ERGODIC PROPERTIES OF EXPANSIVE AUTOMORPHISMS

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INTRODUCTION

In the study of expansive homeomorphisms, first made under the term "unstable homeomorphisms" (see [3], for example), many reports on expansive homeomorphisms and automorphisms have been published. However, in spite of a perhaps interesting topic, ergodic properties of expansive automorphisms are unknown, except in special cases. In this paper, we show that *expansive automorphisms of compact, connected, finite-dimensional abelian groups are* K-*automorphisms.*

1. PRELIMINARY LEMMAS

Throughout this paper, given an automorphism of an abelian group, we shall denote the restrictions on invariant subgroups and the automorphisms induced on factor groups by invariant subgroups by the symbol used for the original automorphism.

LEMMA A. *Let* G *be a countable, torsion-free, discrete, abelian group, and let* U *be an automorphism of* G. *Then there exist a minimal divisible extension* $\overline{G}$ *of* G *and an automorphism* $\overline{U}$ *of* $\overline{G}$ *such that* $\overline{U}$ *is an extension of* U. *Furthermore, if* U *has no finite orbit, then* $\overline{U}$ *has no finite orbit.*

Proof. Write $G = \{h_1, h_2, \cdots\}$. For each integer j and each positive integer n, let $[\hat{h}_{nj}]$ denote the free cyclic group generated by a new element $\hat{h}_{nj}$, and let $\langle U^j h_n \rangle$ denote the cyclic group generated by $U^j h_n$. Then we can construct a natural homomorphism ϕ_n from the direct-product group $W_n = \bigotimes_{j=-\infty}^{\infty} [\hat{h}_{nj}]$ onto the subgroup $\prod_{j=-\infty}^{\infty} \langle U^j h_n \rangle$ of G via the correspondence $\hat{h}_{nj} \to U^j h_n$. Let U_n' denote the automorphism of W_n defined by $\hat{h}_{nj} \to \hat{h}_{nj+1}$; then $\phi_n U_n' = U\phi_n$. Since

$$\prod_{n=1}^{\infty} \prod_{j=-\infty}^{\infty} \langle U^j h_n \rangle = G,$$

there is a homomorphism ϕ from the direct-product group $W = \bigotimes_1^{\infty} W_n$ onto G such that $\phi\{k_n\} = \{\phi_n k_n\}$ for each $\{k_n\} \in W$ and $\phi U' = U\phi$, where $U'\{k_n\} = \{U_n' k_n\}$. It is easy to see that U' is an automorphism of W. Hence (G, U) is a factor of (W, U'), and if we denote by K the kernel of ϕ, then (G, U) is isomorphic to (W/K, U').

Now there is a divisible extension $\overline{W} = \bigotimes_{n=1}^{\infty} \bigotimes_{j=-\infty}^{\infty} Q_{nj}$ of W, where Q_{nj} is an abelian group isomorphic to the additive group consisting of all rational numbers, and since W is torsion-free, there is an automorphism $\overline{U}'$ of $\overline{W}$ that is an extension of U'. Because $\overline{W}/K$ is a divisible extension of W/K, $\overline{U}'$ on $\overline{W}/K$ is an

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extension of U' on W/K, and since G is torsion-free, there is a minimal divisible extension of (G, U), which we denote by $(\overline{G}, \overline{U})$.

Assume that $\overline{U}^n f = f$ for an element f of $\overline{G}$ and an integer $n \neq 0$. Then there exists an integer $m \neq 0$ such that $f^m \in G$. Hence, from the relation $\overline{U}^n f^m = f^m$ we see that $U^n f^m = f^m$, where $f^m \neq 1$ if $f \neq 1$.

LEMMA B. *Let* X *be a compact abelian group, let* T *be an automorphism of* X, *and let* X_1 *be a* T-*invariant* $(TX_1 = X_1)$ *subgroup of* X. *If* $T: X_1 \to X_1$ *and* $T: X/X_1 \to X/X_1$ *are ergodic, then* $T: X \to X$ *is also ergodic.*

Proof. Let G, G_1, and G' be the character groups of X, X_1, and X/X_1, respectively. Assume for an element f of G and an integer $n \neq 0$ that $U^n f = f$, where U is the dual automorphism of T on G defined by $Ug(x) = g(Tx)$. Then, since T is ergodic on X_1 and G_1 is U-invariant $(UG_1 = G_1)$, $f(x) = 1$ on X_1. Hence $f \in G'$. Consequently, because T is ergodic on X/X_1, $f(x) = 1$ on X. Thus T is ergodic on X.

