CONTRACTIVE LINEAR MAPS ON C*-ALGEBRAS

Richard I. Loebl

0. INTRODUCTORY REMARKS

The purpose of this note is to study the interplay and distinctions between contractive and completely contractive linear maps on C*-algebras. Both in spirit and in technique, these results follow the outline given by M.-D. Choi [2].

If $\mathscr{A}$ and $\mathscr{B}$ are C*-algebras with identity, and $\phi: \mathscr{A} \to \mathscr{B}$ is a linear map, then $\phi_n = \phi \otimes \mathrm{id}_n$ is the entry-wise map from the C*-algebra $\mathscr{A} \otimes M_n$ to $\mathscr{B} \otimes M_n$, where M_n denotes the C*-algebra of n-by-n complex matrices. We say that ϕ is *completely positive* if every ϕ_n $(n \geq 1)$ is positive; ϕ is *completely contractive* if $\sup_n \|\phi_n\| \leq 1$; and ϕ is *completely bounded* if $\sup_n \|\phi_n\| < \infty$; see [1]. Note that $\|\phi_n\| \geq \|\phi\|$.

Let $C_k[\mathscr{A}, \mathscr{B}]$ denote the set of all linear maps ϕ from $\mathscr{A}$ to $\mathscr{B}$ such that $\phi_1, \cdots, \phi_k$ are contractive; $C_\infty[\mathscr{A}, \mathscr{B}]$ is then the set of all completely contractive maps. It is easy to see that $C_1 \supseteq C_2 \supseteq \cdots$ and $C_\infty = \bigcap_{k \geq 1} C_k$.

It is known that if $P_k[\mathscr{A}, \mathscr{B}]$ denotes the set of all linear maps ϕ from $\mathscr{A}$ to $\mathscr{B}$ such that $\phi_1, \cdots, \phi_k$ are positive, then $P_1[\mathscr{A}, \mathscr{B}] = P_\infty[\mathscr{A}, \mathscr{B}]$ if either $\mathscr{A}$ or $\mathscr{B}$ is commutative [1, p. 144]. Further, Choi established that $P_1[\mathscr{A}, \mathscr{B}] = P_2[\mathscr{A}, \mathscr{B}]$ implies $\mathscr{A}$ or $\mathscr{B}$ is commutative [2, Theorem 4].

The results we shall establish are analogous: if $\mathscr{B}$ is commutative, then $C_1[\mathscr{A}, \mathscr{B}] = C_\infty[\mathscr{A}, \mathscr{B}]$; and if $C_1[\mathscr{A}, \mathscr{B}] = C_2[\mathscr{A}, \mathscr{B}]$, then $\mathscr{A}$ or $\mathscr{B}$ is commutative. The analogy breaks down drastically in the case of a commutative domain: If $\mathscr{A}$ is commutative, therefore of the form C(X) [5, Theorem 4.2.2] and $C_1[\mathscr{A}, \mathscr{B}] = C_\infty[\mathscr{A}, \mathscr{B}]$, then by Theorem C, the space X contains at most two points! We shall also make some remarks about the case of completely bounded maps.

1. COMPLETELY CONTRACTIVE MAPS

LEMMA 1. *Let $\mathscr{I}$ be a linear subspace of a C*-algebra, and let $\mathscr{C}$ be a commutative C*-algebra. Let $\phi: \mathscr{I} \to \mathscr{C}$ be a linear map. Then $\|\phi\| = \|\phi_n\|$ for* $n = 1, 2, \cdots$.

Proof. We modify [1, Proposition 1.2.2]. Identify $\mathscr{C}$ as C(X), let n be a positive integer, and for $[a_{ij}] \in \mathscr{I} \otimes M_n$, let $\phi(a_{ij}) = f_{ij} \in C(X)$. Then

$$\|[f_{ij}]\| = \sup_x \sup_{\|\zeta\|, \|\eta\| \leq 1} |\langle [f_{ij}(x)]\zeta, \eta \rangle|,$$

where $\zeta, \eta \in \mathbb{C}^n$. However,

Received June 2, 1975.
The author was supported in part by a Wayne State University Faculty Research Award.

Michigan Math. J. 22 (1975).

$$\langle [f_{ij}(x)]\zeta, \eta\rangle = \sum_{ij} f_{ij}(x)\zeta_j\bar{\eta}_i = \left(\sum \phi(a_{ij})\zeta_j\bar{\eta}_i\right)(x)$$

$$= \phi\left(\begin{bmatrix} \bar{\eta}_1 & \cdots & \bar{\eta}_n \\ 0 & \cdots & 0 \\ \cdot & \cdots & \cdot \\ 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix}\begin{bmatrix} \zeta_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \zeta_n & 0 & \cdots & 0 \end{bmatrix}\right)(x) .$$

Hence, $|\langle [f_{ij}(x)]\zeta, \eta\rangle| \leq \|\phi\| \cdot \|\eta\| \cdot \|[a_{ij}]\| \cdot \|\zeta\|$, so that $\|\phi_n\| \leq \|\phi\|$; the opposite inequality is evident.

Notice that Lemma 1 improves the result of [1, Proposition 1.2.11].

