SMOOTH S^1-MANIFOLDS IN THE HOMOTOPY TYPE OF $\mathbb{C}P^3$

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0. INTRODUCTION

Four years ago T. Petrie [6] conjectured that *if* X *is a closed, smooth,* 2n-*dimensional homotopy* $\mathbb{C}P^n$ *that admits a nontrivial action of* S^1, *and if* $h: X \to \mathbb{C}P^n$ *is a homotopy equivalence, then* h *preserves Pontrjagin classes.*

In the present paper we prove the conjecture for the case $n = 3$:

THEOREM 0.1. *Let* X *be a closed, smooth* S^1-*manifold such that* $X^{S^1} \neq X$, *and let* $f: X \to \mathbb{C}P^3$ *be a homotopy equivalence. Then*

$$f^* \hat{\mathscr{A}}(\mathbb{C}P^3) = \hat{\mathscr{A}}(|X|),$$

where $|X|$ *denotes the underlying smooth manifold of* X,

$$\hat{\mathscr{A}}(|X|) = (x_i/2)(\sinh x_i/2)^{-1} \in H^*(|X|; \mathbb{Q}),$$

and the elementary symmetric functions of the x_i^2 *give the Pontrjagin classes of* $|X|$. *In particular,* f *preserves the Pontrjagin classes of* $|X|$.

Furthermore, a theorem of D. Montgomery and C. T. Yang [5] implies that there is a bijective application

$$P: \mathbb{Z} \to \{\text{diffeomorphism classes of smooth manifolds homotopy equivalent to } \mathbb{C}P^3\}$$

such that, for every $\alpha \in \mathbb{Z}$,

$$p_1(P(\alpha)) = (24\alpha + 4)z^2,$$

where p_1 is the first Pontrjagin class and z is a generator of $H^2(\mathbb{C}P^3)$.

THEOREM 0.2. *A closed smooth* S^1 *manifold* X, *homotopy-equivalent to* $\mathbb{C}P^3$ *and such that* $X^{S^1} \neq X$, *is diffeomorphic to* $\mathbb{C}P^3$.

Theorem 0.1 follows from Theorem 2.1, as indicated subsequently. This is intimately related to the proof of Theorem 1.3, which completely determines the rational torsion-free equivariant K-theory of X.

1. EQUIVARIANT COHOMOLOGIES

Let G be a compact abelian Lie group that is topologically cyclic, in other words, such that there exists a dense generator g in G. Let R(G) be the representation ring of G. Let Z be a closed, smooth G-manifold such that $Z^G \neq Z$, and let $\hat{K}_G^*(Z)$ be the quotient of the equivariant K-theory $K_G^*(Z)$ by its R(G)-torsion.

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We assume $K_G^*(Z^G) \neq \emptyset$. Then the R(G)-homomorphism

$$(1.1) \qquad i_Z^*: \hat{K}_G^*(Z) \to K_G^*(Z^G)$$

induced by the inclusion $i_Z: Z^G \to Z$ is a monomorphism, by the Atiyah-Segal localization theorem [6, page 109], and the induced homomorphism (where F is the field of fractions of R(G))

$$(1.1a) \qquad i_Z^* \otimes 1_F: \hat{K}_G^*(Z) \otimes F \to K_G^*(Z^G) \otimes F$$

is an isomorphism. For simplicity of exposition, we consider $\hat{K}_G^*(Z)$ as an R(G)-subalgebra of $K_G^*(Z^G)$, that is, we identify $K_G^*(Z)$ with its image under i_Z^*; thus $\hat{K}_G^*(Z) \otimes F$ is not only a submodule of $K_G^*(Z^G) \otimes F$, but it also coincides with it.

If Z is simply connected and the tangent bundle τZ of Z admits a spin^c structure [6, page 117], let

$$(1.2) \qquad Id_G^Z: \hat{K}_G^*(Z) \to R(G)$$

be the R(G)-module homomorphism defined as the composition

$$\hat{K}_G^*(Z) \xrightarrow{\psi} K_G^*(\tau Z) \xrightarrow{\text{Ind}} R(G),$$

where ψ is the Thom isomorphism of [6, page 119] and Ind is the Atiyah-Singer index homomorphism [1].

According to [6, page 123], Id_G^Z has an algebraic extension R(G) module homomorphism that we denote by

$$(1.3) \qquad Ed_G^Z: K_G^*(Z^G) \to F,$$

and such that, if $\{Z_i^G\}_{i=0}^{\ell-1}$ is the set of components of Z^G, then

$$(1.4) \qquad Ed_G^Z = tr\left[\bigoplus_{i=0}^{\ell-1} Ed_i\right],$$

where $Ed_i: K_G^*(Z_i^G) \to F$ are R(G)-module homomorphisms. If $G = S^1$, let

$$(1.5) \qquad h^*(\,\cdot\,) = \hat{K}_{S^1}^*(\,\cdot\,) \otimes \mathbb{Q}$$

and

$$(1.6) \qquad R = R(S^1) \otimes \mathbb{Q}.$$

(See the universal coefficient theorem (U.C.T.) of [4].) Let

$$(1.7) \quad \Lambda = h^*(Z), \qquad \Theta = h^*(Z^{S^1}), \qquad Id_\Lambda = Id_{S^1}^Z \otimes 1_R, \qquad Ed_\Lambda = Ed_{S^1}^Z \otimes 1_R.$$

THEOREM 1.1. *Let* Γ *be a* Λ*-submodule of* Θ *such that* $Ed_\Lambda(\Gamma) \subset R$. *Then* $\Gamma = \Lambda$.

