# BRANCHED COVERINGS AND ORBIT MAPS

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## 1. INTRODUCTION

Let $f: X \to Y$ be a finite-to-one, closed and open (continuous) map, and consider the question as to when the induced homomorphism in rational (sheaf-theoretic) cohomology

(1.1) $$f^*: H^*(Y; \mathbb{Q}) \to H^*(X; \mathbb{Q}) \text{ is injective}.$$

Questions of this sort seem to have originated in a dual form with Eilenberg [6] and Whyburn [11]. See also [2]. If X and Y are connected orientable manifolds, then a standard Poincaré duality and degree argument shows that (1.1) is true. Also, if f happens to be the orbit map for a finite group action on X, then (1.1) again holds because of the existence of a transfer map in this context [1; II.19].

On the other hand, (1.1) is known to be false in general, since Bredon [2] and others have constructed finite-to-one, open, piecewise linear maps from compact contractible polyhedra onto the 2-sphere.

In this paper, X and Y shall always be assumed to be locally connected Hausdorff spaces. Additional hypotheses on the map f or on the spaces X and Y are then considered which guarantee that (1.1) holds. In Section 2, standard geometric notions of degree and local degree for f are defined and it is shown that if the degree of f is always equal to the sum of the local degrees of f on each point inverse, then (1.1) holds. The proof involves the construction of transfer homomorphisms. Such a map f can be viewed as a generalization of Fox's notion [7] of a branched covering. In Section 3, the concept of a (topological) normal n-circuit is introduced, generalizing that of an n-manifold. It is then shown that if X is a normal n-circuit, then Y is also a normal n-circuit and the degree of f is always equal to the sum of the local degrees in each point inverse of f. Proofs here are based on work of Černavskiĭ [3], [4] and Väisälä [10]. Finally, in Section 4 the automorphism group of f consisting of all homeomorphisms $g: X \to X$ such that $fg = f$ is calculated when the branch set of f does not locally separate X. This result is then used to characterize those maps f which can be identified with orbit maps for finite group actions in the case where X is a simply connected piecewise linear manifold, Y is a polyhedron, and f is a piecewise linear map.

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## 2. FINITE BRANCHED COVERINGS

Throughout this section, X and Y will denote locally connected, connected, Hausdorff spaces and $f: X \to Y$ will be a finite-to-one, closed and open (continuous) map. Thus for any $y \in Y$, $f^{-1}(y)$ is a finite set, and if $A \subset X$ is closed (open), then f(A) is closed (open) in Y.

If S is a finite set, let #S denote the cardinality of S. Define the *degree* of f to be $\deg(f) = \sup\{\#f^{-1}(y): y \in Y\}$, if the supremum exists; otherwise, set $\deg(f) = \infty$. Define the *local degree* of f at $x \in X$ to be

$$\deg(f; x) = \inf_U \sup\{\#f^{-1}f(z) \cap U: z \in U\}$$

(where U ranges over the neighborhoods of x in X) if the supremum exists for some U; otherwise, set $\deg(f; x) = \infty$.

Notice that if $x \in V \subset U$, then

$$\sup\{\#f^{-1}f(z) \cap V: z \in V\} \leq \sup\{\#f^{-1}f(z) \cap U: z \in U\}.$$

Thus, if $\deg(f; x) < \infty$, it follows that $\deg(f; x) = \sup\{\#f^{-1}f(z) \cap U: z \in U\}$ for all sufficiently small neighborhoods U of x. Also, if $\deg(f) < \infty$, then $\deg(f; x) < \infty$ for all $x \in X$.

Define $f: X \to Y$ to be a *finite branched covering* (FBC) if $\deg(f) < \infty$ and for each $y \in Y$, $\deg(f) = \sum_{x \in f^{-1}(y)} \deg(f; x)$.

*Remark.* Among finite-to-one, closed and open maps of locally connected, connected, Hausdorff spaces, finite branched coverings are strictly more general than the branched coverings of Fox [7; p. 250]. His hypotheses easily imply, via covering space theory, the additivity of local degree (the argument is essentially that of Theorem 3.2 below), but exclude from consideration many "folded coverings" which FBC's include (*e.g.*, $S^n \to D^n$, by $(x_0, \cdots, x_n) \mapsto (x_0, \cdots, x_{n-1})$). In particular, the orbit map for *any* finite group action on a locally connected Hausdorff space is an FBC.

If R is a ring with unit, $\mathscr{B}$ is a sheaf of (left) R-modules on a space Z, and $\phi$ is a family of supports in Z, then $H^*_\phi(Z; \mathscr{B})$ denotes the sheaf-theoretic cohomology of Z with supports in $\phi$ and coefficients in $\mathscr{B}$. As far as possible, sheaf-theoretic notation will follow that established by Bredon in [1].

The following is the main result of this section.

THEOREM 2.1. *Let* R *be a ring with unit, let* $\mathscr{B}$ *be a* constant *sheaf of left* R-*modules, let* $\phi$ *be a family of supports on* Y, *and let* $f: X \to Y$ *be a finite branched covering. Then there is a transfer homomorphism*

$$\tau: H^*_{f^{-1}\phi}(X; \mathscr{B}) \to H^*_\phi(Y; \mathscr{B})$$

*such that* $\tau f^* = \deg(f)\,1$.