An automorphism T of a compact metric group X with metric d is called *distal* if for $x \in X$ the relation $\inf_{-\infty < j < \infty} d(T^j x, e) = 0$ implies $x = e$, e being the identity of X.

LEMMA C. *Let* X *be a compact metric abelian group, and let* T *be an automorphism of* X. *If for* $x \in X$ *we define* $Bx = x^{-1} Tx$, *then*

(1) B *is a continuous endomorphism of* X *into* X,

(2) $BT = TB$ *and* $TB(X) = B(X)$,

(3) *if* $B^n(X)$ *is trivial for some* $n > 0$, T *is distal.*

Proof. Since (1) and (2) are clear, we shall prove (3). Let G and U be as in Lemma B, and set $G_i = \operatorname{ann}(B^i(X), G) = \{g \in G: g(y) = 1 \text{ for all } y \in B^i(x)\}$ for $i \geq 1$. Then it follows from $X \supset B(X) \supset \cdots \supset B^n(X) = \{e\}$ that $G_1 \subset G_2 \subset \cdots \subset G_n = G$. If we define inductively

$$H_1 = \{f \in G;\ Uf = f\}$$

and

$$H_{i+1} = \{f \in G;\ Uf = gf \text{ for some } g \in H_i\} \qquad (i = 1, 2, \cdots),$$

then $H_1 \subset H_2 \subset \cdots$, and it is easy to see inductively that each H_i is a U-invariant subgroup of G. By induction, we shall show that $H_i \supset G_i$ $(i = 1, 2, \cdots, n)$. Assume $G_i \subset H_i$ and $f \in G_{i+1}$. Then, since $B(B^i f(x)) = 1$ for $x \in X$, we see that $UB^i f(x) = B^i f(x)$, and therefore $B^i f \in H_1$. Assuming $B^{i-j} f \in H_{j+1}$ for $j = 0, 1, \cdots, i-1$, we have the relation $UB^{i-j} f(x) = g(x) \cdot B^{i-j} f(x)$ for some $g \in H_j$. Hence

$$UB^{i-j-1} f(x) = g^{-1}(x) \cdot UB^{i-j} f(x) \cdot B^{i-j-1} f(x),$$

where $g^{-1} \cdot UB^{i-j} f \in H_{j+1}$. Hence $B^{i-j-1} f \in H_{j+2}$. Consequently, we see by induction that $f \in H_{i+1}$, and it follows that $G_{i+1} \subset H_{i+1}$. Thus, for $i = 1, 2, \cdots, n$, $H_i \supset G_i$, and therefore $G = H_n$.

Now let $\inf_{-\infty < j < \infty} d(T^j x, e) = 0$, where d is the metric of X. If $f \in H_1$, the equation $f(Tx) = f(x)$ implies that $f(T^j x) = f(x)$ for each integer j. Hence, by continuity of f, $f(x) = f(e) = 1$. Assume $f(x) = 1$ for every $f \in H_i$. If $f \in H_{i+1}$, then $f(Tx) = g(x) \cdot f(x)$ for some $g \in H_i$. Hence, because $f(T^j x) = f(x)$ for each j, it follows that $f(x) = 1$. Consequently, since we obtain inductively $f(x) = 1$ for every $f \in G$, x must be the identity. Thus the proof is complete.

2. ERGODICITY OF EXPANSIVE AUTOMORPHISMS

An automorphism T of a compact group X is called *expansive* if there exists a neighborhood V of the identity e of X such that $x \in X - \{e\}$ implies $T^n x \notin V$ for some integer n. It is known that if X admits an expansive automorphism, X is metrizable [2]. We observe that "expansive" is incompatible with "distal" [2].

From now on, let X be a compact, connected, finite-dimensional abelian group, let T be an expansive automorphism of X, and let (G, U) be the dual of (X, T). We note that G is countable and discrete, and that the rank of G is finite.