For reference, we restate the following result [6, Corollary 1].

LEMMA 2. *Let* $\phi: \mathscr{A} \to \mathscr{B}$ *be a linear map of* C*-*algebras such that* $\phi(1) = 1$. *Then* ϕ *is positive if and only if* $\|\phi\| = 1$.

LEMMA 3. *If* $C_1[\mathscr{A}, \mathscr{B}] = C_2[\mathscr{A}, \mathscr{B}]$ *for all* $\mathscr{A}$, *then* $\mathscr{B}$ *is commutative.*

Proof. If $\mathscr{B}$ is not commutative, we shall produce a map $\psi: M_2 \to \mathscr{B}$ with the properties that $\psi(1) = 1$, $\psi \geq 0$, but $\psi_2 \not\geq 0$; thus, by Lemma 2, $\|\psi_2\| > 1$. By [2, Lemma 3], we can assume there are Hermitian operators B_1, $B_2 \in \mathscr{B}$ and nonzero vectors u and v in the underlying Hilbert space of $\mathscr{B}$ such that $B_2 u = 0$ and $B_2 B_1 u = v$. Further, we can assume $\|B_i\| \leq 1/2$, so that $1 - B_1^2 - B_2^2 \geq 0$. Let the map $\Phi: M_2 \to \mathscr{B}$ be defined by

$$\Phi\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}B_1^2 + a_{12}B_1B_2 + a_{21}B_2B_1 + a_{22}B_2^2 + a_{11}(1 - B_1^2 - B_2^2) .$$

If $\theta: M_2 \to M_2$ denotes the transpose, then $\psi = \Phi \circ \theta$ is positive and $\psi(1) = 1$. However, ψ_2 is not positive, since the inner product

$$\left\langle \psi_2\begin{bmatrix} 1 & 0 & \cdot & 0 & 1 \\ 0 & 0 & \cdot & 0 & 0 \\ \cdots & & \cdot & \cdots & \\ 0 & 0 & \cdot & 0 & 0 \\ 1 & 0 & \cdot & 0 & 1 \end{bmatrix}(-\varepsilon v + u), (-\varepsilon v + u)\right\rangle$$

$$= \varepsilon^2 \|B_1 v\|^2 - 2\varepsilon\|v\|^2 + \varepsilon^2(\|v\|^2 - \|B_1 v\|^2 - \|B_2 v\|^2)$$

is negative whenever ε is a sufficiently small positive number.

THEOREM A. *A* C*-*algebra* $\mathscr{B}$ *is commutative if and only if* $\forall\phi: \mathscr{A} \to \mathscr{B}$, $\|\phi_n\| = \|\phi\|$ $(n = 1, 2, \cdots)$.

Scrutiny of the proof of Theorem A reveals that the following result is valid.

COROLLARY. $\mathscr{B}$ *is commutative if and only if* $C_1[\mathscr{S}, \mathscr{B}] = C_\infty[\mathscr{S}, \mathscr{B}]$ *for all subspaces* $\mathscr{S}$ *of a* C*-*algebra* $\mathscr{A}$; *this is so if and only if* $C_1[\mathscr{S}, \mathscr{B}] = C_2[\mathscr{S}, \mathscr{B}]$.

Let us now attempt to reverse the role of range and domain in Theorem A.

LEMMA 4. *If* $C_1[\mathscr{A}, \mathscr{B}] = C_2[\mathscr{A}, \mathscr{B}]$ *for all* $\mathscr{B}$, *then* $\mathscr{A}$ *is commutative.*

Proof. If $\mathscr{A}$ is not commutative, we can assume, as in [2, Lemma 2] that there are self-adjoint $A_1, A_2 \in \mathscr{A}$ and nonzero vectors x and y in the underlying Hilbert space such that $A_1 x = 0$ and $A_1 A_2 x = y$. Without loss of generality, we can take $\|A_i\| \leq 1$ and $\|x\| = 1$. Then $\|y\| \leq 1$ and

$$\langle y, x \rangle = \langle A_1 A_2 x, x \rangle = \langle A_2 x, A_1 x \rangle = 0.$$

Let e be a state of $\mathscr{A}$. Let $\tau: \mathscr{A} \to M_2$ be defined by

$$\tau(A) = \begin{bmatrix} \langle Ax, x \rangle & \langle Ay, x \rangle \\ \langle Ax, y \rangle & \langle Ay, y \rangle \end{bmatrix} + e(A) \begin{bmatrix} 0 & 0 \\ 0 & 1 - \|y\|^2 \end{bmatrix}.$$

Again, let $\theta: M_2 \to M_2$ be the transpose. Then $\sigma = \theta \circ \tau$ satisfies the conditions $\sigma \geq 0$ and $\sigma(1) = 1$, and exactly as in [2, Lemma 2], we conclude that $\sigma_2 \not\geq 0$. Hence $C_1[\mathscr{A}, M_2] \supset C_2[\mathscr{A}, M_2]$.

We can now prove the following analogue of [2, Theorem 4].