Proof. Let $\gamma \in \Gamma$. Define $f \in Hom_R(\Lambda, R)$ by

$$f(x) = Ed_\Lambda(x\gamma).$$

By the U.C.T. of [4], there exists a unique element $\lambda \in \Lambda$ such that

$$f(x) = Id_\Lambda(x_\lambda) = Ed_\Lambda(x_\lambda).$$

Thus $Ed_\Lambda[x(\lambda - \gamma)] = 0$ for every $x \in \Lambda$. On the other hand, Ed_Λ also defines a nondegenerate bilinear form

$$\Theta \otimes F = \Lambda \otimes F \to \mathrm{Hom}_F(\Lambda \otimes F, F)$$

(see (1.1a)). Thus $\lambda = \gamma$.

We recall that an S^1-linear $\mathbb{C}P^n$ is a smooth S^1-manifold Y with underlying manifold $\mathbb{C}P^n$ and such that the action of S^1 over Y is given by

$$t \circ [z] = [w_0; \cdots; w_n],$$

where $t \in S^1$, $[z] = [z_0; \cdots; z_n]$, and $w_i = t^{a_i} z_i$ for $i = 0, \cdots, n$ (with the usual complex operations). We write

(1.8) $$Y = Y(a_0, \cdots, a_n).$$

A closed smooth S^1-manifold, homotopy-equivalent to $\mathbb{C}P^n$ (usually called an S^1-homotopy-$\mathbb{C}P^n$), is said to be S^1-exotic if it is not an S^1-linear $\mathbb{C}P^n$.

THEOREM 1.2. *If a closed smooth S^1-manifold X is homotopy equivalent to $\mathbb{C}P^n$, where $n \leq 3$, and if $X^{S^1} \neq X$, then there exists either an S^1-linear $\mathbb{C}P^n$ (see (1.8)), which we denote by Z_0, or an S^1-exotic $\mathbb{C}P^3$ of the only known type* [4], *denoted by Z_1; in either case, there exists an R-algebra-isomorphism*

(1.9) $$\omega: h^*(Z_j^{S^1}) \to h^*(X^{S^1}),$$

where $j = 0$ or 1, such that

(1.10) $$\omega[h^*(Z_j)] \subset h^*(X)$$

and

(1.11) $$[(Ed_{S^1}^{Z_j} \otimes 1_R) \circ \omega^{-1}][h^*(X)] \subset R.$$

The proof of Theorem 1.2 is carried out in Sections 3 and 4 for the two different possible situations $j = 0, 1$.

The following is an immediate corollary of Theorems 1.1 and 1.2.

THEOREM 1.3. *If X is a closed, smooth S^1-manifold, homotopy-equivalent to $\mathbb{C}P^n$, where $n \leq 3$, and such that $X^{S^1} \neq X$, then $\hat{K}^*_{S^1}(X) \otimes \mathbb{Q}$ is the rational $\hat{K}^*_{S^1}$ of either an S^1-linear $\mathbb{C}P^n$ or an S^1-exotic $\mathbb{C}P^3$ as in* [4].

2. THE DIFFERENTIAL STRUCTURE

Let X be a closed, smooth S^1-manifold in the homotopy type of $\mathbb{C}P^n$, such that $X^{S^1} \neq X$. The set of fixed points X^{S^1} is the disjoint union [2]

(2.1) $$X^{S^1} = \sum_{i=0}^{\ell-1} X_i$$

of the fixed-point set components X_i, such that X_i is a cohomology $\mathbb{C}P^{k_i}$ and

(2.2) $$\sum_{i=0}^{\ell-1} (k_i + 1) = n + 1 .$$

Let η be the equivariant Hopf bundle over X [6, page 132], and let $\nu_X X_i$ be the normal S^1-bundle of X_i in X. Then there are integers x_{ij} and a_i such that as complex S^1-modules, for $x_i \in X_i$

(2.3) $$(\nu_X X_i)_{x_i} = \sum_{j=1}^{n-k_i} t^{x_{ij}} \quad \text{and} \quad (\eta \mid X_i)_{x_i} = t^{a_i} .$$

The exponents a_i are distinct, and they are determined, up to translation by a common integer [6, page 132]. We shall assume throughout that $a_0 = 0$.

We observe that

$$R(S^1)[\eta]/(I) \subset \hat{K}^*_{S^1}(X),$$

where $I = \prod_{i=0}^{\ell-1} (\eta - t^{a_i})^{k_i+1} \in R(S^1)[\eta]$.