COROLLARY 2.2. *If* $f: X \to Y$ *is an* FBC, *then* $f^*: H^*_\phi(Y; \mathbb{Q}) \to H^*_{f^{-1}\phi}(X; \mathbb{Q})$ *is injective.*

COROLLARY 2.3. *If* $f: X \to Y$ *is an* FBC *where* X *and* Y *have the homotopy type of finite* CW *complexes, then* $f_*: H_*(X; \mathbb{Q}) \to H_*(Y; \mathbb{Q})$ *is surjective in rational singular homology.*

Some preliminary facts are needed for the proof of Theorem 2.1. Except when explicitly stated to the contrary, $f\colon X \to Y$ is any finite-to-one, closed and open map.

LEMMA 2.4. *If* $V \subset Y$ *is open and connected and* $U$ *is a component of* $f^{-1}(V)$ *in* $X$, *then* $U$ *is open and* $f \mid U\colon U \to V$ *is a finite-to-one, closed and open, surjective map.*

*Proof.* Since $X$ is locally connected and $f^{-1}(V)$ is open in $X$, it follows that $U$ is open in $X$. That $f \mid U$ is finite-to-one and open is obvious. To see that $f \mid U\colon U \to V$ is also closed, let $A \subset U$ be closed in $U$. Then

$$A = \overline{A} \cap U = \overline{A} \cap f^{-1}(V),$$

where $\overline{A}$ is the closure of $A$ in $X$. It follows that $f(A) = f(\overline{A}) \cap V$, so that $f(A)$ is closed in $V$. Since $f(U)$ is closed and open in $V$, $f(U) = V$.

*Notation.* If $V \subset Y$ is connected, then $U/V$ signifies that $U$ is a component of $f^{-1}(V)$. Similarly, if $y \in Y$ then $x/y$ signifies that $x \in f^{-1}(y)$.

LEMMA 2.5. *Let* $y \in Y$ *and* $f^{-1}(y) = \{x_1, x_2, \cdots, x_k\}$. *Then there exist arbitrarily small connected open neighborhoods* $V$ *of* $y$ *in* $Y$ *such that*

$$f^{-1}(V) = U_1 \coprod U_2 \coprod \cdots \coprod U_k,$$

*the disjoint union of connected open neighborhoods of* $x_1, x_2, \cdots, x_k$, *respectively, and such that* $\deg(f; x_i) = \deg(f \mid U_i)$, *for* $i = 1, 2, \cdots, k$.

*Proof.* Let $W_1, W_2, \cdots, W_k$ be arbitrarily small disjoint open neighborhoods of $x_1, x_2, \cdots, x_k$, respectively, such that $\deg(f; x_i) = \sup\{\# f^{-1} f(z) \cap W_i : z \in W_i\}$ for each $i$. Let $V$ be the component of the open set $Y - f\left(X - \bigcup W_i\right)$ which contains $y$. Clearly by choosing the $W_i$'s arbitrarily small, $V$ also can be made arbitrarily small. Let $U_i$ be the component of $f^{-1}(V)$ containing $x_i$, $i = 1, 2, \cdots, k$. It follows from Lemma 2.4 that $f^{-1}(V) = U_1 \coprod U_2 \coprod \cdots \coprod U_k$, because $f^{-1}(V) \subset \bigcup W_i$, so that the $U_i$'s are pairwise disjoint and map onto $V$. Finally, since $U_i \subset W_i$ for each $i$, $\deg(f; x_i) = \sup\{\# f^{-1} f(z) \cap U_i : z \in U_i\} = \deg(f \mid U_i)$.

If $y \in Y$ and $V$ is an open neighborhood of $y$, $V$ is said to be *good for* $f$ *at* $y$ if $V$ is connected, $f^{-1}(y) = \{x_1, x_2, \cdots, x_k\}$,

$$f^{-1}(V) = U_1 \coprod U_2 \coprod \cdots \coprod U_k,$$

where each $U_i$ is a connected neighborhood of $x_i$, and $\deg(f; x_i) = \deg(f \mid U_i)$, $i = 1, 2, \cdots, k$.

Define an element $y \in Y$ to be a *principal value* for $f$ if $\# f^{-1}(y) = \deg(f)$.

LEMMA 2.6. *If* $\deg(f) < \infty$, *then the set of principal values for* $f$ *is a nonempty open set in* $Y$; *if* $f\colon X \to Y$ *is an* FBC, *then the set of principal values for* $f$ *is also dense in* $Y$.

*Proof.* Let $y \in Y$ be a principal value and let $V$ be an open neighborhood of $y$ which is good for $f$ at $y$. Then $V$ consists entirely of principal values. For it follows immediately that for any $z \in V$, $\# f^{-1}(z) \geq \# f^{-1}(y) = \deg(f)$, so that $\# f^{-1}(z) = \deg(f)$.

Now suppose that $f\colon X \to Y$ is an FBC and let $y \in Y$. Suppose that $V$ is an open neighborhood of $y$ which is good for $f$ at $y$. It suffices to show that $V$ contains principal values. Let $z_0 \in V$ be a point such that $\# f^{-1}(z_0) = \sup\{\# f^{-1}(z): z \in V\}$. Then $\deg(f; x) = 1$ for all $x/z_0$. For let $W$ be a neighborhood of $z_0$ in $V$ which is good for $f$ at $z_0$, so that if $f^{-1}(z_0) = \{x_1, x_2, \cdots, x_k\}$ then

$$f^{-1}(W) = U_1 \coprod U_2 \coprod \cdots \coprod U_k, \quad \text{where } x_i \in U_i \text{ and } U_i/W.$$

Then for any $z \in W$, $\# f^{-1}(z) \geq k$; hence, by maximality, $\# f^{-1}(z) = k$. Therefore $\deg(f \mid U_i) = 1$ for all i, and so $\deg(f; x_i) = 1$ for all i. Then

$$\deg(f) = \sum_{x/z_0} \deg(f; x) = \# f^{-1}(z_0),$$

and $z_0$ is a principal value.

