subseteq I^n$ and $I^{n-m} \subseteq I^n:I^m$. Therefore, $I^m(I^n:I^m) = I^mI^{n-m}$. Now $(0):I = (0)$, so, by [3, Theorems 2 and 3],

$$I^{n-m} \subseteq I^n:I^m \subseteq (I^{n-m})_a = I^{n-m},$$

by hypothesis. Therefore, $I^n:I^m = I^{n-m}$, for all $n \geq m \geq 1$.

(4.8) COROLLARY. *Let* A *and* I *be as in* (4.7). *Then* $u\mathscr{R}(A, I)$ *has no irrelevant prime divisor,* I *satisfies* (PD), *and there exist* k *and* $b \in I^k$ *such that* $I^{n+k}:bA = I^n$, *for all* $n \geq 1$.

Proof. This follows immediately from (4.7) and (3.5.2).

An alternate proof of (4.8) can be given by verifying the following steps: if I is normal, then $(u\mathscr{R})_a = u\mathscr{R}$; hence, every prime divisor of $u\mathscr{R}$ has height one and is relevant.

The last result in this section applies (4.8) to analytically unramified local rings.

(4.9) COROLLARY. *Let* I *be an ideal in an analytically unramified local ring* R *such that* height $I \geq 1$, *fix a large integer* m, *and let* $B = (I^m)_a$. *Then* $u\mathscr{R}(R, B)$ *has no irrelevant prime divisor,* B *satisfies* (PD), *and there exist* k *and* $b \in B^k$ *such that* $B^{n+k}:bA = B^n$, *for all* $n \geq 1$.

Proof. B is normal [13, Theorem 1], and $(0):B = (0)$ (since height $B > 0$ and Rad $R = (0)$), so the conclusions follow from (4.8).

5. NOTES ON $K + I^n$

In (2.18), three equivalences of (2.1) were given, two of which concerned prime divisors of $K + I^n$. In this section, we characterize these prime divisors in (5.2), and then the main result (5.5) gives some further information concerning them. Finally, in (5.10) it is shown that frequently in local rings $p^n \neq p^{(n)}$.

To prove (5.5), we need the following two lemmas. To explain the setting for the first lemma, let I be an ideal in a Noetherian ring A, and let $\mathscr{R} = \mathscr{R}(A, I)$. Then, as has been noted in the proof of (2.11),

$$W = \{p \in \text{Spec } A: p \text{ is a prime divisor of } I^k, \text{ for some } k \geq 1\} \subseteq$$

$$W' = \{P \cap A: P \text{ is a prime divisor of } u\mathscr{R}\}.$$

(5.1) shows that, in fact, $W = W'$.

(5.1) LEMMA. *Let* I *be an ideal in a Noetherian ring* A, *let* $\mathscr{R} = \mathscr{R}(A, I)$, *and let* P *be a prime divisor of* $u\mathscr{R}$. *Then* $p = P \cap A$ *is a prime divisor of* I^k, *for some* $k \geq 1$. *Therefore, with* W *and* W' *as above,* $W = W'$.

Proof. There exists a homogeneous element bt^m ($b \in I^m$) in $\mathscr{R}$ such that $u\mathscr{R}: bt^m \mathscr{R} = P$ (so $m \geq 0$, since $bt^m \notin u\mathscr{R}$). Then

$$p = P \cap A = (u\mathscr{R}: bt^m\mathscr{R}) \cap A = I^{m+1}: bA,$$

so p is a prime divisor of I^{m+1}, for some $m \geq 0$. The last statement follows from this and the comment preceding this lemma.

(5.2) characterizes the set of prime divisors of $K + I^k$ ($k \geq 1$).

(5.2) COROLLARY. *Let* K *and* I *be ideals in a Noetherian ring* A, *let* $\mathscr{R} = \mathscr{R}(A, I)$, *and let* $K^* = KA[t, u] \cap \mathscr{R}$. *Then*

$$\{P \cap A: P \textit{ is a prime divisor of } (K^*, u)\mathscr{R}\}$$

$$= \{p \in \text{Spec } A: p \textit{ is a prime divisor of } K + I^k, \textit{for some } k \geq 1\}.$$

Proof. $\mathscr{R}/K^* \cong \mathscr{R}(A/K, (I + K)/K)$ by (2.4.4), so the conclusion follows from (5.1).