Let ϕ: $R(S^1)[\eta] \to \hat{K}^*_{S^1}(X)$ be the composition

$$R(S^1)[\eta] \to R(S^1)[\eta]/(I) \subset \hat{K}^*_{S^1}(X) .$$

For some positive integer h, let j_α $(\alpha = 1, \cdots, h)$ be an integer such that $0 \leq j_\alpha < \ell$, and such that in $\{j_1, \cdots, j_h\}$ there are at most $k_{j_\alpha} + 1$ distinct appearances of j_α, for each $\alpha = 1, \cdots, h$. We write

(2.4) $$\psi_{j_1,\cdots,j_h}(t) = (\mathrm{Id}^X_{S^1}) \circ \phi \left[I \prod_{j=1}^{h} (\eta - t^{a_{j_\alpha}})^{-1} \right] \in R(S^1) .$$

We point out the following generalizations of [6, Part II, Lemmas 2.1 and 2.3 and Corollary 2.4], from [6, Part I, Proposition 5.2].

(2.5) $$\psi_i(t) = \left[\prod_{j\neq i} (1 - t^{a_i - a_j})^{k_j+1} \right] \left[\prod_{j=1}^{n-k_i} (1 - t^{x_{ij}})^{-1} \right] \in R(S^1)$$

(up to units), and then

(2.5a) $$\psi_i(1) = \pm 1,$$

which implies that

(2.6) $$\prod_{j\neq i} |a_i - a_j|^{k_j+1} = \prod_{j=1}^{n-k_i} |x_{ij}|$$

and

(2.6a) $$\text{g.c.d.}\{|a_i - a_j|;\ j = 0, \cdots, \ell - 1;\ j \neq i\} = \text{g.c.d.}\{|x_{ij}|;\ j = 1, \cdots, n - k_i\}.$$

The numerical value of the last expression is independent of i $(i = 0, \cdots, \ell - 1)$.

We say that the action on X is S^1-*quasi-linear* if for every i such that $0 \leq i < \ell$,

(2.7) $$\{|x_{ij}|;\ j = 1, \cdots, n - k_i\} = \{|a_j - a_i| \text{ repeated } k_j + 1 \text{ times, } j \neq i\},$$

(where, in the right hand side, $j = 0, \cdots, \ell - 1$), or equivalently by (2.5), $\psi_i(t) = \pm t^{N_i}$, where $N_i \in \mathbb{Z}$. Otherwise the action is said to be S^1-*quasi-exotic.*

[6, Part I, Proposition 5.2] implies that if for $i \neq j$, $k_i, k_j \leq 1$, then for some $\lambda \in \mathbb{Z}$,

(2.8) $$\psi_{ij}(t) = (1 - t^{a_i - a_j})^{-1}[\psi_i(t) \pm t^{\lambda}\psi_j(t)] \in R(S^1) \quad \text{(up to units)}.$$

Together with Theorem 1.2, we prove in Sections 3 and 4 the following result.

THEOREM 2.1. $\mathrm{Id}^X_{S^1}(\eta^k)(1) = \pm\mathrm{Id}^Y_{S^1}(\mathscr{H}^k)(1)$, *where*

$$Y = Y(a_i \textit{ repeated } k_i + 1 \textit{ times } (0 \leq i < \ell))$$

(*see* (1.8)), *and where* $\mathscr{H}$ *is the equivariant Hopf bundle over* Y.

Proof of Theorem 0.1. The proof is obtained from Theorem 2.1, in the same way as the one given for the case where X^{S^1} is isolated, in [6, Part II, Corollary 2.12].

Remark 2.2. [6, Part I, Proposition 5.2] implies, with the notation of the present section, that, if X^{S^1} is isolated, then

(2.9) $$\mathrm{Id}^X_{S^1}(u(\eta)) = \sum_{i=0}^{\ell-1} u(t^{a_i})t^{\lambda_i} \prod_{j=1}^{n-k_i} (1 - t^{x_{ij}})^{-1} \in R(S^1),$$

where $u(\eta) \in R(S^1)[\eta]/(I) \subset \hat{K}^*_{S^1}(X)$ and the exponents λ_i are integers. Compare with [4, Part II, Section 10].

When necessary, we shall make use of the distinction $\lambda_i = \lambda_i^X$ and $\psi_{j_1,\cdots,j_h} = \psi^X_{j_1,\cdots,j_h}$.

3. THE QUASI-LINEAR ACTIONS (2.7)

PROPOSITION 3.1. *The statements of Theorems* 1.2 *and* 2.1 *are true for any* $n > 0$, *provided that* X *is* S^1-*quasi-linear* (*see* (2.7)) *and* X^{S^1} *is isolated. More specifically, in this case there exists an* S^1-*linear* $\mathbb{C}P^n$, *denoted by* Z_0, *such that formulas* (1.9), (1.10), (1.11) *hold for* $j = 0$.