LEMMA 2.7. *Let* $f: X \to Y$ *be an* FBC, *let* $y \in Y$, *and let* $V$ *be a neighborhood of* $y$ *in* $Y$ *which is good for* $f$ *at* $y$. *Then for any principal value* $z \in V$ *of* $f$ *and for any* $U/V$, $\deg(f \mid U) = \# f^{-1}(z) \cap U$.

*Proof.* For such a $z \in V$, clearly $\deg(f \mid U) \geq \# f^{-1}(z) \cap U$. Now,

$$\deg(f) = \sum_{x/y} \deg(f; x) = \sum_{U/V} \deg(f \mid U)$$

$$\geq \sum_{U/V} \# f^{-1}(z) \cap U = \# f^{-1}(z) = \deg(f).$$

Therefore, $\deg(f \mid U) = \# f^{-1}(z) \cap U$ for each $U/V$.

PROPOSITION 2.8. *Let* $f: X \to Y$ *be an* FBC, *let* $V \subset Y$ *be open and connected, and let* $U/V$. *Then* $f \mid U: U \to V$ *is an* FBC.

*Proof.* First consider the case where $V$ is good for $f$ at $y \in V$. It must be shown that for any $z \in V$, $\deg(f \mid U) = \sum_{x \in f^{-1}(z) \cap U} \deg(f; x)$.

Let $P$ be a neighborhood of $z$ in $V$ which is good for $f$ at $z$. Then

$$f^{-1}(z) = \{q_1, q_2, \cdots, q_m\} \quad \text{and} \quad f^{-1}(P) = Q_1 \coprod Q_2 \coprod \cdots \coprod Q_m,$$

where $q_j \in Q_j$, $Q_j/P$, and $\deg(f \mid Q_j) = \deg(f; q_j)$ for $j = 1, 2, \cdots, m$. Let $p \in P$ be a principal value for $f$. Then, using Lemma 2.7 twice, we find that

$$\deg(f \mid U) = \# f^{-1}(p) \cap U = \sum_{Q_j \subset U} \# f^{-1}(p) \cap Q_j$$

$$= \sum_{Q_j \subset U} \deg(f \mid Q_j) = \sum_{Q_j \subset U} \deg(f; q_j).$$

This completes the proof of the special case.

Now consider the general case. Once again we must show that

$$\deg(f \mid U) = \sum_{x \in f^{-1}(y) \cap U} \deg(f; x),$$

holds for all $y \in V$. Let $W = \{y \in V: \deg(f \mid U) = \sum_{x \in f^{-1}(y) \cap U} \deg(f; x)\}$. We shall show that $W$ is nonempty, open, and closed in the connected set $V$, so that $W = V$, completing the proof.

To see that $W \neq \emptyset$, let $S = \{y \in V: \deg(f \mid U) = \# f^{-1}(y) \cap U\}$ and $T = \{y \in V: \deg(f) = \# f^{-1}(y)\}$. Then by Lemma 2.6, $S$ is a nonempty open set while $T$ is a dense open set in $V$. Thus, $S \cap T \neq \emptyset$. Let $y \in S \cap T$. Then $\deg(f; x) = 1$ for all $x \in f^{-1}(y)$. Thus

$$\deg(f \mid U) = \# f^{-1}(y) \cap U = \sum_{x \in f^{-1}(y) \cap U} \deg(f; x),$$

and $y \in W$.

To see that $W$ is open, let $y \in W \subset V$ and let $Q$ be a neighborhood of $y$ in $V$ which is good for $f$ at $y$. It suffices to show $Q \subset W$. Let

$$f^{-1}(y) = \{x_1, x_2, \cdots, x_m\} \quad \text{and} \quad f^{-1}(Q) = P_1 \amalg P_2 \amalg \cdots \amalg P_m,$$

where $x_i \in P_i$, $P_i/Q$, and $\deg(f \mid P_i) = \deg(f; x_i)$ for each $i$. By the first case considered $f \mid P_j: P_j \to Q$ is an FBC. Let $z \in Q$. Then, since $y \in W$ and $f \mid P_j$ is an FBC,

$$\deg(f \mid U) = \sum_{x_j \in U} \deg(f; x_j) = \sum_{P_j \subset U} \deg(f; x_j) = \sum_{P_j \subset U} \deg(f \mid P_j)$$

$$= \sum_{P_j \subset U} \sum_{x \in f^{-1}(z) \cap P_j} \deg(f; x) = \sum_{x \in f^{-1}(z) \cap U} \deg(f; x).$$

Thus $z \in W$, as desired.