LEMMA 1. *For* $k \geq 1$ *and* $x \in X$, *define* $B_k x = x^{-1}T^k x$. *Then*

(1) B_k *is a continuous endomorphism of* X *into* X,

(2) $B_k T = TB_k$, $TB_k(X) = B_k(X)$, *and* $B_k B_j = B_j B_k$ *for* $j \geq 1$,

(3) $B_k^n(X)$ *is a nontrivial compact connected subgroup of* X, *for* $n \geq 1$.

Lemma 1 is clear from Lemma C, since T^k is expansive.

We claim that if Y_1 and Y_2 are connected subgroups of X such that $Y_1 \subsetneqq Y_2$, then $\dim Y_1 < \dim Y_2$. For let

$$F_i = \operatorname{ann}(Y_i, G) = \{g \in G: g(y) = 1 \text{ for all } y \in Y_i\}$$

for i = 1, 2; then $F_2 \subsetneqq F_1$ and F_1/F_2 is torsion-free. Hence there is an element $g \in F_1$ such that $g^n \notin F_2$ for all $n \neq 0$. This g can not be represented by a maximal independent system of F_2. This implies that rank $F_2 <$ rank F_1, and therefore rank $G/F_1 <$ rank G/F_2, which shows $\dim Y_1 < \dim Y_2$.

Now we consider the following cases: (i) $B_k(X) = X$ for all $k > 0$, and (ii) $B_k(X) \neq X$ for some $k > 0$. In the case (i), assume for a character f and an integer $n > 0$ that $U^n f = f$. Then, since $f(x^{-1}T^k x) = 1$ for $x \in X$, we see that $f(B_k X) = 1$. Hence $f(X) = 1$, and therefore U has no finite orbit on X, so that T is a K-automorphism. Conversely, if T is a K-automorphism, then it is easy to see that the condition (i) holds.

From now on, assuming the case (ii), we show a contradiction. By Lemma 1 (3), if $B_k(X) \neq X$ for some $k > 0$, then $\dim X > \dim B_k(X)$. Hence, for each $k \geq 1$, $B_k^N(X) = B_k^{N+j}(X)$ $(j > 0)$, where $N = \dim X$. Let k_1 be the least positive integer such that B_{k_1} is not onto, and write $X_1 = B_{k_1}^N(X)$. Then $TX_1 = X_1$, $\dim X_1 < \dim X$, and $B_i(X_1) \subset X_1$ for $i \geq k_1$, since $B_k B_j = B_j B_k$ for $j \geq 1$. But, for $i \leq k_1$, $B_i(X_1) = X_1$. Next, let k_2 be the least positive integer such that B_{k_2} is not onto on X_1, and write $X_2 = B_{k_2}^N(X_1)$. Then $k_2 > k_1$, $X_2 \subset X_1$, $TX_2 = X_2$, $\dim X_2 < \dim X_1$, and $B_i(X_2) \subset X_2$ for $i \geq k_1$, but $B_i(X_2) = X_2$ for $i \leq k_2$. Repeating this process, we obtain positive integers n, $k_1, k_2, \cdots, k_n$ and compact connected subgroups X_0 (= X), $X_1, \cdots, X_n$ of X such that for $i \geq 1$ and $j = 1, 2, \cdots, n$

$$X_j = B_{k_j}^N(X_{j-1}), \qquad TX_j = X_j,$$

$$B_i(X_j) \subset X_j, \qquad B_i(X_n) = X_n.$$

LEMMA 2. $X_n \neq \{e\}$ *and* $T: X_n \to X_n$ *is ergodic.*

Proof. If $X_n = \{e\}$, then

$$X_{n-1} \supset B_{k_n}(X_{n-1}) \supset \cdots \supset B_{k_n}^N(X_{n-1}) = \{e\}.$$

Hence, by Lemma C, T^{k_n} is distal on X_{n-1}, which contradicts the fact that T is expansive on X_{n-1}. Consequently, $X_n \neq \{e\}$.

Ergodicity of $T: X_n \to X_n$ follows from the fact that $B_i(X_n) = X_n$ for all $i > 0$.