Proof. We consider here the Remark 2.2. First, observe from $\psi_{ij} \in R$ $(i \neq j)$ (see (2.8) and (2.9)) that there exists $\varepsilon = \pm 1$ such that

$$\prod_{\substack{j \neq i \\ j=0,\cdots,n}} (a_i - a_j) = \varepsilon \prod_{j=1,\cdots,n-k_i} x_{ij}$$

for $i = 0, 1, \cdots, n$. (This observation implies already Theorem 2.1 in the present case.)

Let Y be as in Theorem 2.1. By [6, page 130], the exponents λ_i^Y are null. Assume that λ_0^X is also zero. Now, if $n > 2$,

$$\varepsilon \psi_{0ij}^X - \psi_{0ij}^Y \in R(S^1)$$

for every i and j such that $i \neq j$ and $i, j \neq 0$, implies that there exists $k \in \mathbb{Z}$ such that $\lambda_j^X = ka_j$, so that

$$\varepsilon \, Id_{S^1}^X (\eta^{-k}) = Id_{S^1}^Y (1).$$

This implies the statement.

In the remainder of this section and in Section 4, we complete the proof of the results stated in Sections 1 and 2.

PROPOSITION 3.2. *Assume that either* $n = 3$ *and* X^{S^1} *is nonisolated, or that* $n < 3$. *Then* X *is* S^1*-quasi-linear.*

Proof. Consider the case $n = \ell = 3$ and $(k_0, k_1, k_2) = (1, 0, 0)$ (see (2.1) and (2.2)).

If $\psi_0 \neq \pm t^N$ for every $N \in \mathbb{Z}$, we see from (2.5) and (2.6) that the assumption $x_{0j} \mid a_j$ for $j = 1, 2$ gives a contradiction; therefore we may assume $x_{01}, x_{02} \mid a_2$, which implies that $a_1 \mid a_2$. By (2.6a),

$$\text{g.c.d.}(|x_{01}|, |x_{02}|) = \text{g.c.d.}(|a_1|, |a_2|) = |a_1|.$$

Since $|x_{01}| \neq |x_{02}|$, we may assume

$$|a_2| = |a_1| pq \quad \text{and} \quad \{|x_{0i}/a_1|\} = \{p, q\},$$

where p and q are coprimes greater than 1. Then

$$|a_2 - a_1| = |a_1|(pq \pm 1).$$

Since $\psi_{02} \in R$, we can assume $|x_{21}| = kp$, and if $k > 1$, then $|x_{22}| = k'p$, where $(k, k') = 1$, and $|x_{02}| = q$, so that (2.6) gives a contradiction; if $k = 1$, then $|x_{22}| = k'q$, and if $k' > 1$, then $|x_{23}| = k'q$, so that (2.6) again gives a contradiction, as in the case $k' = 1$.

If $\psi_1 \neq \pm t^N$ for every $N \in \mathbb{Z}$, then by (2.5) and symmetry we may assume either

(i) $x_{11} \mid a_1$; $x_{12}, x_{13} \mid a_2 - a_1$, or

(ii) $x_{11}, x_{12}, x_{13} \mid a_2 - a_1$, or

(iii) $x_{11}, x_{12}, x_{13} \mid a_1$.

(Observe that (iv) $x_{11}, x_{12} \mid a_1$; $x_{13} \mid a_1 - a_2$ contradicts $\psi_1 \neq \pm t^N$.)

Then, by (2.6) and (2.6a), there are mutually coprime numbers $p_1 > 1$, $p_2 > 1$, $p_3 \geq 1$, such that

$$\text{for (i) and (ii), } |a_2 - a_1| = |a_1| \prod p_j, \quad \{|x_{ij}/a_1|\} = \{p_j\},$$

$$\text{for (iii), } |a_1| = |a_2 - a_1| \prod p_j, \quad \{|x_{ij}/(a_2 - a_1)|\} = \{p_i p_j;\ i \neq j\}.$$

In particular, $\psi_1 \neq \pm t^N$ implies that either $a_1 \mid a_2 - a_1$ or $a_2 - a_1 \mid a_1$. Since a similar conclusion would arise if $\psi_2 \neq \pm t^N$, that is, if either $a_2 \mid a_2 - a_1$ or $a_2 - a_1 \mid a_2$, we see that (i) or (ii) implies $\psi_2 = \pm t^N$, and therefore $\{|x_{2i}|\} = \{|a_2|, |a_2|, |a_2 - a_1|\}$. Then $\psi_{12} \in R$ gives a contradiction.

Recall that if $\psi_0 \neq \pm t^N$, then either $a_1 \mid a_2$ or $a_2 \mid a_1$. This observation and (iii) imply $\{|x_{0i}|\} = \{|a_i|\}$. Then $\psi_{01} \in R$ gives again a contradiction. Similarly for $\psi_2 \neq \pm t^N$.

Remark 3.3. When $n \leq 3$ and X is S^1-quasi-linear, then [6, Part I, Proposition 5.2] implies Theorem 2.1 and 1.2.