Finally, to see that $W$ is closed, let $y \in \overline{W}$ and let $Q$ be a neighborhood of $y$ in $Y$ which is good for $f$ at $y$. Let

$$f^{-1}(y) = \{x_1, x_2, \cdots, x_m\} \quad \text{and} \quad f^{-1}(Q) = P_1 \amalg P_2 \amalg \cdots \amalg P_m,$$

where $x_i \in P_i$, $P_i/Q$, and $\deg(f \mid P_i) = \deg(f; x_i)$ for each $i$. Choose $z \in Q \cap W \subset V$. Then

$$\deg(f \mid U) = \sum_{x \in f^{-1}(z) \cap U} \deg(f; x) \qquad \text{(since } z \in W\text{)}$$

$$= \sum_{P_j \subset U} \sum_{x \in f^{-1}(z) \cap P_j} \deg(f; x)$$

$$= \sum_{P_j \subset U} \deg(f \mid P_j) \qquad \text{(since by the first case } f \mid P_j \text{ is an FBC)}$$

$$= \sum_{P_j \subset U} \deg(f; x_j) = \sum_{x \in f^{-1}(y) \cap U} \deg(f; x).$$

Thus $y \in W$ and $W$ is closed.

*Proof of Theorem* 2.1. Let $\mathscr{A} = f^*\mathscr{B}$ be the induced sheaf on X (necessarily a constant sheaf $X \times B$ where $\mathscr{B} = Y \times B$), and let $f\mathscr{A}$ denote the direct image sheaf associated with the presheaf $f\mathscr{A}(V) = \mathscr{A}(f^{-1}V)$. Here, as usual, we identify a sheaf with its associated presheaf of sections.

Because $\mathscr{B}$ is a constant sheaf on Y, for any connected open set $V \subset Y$, $\mathscr{B}(V)$ can be *canonically* identified with B. In particular, if $V' \subset V$ are both connected, the restriction homomorphism $\mathscr{B}(V) \to \mathscr{B}(V')$ is identified with the identity $B \to B$. A similar observation applies to $\mathscr{A}$ on X.

Since f is finite-to-one and closed, the Vietoris-Begle theorem [1; II.11.1] implies that the natural homomorphism $f^+: H^*_\phi(Y; f\mathscr{A}) \to H^*_{f^{-1}\phi}(X; \mathscr{A})$ is an isomorphism. Thus, to define a transfer map $\tau: H^*_{f^{-1}\phi}(X; \mathscr{A}) \to H^*_\phi(Y; \mathscr{B})$ such that $\tau f^* = \deg(f)\,1$, where $f^*: H^*_\phi(Y; \mathscr{B}) \to H^*_{f^{-1}\phi}(X; \mathscr{A})$ is the induced map on cohomology, it suffices to find a sheaf homomorphism $\sigma: f\mathscr{A} \to \mathscr{B}$ such that if $h: \mathscr{B} \to ff^*\mathscr{B} = f\mathscr{A}$ is the natural homomorphism, then $\sigma h = \deg(f)\,1$. For $\tau$ can then be defined to be the composition

$$\sigma_*(f^+)^{-1}: H^*_{f^{-1}\phi}(X; \mathscr{A}) \to H^*_\phi(Y; f\mathscr{A}) \to H^*_\phi(Y; \mathscr{B}),$$

since $f^*$ is precisely the composition

$$f^+h_*: H^*_\phi(Y; \mathscr{B}) \to H^*_\phi(Y; f\mathscr{A}) \to H^*_{f^{-1}\phi}(X; \mathscr{A}).$$

One defines $\sigma: f\mathscr{A} \to \mathscr{B}$ on fibers as follows. Note that $\mathscr{B}_y = B$ while $(f\mathscr{A})_y = \bigoplus_{x/y} \mathscr{A}_x = \bigoplus_{x/y} B$. Then if $c \in (f\mathscr{A})_y$, one can write $c = \bigoplus_{x/y} c_x$, $c_x \in B$. Define $\sigma_y: (f\mathscr{A})_y \to \mathscr{B}_y$ by $\sigma_y(\bigoplus_{x/y} c_x) = \sum_{x/y} \deg(f; x)\, c_x$. Clearly this defines a function $\sigma: f\mathscr{A} \to \mathscr{B}$ which, on fibers, is a homomorphism. Moreover, on fibers, the natural homomorphism $h: \mathscr{B} \to f\mathscr{A}$ is the diagonal map $h_y: B \to \bigoplus_{x/y} B$. Thus, for $c \in \mathscr{B}_y$,

$$\sigma h(c) = \sigma \bigoplus_{x/y} c = \sum_{x/y} \deg(f; x)\, c = \deg(f)\, c\ .$$

Therefore the crux of the proof is to show that $\sigma$ is continuous. This will be accomplished by showing that $\sigma$ is induced by a homomorphism of the associated presheaves of sections of the respective sheaves $\mathscr{B}$ and $f\mathscr{A}$.

For any open set $V \subset Y$, define a homomorphism $\sigma_V: f\mathscr{A}(V) \to \mathscr{B}(V)$. Now if $V = \coprod_{\alpha \in A} V_\alpha$, where each $V_\alpha$ is connected, then each $V_\alpha$ is open, since Y is locally connected and we have $f\mathscr{A}(V) = \prod_{\alpha \in A} f\mathscr{A}(V_\alpha)$ and $\mathscr{B}(V) = \prod_{\alpha \in A} \mathscr{B}(V_\alpha)$. Therefore, we may assume that V is connected. Then

$$f\mathscr{A}(V) = \mathscr{A}(f^{-1}V) = \bigoplus_{U/V} \mathscr{A}(U) = \bigoplus_{U/V} B_U\,,$$

where $B_U = B$, since $\mathscr{B}$ and $\mathscr{A}$ are constant sheaves. Also, $\mathscr{B}(V) = B$. Thus if $c \in f\mathscr{A}(V)$, one can write $c = \bigoplus_{U/V} c_U$, $c_U \in B$. Define

$$\sigma_V(c) = \sum_{U/V} \deg(f \mid U)\, c_U \quad \text{in } B = \mathscr{B}(V).$$

Each $\sigma_V$ is clearly a homomorphism, being a linear combination of projections. Also, if the family $\{\sigma_V\}$ for all open $V \subset Y$ can be shown to be compatible (and so to define a presheaf homomorphism), it is evident that $\sigma$ is induced by $\{\sigma_V\}$ and so is continuous.