The following lemma is a useful result. Its proof, from the third sentence on, is essentially the proof given by D. G. Northcott and D. Rees for [4, Lemma 6].

(5.3) LEMMA. *Let* z *be a prime divisor of* (0) *in a Noetherian ring* A, *and let* b *be a regular element in* A *such that* $I = (z, b)A \neq A$. *Then, for each minimal prime divisor* p *of* I, p *is a prime divisor of* bA.

Proof. The hypotheses continue to hold in A_p, so it may be assumed that A is a local ring and p is the maximal ideal in A. Suppose that p is not a prime divisor of bA, so that $bA = bA: p$. Hence, $bA = bA: z$ (since $(z, b)A$ is p-primary). Therefore, there exists $x \in z$ such that $bA: xA = bA$. Then $xy = 0 \in bA$, for some $0 \neq y \in A$, so $y \in bA$; say $y = by_1$. Then $0 = xy = xby_1$, so $xy_1 = 0$; hence, $y_1 \in bA$, whence $y \in b^2 A$. Repeating this procedure, we see that $y \in b^n A$, for all n. Thus, $y = 0$ (since A is local); contradiction. Therefore, p is a prime divisor of bA. *q. e. d.*

The following corollary to (5.3) sharpens [17, Lemma 1, p. 394].

(5.4) COROLLARY. *Let* K *be an ideal in a Noetherian ring* A, *and let* $b \in A$ *such that* $K: bA = K$. *Then, for each prime divisor* q *of* K *and each prime ideal* p *in* A *which is a minimal prime divisor of* $(q, b)A$, p *is a prime divisor of* $(K, b)A$.

Proof. This follows immediately from (5.3) on considering A/K.

We can now prove the main result in this section.

(5.5) THEOREM. *Let* K *and* I *be ideals in a Noetherian ring* A, *and let* $K = \bigcap_1^g q_i$ *be a normal primary decomposition of* K, *where* q_i *is* p_i*-primary. Then the following ideals are equal:*

(5.5.1) $I_1 = \bigcap (K + I^n)$.

(5.5.2) $I_2 = \bigcap \{q_i : p_i + I \neq A\}$.

(5.5.3) $I_3 = \bigcap \{q_i$: *there exist* $k \geq 1$ *and a prime divisor* p *of* $K + I^k$ *such that* $p_i \subseteq p\}$.

Proof. $I_1 = I_2$ by [16, Theorem 13, pp. 217-218], and it is clear that $I_2 \subseteq I_3$. Finally, let i be such that $p_i + I \neq A$, and let

$$p_i^* = p_i A[t, u] \cap \mathscr{R}, \quad \text{where } \mathscr{R} = \mathscr{R}(A, I).$$

Then p_i^* is a prime divisor of $K^* = KA[t, u] \cap \mathscr{R}$ by (2.4.5). Let $Z = (p_i^*, u)\mathscr{R}$. Then $Z_{[0]} = p_i + I \neq A$ (see (2.4.6)), so Z is proper. Therefore, since $K^* : u\mathscr{R} = K^*$, (5.4) says that there exists a prime divisor P of $(K^*, u)\mathscr{R}$ such that $Z \subseteq P$. Thus, $p_i^* \subseteq P$, so $p_i = p_i^* \cap A \subseteq P \cap A$, and $P \cap A$ is a prime divisor of $K + I^k$, for some $k \geq 1$ by (5.2). Therefore $I_3 \subseteq I_2$.

(5.6) REMARK. *The proof of* (5.5) *shows that if* p_i *is a prime divisor of* K *such that* $p_i + I \neq A$, *then there exist* $k \geq 1$ *and a prime divisor* p *of* $K + I^k$ *such that* $p_i \subseteq p$ (*and conversely*).

(5.7) COROLLARY. *With the notation of* (5.5), $\bigcap (K + I^n) = K$ *if and only if, for each* $i = 1, \cdots, g$, *there exist* $k \geq 1$ *and a prime divisor* p *of* $K + I^k$ *such that* $p_i \subseteq p$ (k *and* p *depend on* i).