For example, if $\ell = 3$ and $(k_0, k_1, k_2) = (1, 0, 0)$, we can assume $\varepsilon_j = a_j/x_{0j}$, for $j = 1, 2$. Then $\psi_{00} \in R(S^1)$ implies $\langle z_0 + \varepsilon_j \xi_{0j}, [X_0] \rangle = 0$, where z_0 is the first Chern class of $\eta \mid X_0$ (as a vector bundle in the nonequivariant sense) and the various ξ_{0j} are the formal roots of the total Chern class of the direct sum of the components of $\nu_{|X|} X_0$ with real S^1-representation $t^{x_{0j}}$, and $[X_0]$ is the orientation class of X_0. It turns out that there exists $\mu_j \in \mathbb{Z}$ such that

$$\pm t^{N_i} \psi_{00i} = (t^{\mu_j} - 3t^{a_j} + 2)(1 - t^{a_j})^{-2} \in R(S^1),$$

for $i, j = 1, 2$ $(i \neq j)$, which only happens when $\mu_j = 3a_j$. This implies our claims.

If $\ell = 2$ and $(k_0, k_1) = (1, 1)$, then $\langle z_0, [X_0] \rangle = \langle a_1, [X_1] \rangle$ (where z_1 is the first Chern class of $\eta \mid X_1$ and $[X_1]$ is the orientation class of X_1), because there is an isomorphism $(\iota_{X_i})_2\colon H_2(X_i) \to H_2(|H|)$ with $(\iota_{X_i})_2 [X_i] = z^2 \cap [|X|]$. Then $\psi_{01} \in R(S^1)$ implies $x_{01} x_{02} = x_{11} x_{22}$, and $\psi_{00j} \in R(S^1)$ for $j = 1, 2$ implies our claims.

If $\ell = 2$ and $(k_0, k_1) = (2, 0)$, then $(\iota_{X_0})_4\colon H_4(X_0) \to H_4(|X|)$ is an isomorphism given by $(\iota_{X_0})_4[X_0] = \pm z \cap [|X|]$. Since X_0 is of codimension 2 in $|X|$, we can extend $\nu_{|X|} X_0$ to a bundle ν over X such that

$$(\iota_{X_0})^2(c_1(\nu)) = c_1(\nu_{|X|} X_0) \quad \text{and} \quad (\iota_{X_0})_4[X_0] = \pm c_1(\nu) \cap [|X|].$$

Then, $p_1(X_0) = (24\alpha + 3) z_0^2$ can be substituted in $\hat{\mathscr{A}}(X_0) = 1 - p_1(X_0)/24 + \cdots$, and by means of $\psi_{00} \in R(S^1)$ and $\psi_{000} \in R(S^1)$ we get our claims.

4. S^1-QUASI-EXOTIC HOMOTOPY $\mathbb{C}P^3$'S

According to Section 3, to prove the results stated in Sections 1 and 2 it is enough to complete their proofs under the following assumption.

Let $n = 3$, let X^{S^1} be isolated, and let X be S^1-quasi-exotic (see (2.7)).
(4.1) Without loss of generality, we can assume $\psi_0 \neq \pm t^m$ for every $m \in \mathbb{Z}$ (see (2.4)).

PROPOSITION 4.1. *There is exactly one* a_i *that is divisible by none of the* x_{0j}, *for* $j = 1, 2, 3$.

Proof. To prove that at least one a_i is not divisible by any of the x_{0j}, for $j = 1, 2, 3$, we assume to the contrary that each a_i is divisible by some x_{0j}. By symmetry, we consider three cases:

(i) $x_{03} \mid a_1, a_2$; $x_{01} \mid a_3$; (ii) $x_{03} \mid a_1, a_2, a_3$, (iii) $x_{0i} \mid a_i$ for $i = 1, 2, 3$.

(iii) and (2.6) imply $\psi_0(t) = \pm t^N$, a contradiction. By (2.6), each x_{0j} divides some a_i. This observation, symmetry, and the recent rejection of (iii) allow us to assume, for (i), that $x_{02} \mid a_3$, and for (ii), that $x_{01}, x_{02} \mid a_3$. Under these assumptions, we see that (ii) is a particular case of (i). Moreover, (2.6) implies $x_{03} \mid a_j$, $x_{0k} \mid a_3$, for $j = 1$ and 2, $k = 1$ and 2. Then (2.6a) implies $(a_1, a_2) = x_{03}$; replacing t by $t^{x_{03}}$, we may assume $x_{03} = 1$, $(a_1, a_2) = 1$, and $|x_{0j}| = b_j |a_1 a_2|$ for $j = 1, 2$. Again by (2.6), $|a_3| = b_1 b_2 |a_1 a_2|$, so that (because $\psi_0 \in R$) we may assume $|a_2| = 1$ and $(b_1, b_2) = 1$, and because $\psi_0 \neq \pm t^N$, we see that $b_1 > 1$ and $b_2 > 1$; therefore $\psi_{03} \in R$ and (2.6) give a contradiction.