Therefore the proof of Theorem 2.1 is reduced to showing that if $V' \subset V$ are open sets in Y, then the diagram

$$\begin{array}{ccc} f\mathscr{A}(V) & \xrightarrow{\sigma_V} & \mathscr{B}(V) \\ \downarrow r & & \downarrow r \\ f\mathscr{A}(V') & \xrightarrow{\sigma_{V'}} & \mathscr{B}(V') \end{array}$$

(in which vertical homomorphisms are restrictions) is commutative. As in the definition of $\sigma_V$, it may be assumed that V and V' are both connected.

Let $c \in f\mathscr{A}(V) = \bigoplus_{U/V} \mathscr{A}(U)$ and write $c = \bigoplus_{U/V} c_U$, $c_U \in \mathscr{A}(U) \approx B$. Then

$$r\sigma_V(c) = r \sum_{U/V} \deg(f \mid U)\, c_U = \sum_{U/V} \deg(f \mid U)\, c_U,$$

since $r: \mathscr{B}(V) \to \mathscr{B}(V')$ is naturally identified with the identity $1: B \to B$.

On the other hand, $r(c) = r(\oplus c_U) = \bigoplus_{U/V} \bigoplus_{\substack{U'/V' \\ U' \subset U}} r_{UU'}(c_U)$, where $r_{UU'}: \mathscr{A}(U) \to \mathscr{A}(U')$ is restriction, again identified with the identity $1: B \to B$. Therefore, by Proposition 2.8,

$$\sigma_{V'} r(c) = \sigma_{V'}\Big( \bigoplus_{U/V} \bigoplus_{\substack{U'/V' \\ U' \subset U}} r_{UU'}(c_U)\Big) = \sigma_{V'}\Big( \bigoplus_{U/V} \bigoplus_{\substack{U'/V' \\ U' \subset U}} c_U\Big)$$

$$= \sum_{U/V} \sum_{\substack{U'/V' \\ U' \subset U}} \deg(f \mid U')\, c_U = \sum_{U/V} \Bigg( \sum_{\substack{U'/V' \\ U' \subset U}} \deg(f \mid U') \Bigg) c_U = \sum_{U/V} \deg(f \mid U) c_U$$

as desired.

*Remark.* Since the transfer map in Theorem 2.1 was constructed using a presheaf homomorphism, it is evident that the analogue of Theorem 2.1 also holds for general locally connected Hausdorff spaces and cohomology defined via the Čech construction.

## 3. NORMAL CIRCUITS

Throughout this section, $f: X \to Y$ denotes a finite-to-one, closed and open map between connected, locally connected Hausdorff spaces. Additional conditions are given which guarantee that f be a finite branched covering. In particular, a large class of spaces X is defined for which f must necessarily be an FBC.

Let $B_f$ denote the *branch set* of f, the closed set of points of X at which f fails to be a local homeomorphism.

LEMMA 3.1. *If* $\deg(f) < \infty$, *then* $\operatorname{int} fB_f = \emptyset$.

*Proof.* Suppose $\operatorname{int} fB_f \neq \emptyset$. Choose $y \in \operatorname{int} fB_f$ such that

$$\# f^{-1}(y) = \sup\{\# f^{-1}(z) : z \in \operatorname{int} fB_f\}.$$

Clearly, such a y exists and $\# f^{-1}(y) \leq \deg(f)$. Choose an open neighborhood V of y inside $\operatorname{int} fB_f$ which is good for f at y. Then for each U/V in X, $f \mid U: U \to V$ is a homeomorphism, being a one-to-one closed surjection. But this implies that $fB_f \cap V = \emptyset$, a contradiction.

Recall that a subspace A of X *separates* X if X - A is not connected, and A *locally separates* X at $x \in A$ if for arbitrarily small connected neighborhoods V of x in X, $V \cap A$ separates V. If $\operatorname{int} A = \emptyset$ and $V \cap A$ separates V, then it is easily seen that A locally separates X at some point of $V \cap A$.

THEOREM 3.2. *If* $\deg(f) < \infty$ *and* $fB_f$ *neither separates nor locally separates* Y, *then* $f: X \to Y$ *is an* FBC.

*Proof.* The restriction $f \mid X - f^{-1}fB_f : X - f^{-1}fB_f \to Y - fB_f$ is a finite-to-one, closed, local homeomorphism and hence a covering map, with connected base $Y - fB_f$. Since the set of principal values for f is open and $\operatorname{int} fB_f = \emptyset$, deg(f) is the number of sheets in this covering. Let $y \in Y$, and let V be a neighborhood of y which is good for f at y.

