SPECTRA OF OPERATOR EQUATIONS

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Halpern [2] and Akemann and Ostrand [1] have recently characterized the spectra of operator equations on W*- and C*-algebras in terms of the spectra of elements in suitable homomorphic images of the algebra in question. The purpose of this note is to show that in certain cases of special interest (*e.g.*, derivations and inner automorphisms), the spectrum of the operator equation on a W*-algebra can be described in an "internal" fashion.

If a is an $n \times n$ complex matrix and δ denotes the inner derivation induced by a on the algebra of all $n \times n$ complex matrices, it is well known that

$$\operatorname{sp}\delta = \{\alpha - \beta \mid \alpha, \beta \in \operatorname{sp} a\}.$$

(Indeed, the same proof applies if the matrix algebra is replaced by any prime algebra over a field F and a is any algebraic element of the algebra such that the minimum polynomial of a splits into linear factors in F.) In a more general setting, this statement may not be true (*e.g.*, [1, Example 2] or, more trivially, a commutative algebra). However, the same derivation may be induced by many different elements; in particular, if z is central, a and $a + z$ induce the same derivation. Corollary 1 says that on a W*-algebra, $\operatorname{sp}\delta$ is the intersection over all a which induce δ of $\{\alpha - \beta \mid \alpha, \beta \in \operatorname{sp} a\}$.

Let A be a W*-algebra with center Z. Let $a, b \in A$ and let ϕ_i, ψ_i be holomorphic functions on domains containing $\operatorname{sp} a$ and $\operatorname{sp} b$, respectively. Define the operator T in A by

$$Tx = \sum \phi_i(a)\, x\, \psi_i(b).$$

The following lemma is easily extracted from [2, Proposition 6] and its proof.

LEMMA. *Given $\gamma \notin \operatorname{sp} T$, there exists a finite set of mutually orthogonal projections $e_1, \cdots, e_n \in Z$ with $\sum e_j = 1$, such that γ is not in*

$$\left\{ \sum_i \phi_i(\alpha)\, \psi_i(\beta) \;\middle|\; \alpha \in \operatorname{sp}_{Ae_j} ae_j,\ \beta \in \operatorname{sp}_{Ae_j} be_j \right\}$$

for any $j = 1, \cdots, n$.

THEOREM. (i) *If* $Tx = ax + xb$, *then*

$$\operatorname{sp} T = \bigcap \{\alpha + \beta \mid \alpha \in \operatorname{sp}(a+z),\ \beta \in \operatorname{sp}(b - z)\},$$

where the intersection is taken over all $z \in Z$.

(ii) *If* $Tx = axb$, *then*

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$$\mathrm{sp}\,T = \bigcap \{\alpha\beta \mid \alpha \in \mathrm{sp}(az),\ \beta \in \mathrm{sp}(bz^{-1})\},$$

where the intersection is taken over all invertible $z \in Z$.

Proof. In each case, sp T is contained in the indicated set, by [4, Theorem 5].

Suppose $\gamma \notin \mathrm{sp}\,T$. Choose $e_1, \cdots, e_n$ as in the lemma. For (i), choose complex numbers $\mu_1, \cdots, \mu_n$ subject to the constraint

$$\gamma - \mu_i + \mu_j \notin \mathrm{sp}\,a + \mathrm{sp}\,b \qquad \text{for } i \neq j,$$

(*e.g.*, let $\mu_i = i\nu$, where $\nu > |\gamma| + \|a\| + \|b\|$). Let $z = \sum \mu_j e_j$. If $\alpha \in \mathrm{sp}(a+z)$ and $\beta \in \mathrm{sp}(b-z)$, then for some i and j, $\alpha = \alpha' + \mu_i$ and $\beta = \beta' - \mu_j$, where $\alpha' \in \mathrm{sp}_{Ae_i} ae_i$ and $\beta' \in \mathrm{sp}_{Ae_j} be_j$. Thus

$$\alpha + \beta = \alpha' + \beta' + \mu_i - \mu_j,$$

which by the choice of the μ's cannot equal γ if $i \neq j$. If $i = j$, $\alpha + \beta \neq \gamma$ by the choice of the e's.