To prove the rest of the statement, assume that a_1 and a_2 are not divisible by any of the x_{0j}. Then $\psi_{03} \in R$ implies that $a_1 - a_3$ and $a_2 - a_3$ are not divisible by any of the x_{3j}, and

(*) $$\{|x_{0j}|\} = \{|x_{3j}|\}.$$

On the other hand, it is easy to see the existence of the decompositions

$$|x_{01}| = m p_3 p_2 q_1, \quad |x_{02}| = m p_3 p_1 q_2, \quad |x_{03}| = m p_1 p_2 q_3$$

into positive factors. Then our present assumption says that $|a_3| = m pq$, where $p = p_1 p_2 p_3$ and $q = q_0 q_1 q_2 q_3$. Because of (2.6) we have the relation $a_1 a_2 = m^2 p$. Because of (2.6a), $|a_1| = mp'$ and $|a_2| = m p''$, where $p' p'' = p$. Therefore, (*) implies

$$(p'' q \pm 1)(p' q \pm 1) = 1,$$

which is possible only if $|a_3| = 2|a_1| = 2|a_2|$, which in turn is absurd.

COROLLARY 4.1a. *We can assume*

$$x_{0j} \nmid a_1 \textit{ for } j = 1, 2, 3, \quad x_{01}, x_{02} \mid a_2, \quad x_{03} \mid a_3,$$

and write

$$|x_{01}| = \gamma p, \quad |x_{02}| = \gamma q, \quad |a_2| = \gamma pq\alpha, \quad |a_3| = |x_{03}|\beta,$$

where p *and* q *are positive coprimes and* $\alpha, \beta, \gamma > 0$.

PROPOSITION 4.2. (a) $\alpha = \beta = 1$.

(b) $a_1 \mid x_{01}, x_{02}, x_{03}$.

(c) $p > 1$, $q > 1$.

(d) $\psi_2 \neq \pm t^N$, *for every* $N \in \mathbb{Z}$.

Proof. In the first place, observe that (2.6) implies

$$\gamma = |a_1| \alpha\beta,$$

and (2.5) implies that the common zeros of $1 - t^{|x_{01}|}$ and $1 - t^{|x_{02}|}$ are the zeros of $1 - t^\gamma$. Because of these facts together with $\psi_0 \neq \pm t^N$, we see that either (i) $\gamma \mid a_1$, or (ii) $\gamma \mid a_3$. We shall prove the proposition for each of these two situations.

(i) $\gamma \mid a_1$. Here (a) and (c) hold. If $a_1 \nmid x_{03}$, by $\psi_{02} \in R$ and (2.6), we can assume

$$\{|x_{2i}|\} = \{|a_1| p, |a_1| q, (pq \pm 1) | a_1 pq \pm x_{03}|\},$$

which is possible only if $a_1 pq \pm x_{03} = \pm 1$; therefore $\psi_{12} \in R$ and (2.6) give

$$\{|x_{1i}|\} = \{|a_1|, |a_1|, pq \pm 1\},$$

which is possible only if $a_1 \pm x_{03} = \pm 1$, which is absurd. This proves (b), and $\psi_{02} \in R$ justifies (d).

(ii) $\gamma \mid a_3$. Here $a_1 \alpha \mid x_{01}, x_{02}, x_{03}$. It follows, by (2.6a), that $\alpha = 1$; therefore (b) holds. To verify (c), let us assume that to the contrary $p = 1$. Then $\psi_{03} \in R$ and (2.6) shows that

$$\{|x_{3i}/a_1|\} = \{k\beta, k'\beta, k''(\beta x \pm 1)x\},$$

where $x = |x_{03}/a_1|$, and $kk'k'' = |q \pm x|$, which is possible only if $x = 1$, and this contradicts $\psi_0 \neq \pm t^N$. This proves (c), and $\psi_{02} \in R$ justifies (d), thus allowing us to verify that $\beta = 1$ by means of the fact that (b) holds at the component X_2, in the following way: By (b) at X_0, $\delta' = |a_3 a_1^{-1} \beta^{-1}| \in \mathbb{Z}$. If $\beta > 1$, then

$$\text{g.c.d.}(\beta pq, \beta\,|pq \pm \delta'|)$$

can be determined to be the g.c.d. of two of $b_j = |a_j - a_2|/|a_1|$ for $j \neq 2$. This only happens if the third b_j, that is, $\beta pq \pm 1$, is equal to 1, absurd.

PROPOSITION 4.3. *Let* $\delta \in \mathbb{Z}$ *be determined by*

$$|a_1 \delta| = |a_3| \quad \textit{and} \quad |a_1|\,|\delta - pq| = |a_3 - a_2|.$$

Then $|\delta - pq| = 1$, *so that we can write*

$$\varepsilon_0 a_1 = |a_1| \varepsilon, \quad \varepsilon_0 a_2 = |a_1| pq, \quad \varepsilon_0 a_3 = |a_1| (pq + \varepsilon'),$$

where $\varepsilon_0, \varepsilon, \varepsilon' = \pm 1$ *and*

$$\{|x_{0j}/a_1|\} = \{p, q, pq + \varepsilon'\}, \quad \{|x_{2j}/a_1|\} = \{p, q, pq - \varepsilon\}.$$

Proof. Part (b) of Proposition 4.2 applied to $\psi_2 \neq \pm t^N$ and the remark below (2.6a) imply $|\delta - pq| = 1$. The rest comes from $\psi_{02} \in R$.