Now define $P = B_{k_n}^N \cdots B_{k_1}^N$, then P is a continuous homomorphism from X onto X_n, and $TP = PT$. Now denote by X' the kernel of P, then $TX' = X'$ and $(X/X', T)$ is isomorphic to (X_n, T). Hence by Lemma 2, $T: X/X' \to X/X'$ is ergodic.

Write

$$G(X') = \text{ann}(X', G) = \{g \in G: g(x) = 1 \text{ for all } x \in X'\}$$

and

$$\tilde{G}(X') = \{g \in G;\ g^n \in G(X') \text{ for some integer } n \neq 0\},$$

then $G(X')$ and $\tilde{G}(X')$ are U-invariant subgroups of G. If we set $\tilde{X}^{(1)} = \text{ann}(\tilde{G}(X'), X)$, then, since the character group of $\tilde{X}^{(1)}$ is $G/\tilde{G}(X')$ and $G/\tilde{G}(X')$ (which may be trivial) is torsion-free, the T-invariant subgroup $\tilde{X}^{(1)}$ of X' is connected. Further, because $\tilde{X}^{(1)} \neq X$, it follows that $\dim \tilde{X}^{(1)} < \dim X$.

Since T is ergodic on X/X' and $G(X')$ is the character group of X/X', U has no finite orbit on $G(X')$. Hence, if $(\overline{G}(X'), \overline{U})$ denotes a minimal divisible extension of $(G(X'), U)$, then $\overline{U}$ has no finite orbit on $\overline{G}(X')$, by Lemma A. Therefore, since $\tilde{G}(X') \subset \overline{G}(X')$, U also has no finite orbit on $\tilde{G}(X')$. Consequently, since $\tilde{G}(X')$ is the character group of $X/\tilde{X}^{(1)}$, $T: X/\tilde{X}^{(1)} \to X/\tilde{X}^{(1)}$ is ergodic.

If $\tilde{X}^{(1)} \neq \{e\}$, an argument similar to that above shows that there exists a T-invariant subgroup X" of $\tilde{X}^{(1)}$ such that $T: \tilde{X}^{(1)}/X'' \to \tilde{X}^{(1)}/X''$ is ergodic. We write $G(X'') = \text{ann}(X'', G/\tilde{G}(X'))$,

$$\tilde{G}(X'') = \{\bar{g} \in G/\tilde{G}(X');\ \bar{g}^n \in G(X'') \text{ for some integer } n \neq 0\},$$

and $\tilde{X}^{(2)} = \text{ann}(\tilde{G}(X''), \tilde{X}^{(1)})$; then $\tilde{X}^{(2)}$ is a T-invariant connected subgroup of $\tilde{X}^{(1)}$, $\dim \tilde{X}^{(1)} > \dim \tilde{X}^{(2)}$, and further $T: \tilde{X}^{(1)}/\tilde{X}^{(2)} \to \tilde{X}^{(1)}/\tilde{X}^{(2)}$ is ergodic.

Since $\dim X < \infty$, a repetition of these arguments shows that we have a positive integer m and T-invariant connected subgroups $\tilde{X}^{(0)}, \tilde{X}^{(1)}, \tilde{X}^{(2)}, \cdots, \tilde{X}^{(m)}$ of X such that $\tilde{X}^{(0)} = X$, $\tilde{X}^{(m)} = \{e\}$, and $T: \tilde{X}^{(i)}/\tilde{X}^{(i+1)} \to \tilde{X}^{(i)}/\tilde{X}^{(i+1)}$ $(i = 0, 1, \cdots, m-1)$ are ergodic. Consequently, applying Lemma B inductively, we see that $T: X \to X$ is ergodic, so that it is a K-automorphism, which shows that (i) holds. This is inconsistent with our assumption. Thus we obtain our result:

THEOREM. *Expansive automorphisms of compact, connected, finite-dimensional abelian groups are* K-*automorphisms.*

Remark. Recently, the first author has shown that compact, connected groups that admit expansive automorphisms are finite-dimensional and abelian [1]. Hence we can improve our Theorem as follows:

Expansive automorphisms on compact connected groups are Kolmogorov automorphisms.

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