Proof. Clear by (5.5), since $I_1 = I_3$.

(5.8) COROLLARY. *With the notation of* (5.5), *assume that* A *is a semi-local ring and* $I \subseteq J$, *the Jacobson radical of* A. *Then, for each* $i = 1, \cdots, g$, *there exist* $k \geq 1$ *and a prime divisor* p *of* $K + I^k$ *such that* $p_i \subseteq p$ (k *and* p *depend on* i).

Proof. Since A is semi-local and $I \subseteq J$, $I_3 = I_1 = K$, so the conclusion follows from (5.7).

The following corollary, which generalizes [17, Lemma 1, p. 394], is a somewhat unexpected result. ((5.9.3) applied to $I = bR$, with b regular, gives [17, Lemma 1, p. 394], since the prime divisors of bR and $b^{mn}R$ are the same.)

(5.9) COROLLARY. *Let* I *be an ideal in a local ring* R. *Then, for each prime divisor* z *of* (0) *in* R:

(5.9.1) *There exist* $k \geq 1$ *and a prime divisor* p *of* I^k *such that* $z \subseteq p$.

(5.9.2) *With* m *and* n *large and* $B = I^{mn}$, *there exists a prime divisor* p *of* B *such that* $z \subseteq p$.

(5.9.3) *If* $(0) : I = (0)$, *then* $z \subset p$ *in* (5.9.1) *and* (5.9.2).

Proof. (5.9.1) follows immediately from (5.8) with $K = (0)$.

(5.9.2) By (2.13.2), a prime ideal p in R is a prime divisor of B^k, for some $k \geq 1$, if and only if p is a prime divisor of B^j, for all $j \geq 1$. Therefore, the conclusion follows from (5.9.1).

(5.9.3) is clear from (5.9.1) and (5.9.2), since $z \subseteq p$ and p contains regular elements.

This paper will be closed with the following result.

(5.10) PROPOSITION. *Let* A *be a semi-local ring, and let* $z_1, \cdots, z_g$ *be the prime divisors of* (0) *in* A. *Let* p *be a prime ideal in* A *such that* $p \subseteq J$, *the Jacobson radical of* A, *and* $\bigcup z_i \not\subseteq p$. *Then:*

(5.10.1) $p^n \neq p^{(n)}$, *for some* $n > 1$.

(5.10.2) *If* $p \not\subseteq \bigcup z_i$, *then* $p^n \neq p^{(n)}$, *for all large* n.

Proof. Let $\mathscr{R} = \mathscr{R}(A, p)$, and let $Z = (z^*, u)\mathscr{R}$, where z is a prime divisor of (0) in A such that $z \not\subseteq p$. Then Z is proper, as in the proof of (5.5). Let P be a minimal prime divisor of Z. Then P is a prime divisor of $u\mathscr{R}$ by (5.3), so $Q = P \cap A$ is a prime divisor of p^k, for some $k \geq 1$ by (5.1). Also, $z \subseteq Q$, so $Q \supset p$; hence, $k > 1$ and $p^k \neq p^{(k)}$. Thus (5.10.1) holds.

(5.10.2) Assume that $p \not\subseteq \bigcup z_i$, and suppose that every minimal prime divisor P of Z is irrelevant. Then $tp \subseteq \text{Rad } Z$, so $t^i p^i \subseteq Z$, for some $i \geq 1$. Therefore

$$p^i = (t^i p^i)_{[i]} \subseteq Z_{[i]} = (z \cap p^i) + p^{i+1} \subseteq (z \cap p^i) + Jp^i \subseteq p^i,$$

so by the lemma of Krull-Azumaya [2, (4.1)], $p^i = z \cap p^i \subseteq z$, a contradiction. Therefore, some P as above is relevant, so that $Q = P \cap A$ is a prime divisor of p^n, for all large n by (2.7). Thus, $p^n \neq p^{(n)}$, for all large n. *q. e. d.*

It is clear that if A is a local ring which has at least two prime divisors of zero and if no prime divisor of zero has height greater than 1, then all but a finite number of height-one prime ideals in A satisfy the conditions on p in (5.10).

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