Subsequently, we shall make use of Section 5, in which we find a polynomial expression and properties of the element

(4.3a) $$\phi_{p,q}(t) = (1 - t^{pq})(1 - t)(1 - t^p)^{-1}(1 - t^q)^{-1} \in R(S^1).$$

PROPOSITION 4.4. (i) $\varepsilon = \varepsilon'$.

(ii) $\{|x_{1j}|\} = \{|x_{2j}|\}, \quad \{|x_{0j}|\} = \{|x_{3j}|\}$.

(iii) $\Pi x_{1j} = -\Pi x_{2j}, \quad \Pi x_{0j} = -\Pi x_{3j}$.

Proof. These conclusions are obtained by means of $\psi_{12} \in R$, $\psi_{03} \in R$, Proposition 4.3, Corollary 5.5, and the fact that the numerator in (2.8) for the data of Proposition 4.3 necessarily takes the form $\phi_{p,q} - t^\lambda \phi_{p',q'}$ for some integers λ, $p' > 0$, $q' > 0$ such that $(p', q') = 1$.

PROPOSITION 4.5. *We can assume, if necessary by permuting the subindexing of the fixed-point set components* X_i, *that*

$$a_1 = |a_1|, \quad a_2 = |a_1|\, pq, \quad a_3 = |a_1|\,(pq + 1).$$

Then there exists an orientation class (see the Thom isomorphism of [6, page 119], *or Section* 1)

$$\partial^X_{S^1} \in K^*_{S^1}(\tau X)$$

such that for every $u \in K^*_{S^1}(X)$ *with (see* (1.1))

$$i^*_X(u) = (u_1, u_2, u_3, u_4) \in K^*_{S^1}(X^{S^1}) = \prod_{j=0}^{3} K^*_{S^1}(X_i)$$

there exists $\varepsilon_0 = \pm 1$ *such that, if* $\sigma = t^{a_i}$,

$$\varepsilon_0 \operatorname{Ind}(\partial^X_{S^1} u)(t)$$
$$= (1 - \sigma^p)^{-1}(1 - \sigma^q)^{-1}[(1 - \sigma^{pq+1})^{-1}(u_3 - u_0) - (\sigma - \sigma^{pq})^{-1}(u_2 - u_1)].$$

(Theorem 2.1 will follow.)

Proof. We assume for simplicity that $a_1 = 1$. As in Proposition 4.4, (iii), by means of $\psi_{ij} \in R$ for $i \neq j$, we get relations of sign among the integers x_{ij} that permit us to obtain the following formula for the Index, where $\delta^X_{S^1} = \psi(1) \in K^*_{S^1}(\tau X)$ is the orientation class of Section 1.

$$\varepsilon_0 \operatorname{Ind}(\delta^X_{S^1} u)$$
$$= (1 - t^p)^{-1}(1 - t^q)^{-1}[(1 - t^{pq+1})^{-1}(t^{\lambda_3} u_3 - t^{\lambda_0} u_0) - (1 - t^{pq-1})^{-1}(t^{\lambda_2} u_2 - t^{\lambda_1} u_1)],$$

where the exponents λ_i are integers. (This implies Theorem 2.1.) Moreover,

$$pq + 1 \mid \lambda_3 - \lambda_0, \quad pq - 1 \mid \lambda_2 - \lambda_1, \quad \text{and } (pq + 1)^{-1}(\lambda_3 - \lambda_0) = (pq - 1)^{-1}(\lambda_2 - \lambda_1).$$

We use $\mathrm{Ind}(\delta^X_{S^1}\, \eta^k t^\ell) \in R$ for integers k and ℓ such that

$$\lambda_0 + \ell = 0 \qquad \text{and} \qquad \lambda_3 + k(pq+1) + \ell = 0$$

to conclude $\lambda_1 + k + \ell = \lambda_2 + kpq + \ell$. Then $\mathrm{Ind}(\delta^X_{S^1}\, \eta^{k+1} t^\ell) \in R$ shows that $\lambda_1 + k + \ell = -1$. Since

$$\partial^X_{S^1} = \eta^k t^\ell \delta^X_{S^1} \in K^*_{S^1}(\tau X)$$

is also an orientation class, the Proposition is proved.

PROPOSITION 4.6. *Theorem* 1.2 *holds with* $j = 1$ *when* X *is* S^1-*quasi-exotic.*

Proof. Consider the R-module homomorphism

$$f\colon h^*(X) \to R$$

given by $f(u) = (u_1 - u_2)(\sigma - \sigma^{pq})^{-1}$. By the U.C.T. of [4], we have an epimorphism

$$\Psi\colon (X) \to \mathrm{Hom}_R(h^*(X), R)$$

given by $\Psi(u)[v] = \mathrm{Ind}(\partial^X_{S^1}\, uv)$. Therefore, there exists $\gamma \in h^*(X)$ such that $\Psi(\gamma) = f$. Comparing f and Ψ (see Proposition 4.5), we find that if $i^*_X(\gamma) = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$, then $\gamma_0 = \gamma_3 = 0$ and $\gamma_1 = \gamma_2 = (1 - \sigma^p)(1 - \sigma^p)$.