If U/V, then $\deg(f \mid U)$ is similarly the degree of the covering

$$U - (f^{-1}fB_f \cap U) \to V - fB_f \cap V.$$

Let $z \in V - fB_f \cap V$. Then

$$\deg(f) = \# f^{-1}(z) = \sum_{U/V} \# f^{-1}(z) \cap U = \sum_{U/V} \deg(f \mid U) = \sum_{x/y} \deg(f; x),$$

since V is good for f at y.

If X is any space of (covering) dimension n, let

$$EX = \{x \in X : x \text{ has no neighborhood homeomorphic to } \mathbb{R}^n\}$$

denote the "intrinsic (n - 1)-skeleton" of X. Clearly, EX is closed in X. A space X is called a *normal* n-*circuit* if X is a connected, locally connected, Hausdorff space of dimension n such that $\operatorname{int} EX = \emptyset$ and EX does not locally separate X at any point (equivalently, X - EX is dense in X and "locally connected in X"). Then X - EX is a connected topological n-manifold. Note that any connected open subspace of a normal n-circuit is again a normal n-circuit.

The introduction of normal circuits is justified by Theorem 3.5 below, in which it is shown that the finite-to-one, closed and open image of a normal circuit is again a normal circuit. This result is an appropriate generalization of Whyburn's theorem [12; pp. 194 ff.] that the light open image of a compact 2-manifold is again a 2-manifold, since the obvious analogue for n-manifolds, $n > 2$, of the latter assertion, is, of course, false. Examples of normal circuits include manifolds, manifolds with boundary, polyhedral homology manifolds, joins of compact manifolds (more generally of normal circuits), complex algebraic varieties, and end point compactifications of connected manifolds.

LEMMA 3.3. *A finite union of closed nonlocally separating sets with empty interior in* X *must be nonlocally separating.*

*Proof.* An obvious inductive argument shows that it suffices to consider a union $A = A_1 \cup A_2$ of closed nonlocally separating sets with empty interior. Suppose A does in fact locally separate X at x. Then there is a neighborhood V of x such that $V - V \cap A$ is not connected, while $U = V - V \cap A_1$ is connected. Now

$$V - V \cap A = U - U \cap A_2,$$

so $U \cap A_2$ separates U. Since $U \cap A_2$ is closed and has empty interior in U, $U \cap A_2$ must therefore locally separate U, which it clearly does not.

LEMMA 3.4. *Suppose* $\deg(f) < \infty$. *If* $A \subset X$ *is a closed subspace such that* $\operatorname{int} A = \emptyset$ *and such that* A *does not locally separate* X *at any point, then* f(A) *does not locally separate* Y *at any point.*

*Proof.* First consider the special case when $f^{-1}f(A) = A$. If $x \in A$ and V is a connected neighborhood of f(x) in Y such that $V - (f(A) \cap V) = V_1 \coprod V_2$, where $V_1$ and $V_2$ are disjoint nonempty open sets, let U be the component of $f^{-1}(V)$ containing x. Then $U - (A \cap U) = U - (f^{-1}f(A) \cap U) = f^{-1}(V_1) \cap U \coprod f^{-1}(V_2) \cap U$. Thus, A must locally separate X at x, a contradiction.

The general case is now reduced to the special case. Suppose that f(A) locally separates Y at some point and let $C \subset f(A)$ be the set of all points at which f(A) locally separates Y. Then the closure $\overline{C}$ locally separates Y at each point of C. For if $c \in C$ and V is a connected neighborhood of c in Y such that

$$V - (f(A) \cap V) = V_1 \coprod V_2,$$

the disjoint union of nonempty open sets, let $W_i = \operatorname{int}(\overline{V}_i - \overline{C} \cap \overline{V}_i)$. Then

$$V - V \cap \overline{C} = W_1 \coprod W_2,$$

since $\operatorname{int} f(A) = \emptyset$. (Compare [10; 2.1].) Thus, replacing A by $f^{-1}(\overline{C}) \cap A$ if necessary, we may assume that $f(A) = \overline{C}$.

Now choose $y \in C \subset f(A)$ so that $\#f^{-1}(y) = \sup\{\#f^{-1}(z): z \in C\}$. Then also $\#f^{-1}(y) = \sup\{\#f^{-1}(z): z \in \overline{C} = f(A)\}$. Let $f^{-1}(y) = \{x_1, x_2, \cdots, x_k\}$, and choose a small neighborhood V of y which is good for f at y and such that $f(A) \cap V$ separates V. Then $f^{-1}(V) = U_1 \coprod U_2 \coprod \cdots \coprod U_k$, where each $U_i$ is connected and open and $x_i \in U_i$, $i = 1, 2, \cdots, k$. Consider the restrictions $f \mid U_i: U_i \to V$, each of which is a finite-to-one, closed and open map of finite degree. By the maximality of $\#f^{-1}(y)$, $f \mid f^{-1}f(A) \cap U_i$ is one-to-one for each $i = 1, 2, \cdots, k$.

Since $(f \mid U_i)^{-1} f(A \cap U_i) = f^{-1}f(A \cap U_i) \cap U_i \subset f^{-1}f(A) \cap U_i$, it follows that $f \mid f^{-1}f(A \cap U_i) \cap U_i$ is also one-to-one. Therefore, $(f \mid U_i)^{-1} f(A \cap U_i) = A \cap U_i$. Hence, by the special case, $f(A \cap U_i)$ does not locally separate V. Then, by Lemma 3.3, $f(A) \cap V = \bigcup_i f(A \cap U_i)$ does not locally separate V. This is a contradiction since $\operatorname{int} f(A) = \emptyset$ and f(A) separates V, so that in fact f(A) must also locally separate V.