For (ii), note that the lemma implies that if either $0 \in \mathrm{sp}\,a$ or $0 \in \mathrm{sp}\,b$, then $0 \in \mathrm{sp}\,T$. Thus we may proceed as above, choosing $\mu_1, \cdots, \mu_n$ to be nonzero and such that

$$\gamma \mu_j \mu_i^{-1} \notin (\mathrm{sp}\,a)(\mathrm{sp}\,b) \qquad \text{for } i \neq j.$$

COROLLARY 1. *Let* A *be a* W*-*algebra with center* Z, $a \in A$.

(i) *If* δ *is the derivation* $\delta x = ax - xa$ *on* A, *then*

$$\mathrm{sp}\,\delta = \bigcap_{z \in Z} \{\alpha - \beta \mid \alpha, \beta \in \mathrm{sp}(a+z)\} = \bigcap_{b \in D} \{\alpha - \beta \mid \alpha, \beta \in \mathrm{sp}\,b\},$$

where $D = \{b \in A \mid \delta x = bx - xb,\ \forall\, x \in A\}$.

(ii) *If* a *is invertible and if* σ *is the inner automorphism* $\sigma x = axa^{-1}$ *on* A, *then*

$$\mathrm{sp}\,\sigma = \bigcap_{z \in U(Z)} \{\alpha\beta^{-1} \mid \alpha, \beta \in \mathrm{sp}(az)\} = \bigcap_{b \in S} \{\alpha\beta^{-1} \mid \alpha, \beta \in \mathrm{sp}\,b\},$$

where U(Z) (*respectively,* U(A)) *denotes the group of invertible elements of* Z (*respectively,* A), *and* $S = \{b \in U(A) \mid \sigma x = bxb^{-1},\ \forall\, x \in A\}$.

If in either case $a = a^*$, *then the corresponding intersections may be restricted to* z (*respectively,* b) *such that* $z = z^*$ (*respectively,* $b = b^*$).

Proof. The last statement follows since the μ's in the proof of the theorem may be chosen to be positive real numbers.

If B is a C*-algebra acting faithfully as operators on a Hilbert space and if A is the W*-algebra generated by B, then any derivation δ on B is of the form $\delta x = ax - xa$ for some $a \in A$ ([3], [6]).

COROLLARY 2. (*Cf.* [1, Theorem 1] and [2, Example 10].) *If* B, A, δ *and* a *are as above and if* $a = a^*$, *then* sp δ *is given by the formula in Corollary* 1(i).

Proof. If δ' denotes the extension of δ to A determined by a, then by Corollary 1,

$$\operatorname{sp}\delta' = \bigcap_{z=z^*\in Z} \{\alpha - \beta \mid \alpha, \beta \in \operatorname{sp}_A(a+z)\},$$

which is a subset of the reals and hence fails to separate the plane. The result now follows from [5, 1.6.13] as in the proof of [2, Corollary 9].

It would be of interest to know whether Corollary 2 is true without any restriction on a.

Not every operator equation involving a single element a can be treated in the manner of Corollary 1. For example, let $A = \mathbb{C} \oplus \mathbb{C}_2$ (where $\mathbb{C}$ denotes the complex field), and let $a = \alpha \oplus a_1$, where $a_1 = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}$ and where α and β are nonzero and $\alpha \neq \pm\beta$. If $Tx = axa$, then the only elements $b \in A$ with $Tx = bxb$, $\forall x \in A$, are a, $-\alpha \oplus a_1$, $-\alpha \oplus -a_1$, and $\alpha \oplus -a_1$. For each such b, $\alpha\beta \in (\operatorname{sp} b)(\operatorname{sp} b)$, but $\alpha\beta \notin \operatorname{sp} T$.

It is doubtful that the methods of the theorem would be useful for operator equations not exhibiting a suitable form of homogeneity.

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