On the other hand, [6] gives explicitly an example of an S^1-exotic $\mathbb{C}P^3$ (we denote it by Z_1), whose equivariant K-theory as an $R(S^1)$ algebra is

$$K^*_{S^1}(Z_1) = R(S^1)[\eta, \eta^{-1}, \gamma]/J,$$

where J is the ideal generated by the elements

$(\eta - \sigma)(\eta - \sigma^{pq})$, $\gamma^2 - \gamma(1 - \sigma^p)(1 - \sigma^q)$, $(\eta - 1)(\eta - \sigma^{pq+1}) + \phi_{p,q}(\sigma)\gamma\eta$ (see (4.3a)).

If $Z_1^{S^1} = \sum_{i=0}^{3} Z_{1,i}$ is the union of the isolated fixed points, then the Atiyah-Segal localization theorem [6, page 109] implies that $i^*_{Z_1} K^*_{S^1}(Z_1)$ is generated as a submodule of

$$K^*_{S^1}(Z_1^{S^1}) = \prod_{i=0}^{3} K^*_{S^1}(Z_{1,i}),$$

by $i^*_{Z_1}(\gamma)$, $i^*_{Z_1}(\eta)$, and $i^*_{Z_1}(\eta^{-1})$, with the same coordinates as those of $i^*_X(\gamma)$, $i^*_X(\eta)$, and $i^*_X(\eta^{-1})$, respectively. These facts imply the existence of an R-algebra homomorphism

$$\omega\colon h^*(Z_1^{S^1}) \to h^*(X^{S^1})$$

such that $\omega(h^*(Z_1))$ is the R-subalgebra of $h^*(X)$ generated by η, η^{-1}, and γ.

The preceding conclusions together with Proposition 4.5 applied to X and to Z_1 imply the statement.

Remark. Theorems 1.1, 1.2, and 1.3 can be extended to smooth torus actions, as can be seen in [3].

5. APPENDIX. THE POLYNOMIAL STRUCTURE OF $\phi_{p,q}$ (4.3a)

LEMMA 5.1. *Let* p *and* q *be coprimes greater than* 1. *Then, there exists a unique pair of nonnegative integers* m *and* n *such that*

$$-mp + nq = 1, \qquad m < q, \qquad n < q.$$

LEMMA 5.2. *If* $\varepsilon = \pm 1$, *there exists exactly one pair of nonnegative integers* m_ε *and* n_ε *such that*

$$m_\varepsilon p + n_\varepsilon q = pq - p - q + \varepsilon .$$

Proof. For $\varepsilon = \pm 1$, we define

$$m_\varepsilon = -\varepsilon m - 1 + (1/2)q(1+\varepsilon),$$

$$n_\varepsilon = \varepsilon n - 1 + (1/2)p(1-\varepsilon).$$

THEOREM 5.3. *We have the equation* $\phi_{p,q} = \sum_{a \in C_1} t^a - \sum_{b \in C_{-1}} t^{b+1}$, *where*

$$C_\varepsilon = \{jp + kq;\ 0 \leq j \leq m_\varepsilon;\ 0 \leq k \leq n_\varepsilon\} \quad \textit{for } \varepsilon = \pm 1 .$$

Moreover, the elements of $C_1 \cup \{b+1,\ b \in C_{-1}\}$ *are pairwise distinct integers.*

A proof of Theorem 5.3 is given in [3].

Example 5.4. We illustrate Theorem 5.3.

The sets C_1 and C_{-1} can be arranged as matrices, partially superposed. For example, let $p = 7$ and $q = 10$. Observe the arrangement

	0	7	14	21	28	35	42
	10	17	24	31	38	45	52
(**)	20	27	34				
	30	37	44				
	40	47	54				

obtained by restriction from the matrix $\{a_{jk},\ 0 \leq j, k\}$ defined by $a_{jk} = j \cdot 10 + k \cdot 7$. Then C_1 (respectively, C_{-1}) is the maximal submatrix of (**) with vertex $(p-1)(q-1) = 54$ (respectively, $(p-1)(q-1) - 2 = 52$). This method, applied to different selections of p and q, produces quite different pairs (C_1, C_{-1}).

COROLLARY 5.5. *If* $\varepsilon = \pm 1$ *and if* p, q, p', q' *are integers greater than zero such that* $(p, q) = (p', q') = 1$, $\{p, q\} \neq \{p', q'\}$, *and* $p'q' = pq + \alpha$, *where* $|\alpha| \leq 2$, *then*

$$1 - t^{pq+\varepsilon} \nmid \phi_{p',q'} - t^\lambda \phi_{p,q}, \textit{ for any } \lambda \in \mathbb{Z}.$$

The proof of Corollary 5.5 is given in [3].

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