The following result will be derived from a theorem of Černavskiĭ [3], [4].

THEOREM 3.5. *If* $X$ *is a normal* $n$*-circuit, then* $\deg(f) < \infty$, $fB_f$ *does not locally separate* $Y$, *and* $Y$ *is a normal* $n$*-circuit.*

*Proof.* That $Y$ is a locally connected, connected Hausdorff space of dimension $n$ follows easily.

First consider the special case where $X$ is actually an $n$-manifold. In the present context, the result of Černavskiĭ can be stated as follows: $\deg(f) < \infty$ and the set of principal values of $f$ is open and dense in $Y$.

Let $A \subset fB_f$ be the closure of the set of points at which $fB_f$ locally separates $Y$, and proceed by induction on $\sup \{\# f^{-1}(z): z \in A\} \leq \deg(f)$.

Suppose that $A \neq \emptyset$ and that $\# f^{-1}(z) = 1$ for all $z \in A$. Let $z \in A$, and let $V$ be an open neighborhood of $z$ such that $V \cap fB_f$ separates $V$. Then $V \cap A$ also separates $V$. For if $V - (V \cap fB_f) = W_1 \coprod W_2$, set $V_i = \operatorname{int}(\overline{W}_i - A)$ for $i = 1, 2$. Then $V - V \cap A = V_1 \coprod V_2$, since $\operatorname{int} fB_f = \emptyset$. Let $U = f^{-1}(V)$ and $U_i = f^{-1}(V_i)$, $i = 1, 2$. Form $Z = U_1 \cup_A V_2$ and define $g: U \to Z$ to be $1 \mid U_1 \cup f \mid U_2$. Then $g$ is a finite-to-one, closed and open surjection, and $U$ is a manifold. By the result of Černavskiĭ, $g$ must be one-to-one and hence a homeomorphism. But then $fB_f \cap V = \emptyset$, contradicting the original choice of $z \in A \cap V$.

Now suppose that $A \neq \emptyset$ and that $\# f^{-1}(z) > 1$ for some $z \in A$. Choose $z \in A$ so that $\# f^{-1}(z)$ is maximal. Let $V$ be a small neighborhood of $z$ which is good for $f$ at $z$. Then for each $U/V$, $f \mid U: U \to V$ is a finite-to-one, closed and open surjection, where $U$ is an $n$-manifold by Lemma 2.4. By the maximality of $\# f^{-1}(z)$, $\# f^{-1}(y) \cap U = 1$ for all $y \in A \cap V$. By the previous case, $fB_{f|U}$ does not locally separate $V$; hence, the finite union of closed subsets of $V$, $\bigcup_{U/V} fB_{f|U} = fB_f \cap V$, also does not locally separate $V$. But this is a contradiction, since by assumption, $fB_f$ locally separates $Y$ at a point of $V$. Thus $A = \emptyset$, and $fB_f$ does not locally separate $X$ if $X$ is a manifold.

Now suppose that $X$ is a normal $n$-circuit. By Lemma 3.4, $f(EX)$ does not locally separate $Y$. By the case where the domain is a manifold,

$$C = f(B_f \cap (X - f^{-1} f(EX)))$$

does not locally separate $Y - f(EX)$. Therefore, $C \cup f(EX)$ does not locally separate $Y$, since $f(EX)$ is closed in $Y$. But clearly $fB_f \subset C \cup f(EX)$, so that $fB_f$ cannot locally separate $Y$ either.

To see that $Y$ is a normal $n$-circuit, observe that also $EY \subseteq fB_f \cup f(EX)$, so that $EY$ does not locally separate or separate $Y$.

Finally, $\deg(f) = \deg(f \mid X - f^{-1} fB_f) < \infty$.

COROLLARY 3.6. *Let* $X$ *be a normal* $n$*-circuit. Then* $f$ *is a finite branched covering.*

*Proof.* This follows immediately from Lemma 3.1 and Theorem 3.5.

## 4. AUTOMORPHISM GROUPS

Let $X$ and $Y$ be locally path-connected, connected Hausdorff spaces and let $f: X \to Y$ be a finite-to-one, closed and open map of finite degree. Define $\operatorname{Aut}(f)$, the *automorphism group* of $f$, to be the group of all homeomorphisms $g: X \to X$ such that $fg = f$.

THEOREM 4.1. *Suppose that* $f^{-1} fB_f$ *does not locally separate* X. *Then*

$$\mathrm{Aut}(f) \approx N_G(H)/H,$$

*where* $G = \pi_1(Y - fB_f, *)$ *and* $H = f_{\#} \pi_1(X - f^{-1} fB_f, *)$, *and* $N_G(H)$ *is the normalizer of* H *in* G.

*Proof.* By Lemma 3.1, int $f^{-1} fB_f = \emptyset$. By covering space theory,

$$\mathrm{Aut}(f \mid X - f^{-1} fB_f) \approx N_G(H)/H .$$

Therefore, it suffices to show that each

$$g: X - f^{-1} fB_f \to X - f^{-1} fB_f$$

in $\mathrm{Aut}(f \mid X - f^{-1} fB_f)$ extends (necessarily uniquely) to a homeomorphism $g': X \to X$. For then such a $g'$ clearly lies in Aut(f).

To this end, let $g \in \mathrm{Aut}(f \mid X - f^{-1} fB_f)$, let $x \in f^{-1} fB_f$, and let V be a neighborhood of f(x) which is good for f at f(x). Let U/V be such that $x \in U$. Then $U - f^{-1} fB_f \cap U$ is connected, which implies that $g(U - f^{-1} fB_f \cap U) = U' - f^{-1} fB_f \cap U'$ for some U'/V. Define g(x) to be x', where $f^{-1} f(x) \cap U' = \{x'\}$. Doing this for each $x \in f^{-1} fB_f$ defines a function $g': X \to X$.

Clearly, g' is well-defined, one-to-one, and onto. Continuity at $x \in f^{-1} fB_f$ follows since sets of the form U' provide a neighborhood basis at x' and $g^{-1} U' = U$. Finally, $(g')^{-1}$ is clearly just $(g^{-1})'$, and so must also be continuous, so that g' is a homeomorphism.

The map $f: X \to Y$ is called an *orbit map* if there is a finite group $\pi$ acting on X, covering the identity on Y such that the induced map of the orbit space $X/\pi \to Y$ is a homeomorphism.

COROLLARY 4.2. *If* $f^{-1} fB_f$ *does not locally separate* X, *then* f *is an orbit map if and only if the covering* $X - f^{-1} fB_f \to Y - fB_f$ *is regular* (*that is,* $f_{\#} \pi_1(X - f^{-1} fB_f, *)$ *is normal in* $\pi_1(Y - fB_f, *)$).

*Proof.* Clearly, f is an orbit map if and only if $f \mid X - f^{-1} fB_f$ is, by the argument of Theorem 4.1. But $f \mid X - f^{-1} fB_f$ is an orbit map if and only if it is regular, by standard covering space results [9].

Corollary 4.2 can often be applied, for example, when X and Y are both n-manifolds. For according to Väisälä [10; 5.4], $\dim(B_f) \leq n - 2$ in this case. Then by a result of Church and Hemmingsen [5; 2.1],

$$\dim(B_f) = \dim(fB_f) = \dim(f^{-1} fB_f) \leq n - 2 .$$

Thus, $f^{-1} fB_f$ cannot locally separate X. This observation generalizes almost immediately to the case where X and Y are normal n-circuits with $\dim(EX) \leq n - 2$ and $\dim(EY) \leq n - 2$. Further details are omitted.

As an application of these notions, consider the case where X is a connected, simply connected, piecewise linear (PL) n-manifold, Y is a connected polyhedron, and $f: X \to Y$ is a PL finite-to-one, closed and open map.

THEOREM 4.3. (a) *If* $\dim B_f \leq n - 3$, *then* f *is an orbit map.*

(b) *If* $\dim B_f \leq n - 2$, *then* f *is an orbit map if and only if for each* $y \in Y$ *the function* deg(f; x) *is constant on* $f^{-1}(y)$.

*Proof.* (a) Since $\dim B_f \leq n - 3$, it follows that $\dim f^{-1} fB_f \leq n - 3$ also. By general position, $\pi_1(X - f^{-1} fB_f) = 0$. Hence, f is an orbit map.

(b) Clearly, the condition on deg(f; x) is necessary for f to be an orbit map. To prove sufficiency, we may assume that $n \geq 2$. It is easy to see that $f^{-1} fB_f$ is a subpolyhedron of X of dimension at most n - 2. We can assume that X and Y are triangulated so that f is simplicial and so that $f^{-1} fB_f$ is a subcomplex of X. According to Corollary 4.2, it suffices to show that $f_{\#} \pi_1(X - f^{-1} fB_f, *)$ is normal in $\pi_1(Y - fB_f, *)$.

Let $\alpha$ be a loop in $X - f^{-1} fB_f$ based at *. It must be shown that every lift of the loop $f\alpha$ in $Y - fB_f$ back to $X - f^{-1} fB_f$ is again a loop.

By general position in the PL n-manifold X, $\alpha$ is homotopic to a sum of loops of the form $\dot{D}(\sigma, X)$, the boundary of the dual cell [8; p. 29] to an (n - 2)-simplex $\sigma$ of $f^{-1} fB_f$, suitably connected to * by a path. Now $f(\dot{D}(\sigma, X)) = \dot{D}(f\sigma, Y)$, since f is open, and

$$f^{-1}(\dot{D}(f\sigma, Y)) = \dot{D}(\sigma_1, X) \coprod \dot{D}(\sigma_2, X) \coprod \cdots \coprod \dot{D}(\sigma_k, X),$$

where $\sigma_1, \sigma_2, \cdots, \sigma_k$ are the (n - 2)-simplices of $f^{-1} f(\sigma)$.

Since Y is a normal n-circuit and f is a covering map away from the (n - 2)-skeleton, it is not hard to see that $\dot{D}(f\sigma, Y)$ is also a circle.

Now $f_{\#}[\dot{D}(\sigma, X)] = d[\dot{D}(f\sigma, Y)]$, where d is the degree of $\dot{D}(\sigma, X) \to \dot{D}(f\sigma, Y)$. By hypothesis, each restriction $\dot{D}(\sigma_i, X) \to \dot{D}(f\sigma, Y)$ also has degree d. Thus each lift of $d[\dot{D}(f\sigma, Y)]$ must be a loop. Hence, each lift of $f\alpha$ must also be a loop.

*Remark.* A similar argument also applies when X is a normal n-circuit and Y is a simply connected PL n-manifold.

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