THEOREM B. *If* $C_1[\mathscr{A}, \mathscr{B}] = C_2[\mathscr{A}, \mathscr{B}]$, *then either* $\mathscr{A}$ *or* $\mathscr{B}$ *is commutative.*

Proof. If not, repeat the argument of [2, Theorem 4] applied to $\psi \circ \tau$, where ψ is the map of Lemma 3 and τ is the map of Lemma 4; we obtain the relations $(\psi \circ \tau) \geq 0$, $(\psi \circ \tau)(1) = 1$, and $(\psi \circ \tau)_2 \not\geq 0$.

Having in hand Theorems A and B and the results of [2], one now expects the following to be true:

Statement. If $\mathscr{A}$ is commutative, then for all $\mathscr{B}$, $C_1[\mathscr{A}, \mathscr{B}] = C_\infty[\mathscr{A}, \mathscr{B}]$.

This is not so! Instead, the following rather surprising result holds.

THEOREM C. *If* $C_1[\mathscr{A}, \mathscr{B}] = C_2[\mathscr{A}, \mathscr{B}]$ *for all* $\mathscr{B}$, *then* $\mathscr{A}$ *is commutative and at most two-dimensional. That is,* $\mathscr{A} = C(X)$ *and* $X \subseteq \{x_1, x_2\}$.

Proof. Since $C_1 = C_2$, we can apply Lemma 4 and assert that the algebra $\mathscr{A}$ is commutative; hence $\mathscr{A} = C(X)$, where X is a compact Hausdorff space. If X has more than two points, there are three positive linear functionals on C(X) with disjoint closed supports and norm $1/\sqrt{6}$. Let $A_1, A_2, A_3 \in M_2$ be the following:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Then we have the relations

(a) $A_i = A_i^*$,

(b) $A_i^2 = I_2$, where I_2 is the 2-by-2 identity matrix,

(c) $A_i A_j + A_j A_i = 2\delta_{ij} I_2$.

If $f \in C(X)$ and $\{\rho_i\}$ is the set of functionals above, let

$$\psi(f) = \rho_1(f) A_1 + \rho_2(f) A_2 - \rho_3(f) A_3.$$

Then, as in [4, Theorem 2.2], we see that $\|\psi(f)\| \leq \sqrt{2}\sqrt{\sum |\rho_i(f)|^2} \leq \|f\|$, so that ψ is a contraction.

We claim that $\|\psi_2\| > 1$. To show this, let $F(x) \in C(X) \otimes M_2$ be such that on the support K_i of ρ_i, $F(x) = A_i$, and such that otherwise $F(x)$ is a convex combination of the A_i. Then $\|F\| = \sup_x \|F(x)\| = 1$.

We see that

$$\psi_2(F) = (\psi \otimes \mathrm{id}_2)(F) = ((\rho_1 \otimes A_1 + \rho_2 \otimes A_2 - \rho_3 \otimes A_3) \otimes \mathrm{id}_2)(F)$$

$$= \|\rho_1\| A_1 \otimes A_1 + \|\rho_2\| A_2 \otimes A_2 - \|\rho_3\| A_3 \otimes A_3 .$$

Notice that $\psi_2(F)$ is self-adjoint, so that $\|\psi_2(F)\| = \sup_{\|z\|=1} |\langle \psi_2(F)z, z\rangle|$.

We claim that there is a unit vector $z \in \mathbb{C}^4$ such that $(A_1 \otimes A_1)(z) = z$, $(A_2 \otimes A_2)(z) = z$, but $(A_3 \otimes A_3)(z) = -z$. For such a vector z,

$$\langle \psi_2(F)z, z\rangle = \|\rho_1\| + \|\rho_2\| + \|\rho_3\| = 3/\sqrt{6} > 1,$$

so that $\|\psi_2\| > 1$. In fact, $\|\psi_2\| = \|\rho_1\| + \|\rho_2\| + \|\rho_3\|$. To find the vector z, compute the matrices $A_i \otimes A_i$, and notice that the vector $(1, 0, 0, 1)$, when normalized, does the trick.

We conjecture that Theorem C can be improved to say that $X = \{x_1\}$, but at the present we do not see how to do this. It follows from Lemma 1 that $C_1[\mathbb{C}, \mathscr{B}] = C_\infty[\mathbb{C}, \mathscr{B}]$ for all $\mathscr{B}$.

2. COMPLETELY BOUNDED MAPS

Now let $B_\infty[\mathscr{A}, \mathscr{B}]$ denote the family of all completely bounded maps from $\mathscr{A}$ to $\mathscr{B}$, and let $B_1[\mathscr{A}, \mathscr{B}]$ denote the bounded maps. We can then restate Lemma 1 in the following form.

LEMMA 1'. *If $\mathscr{B}$ is commutative, then* $B_1[\mathscr{I}, \mathscr{B}] = B_\infty[\mathscr{I}, \mathscr{B}]$.

Since each ϕ_n is bounded, the question is how rapidly the sequence $\{\|\phi_n\|\}$ grows. We can answer this in some special cases. An easy application of Lemma 1' yields the following result.

LEMMA 5. *If $\mathscr{A}$ is finite-dimensional, then* $B_1[\mathscr{A}, \mathscr{B}] = B_\infty[\mathscr{A}, \mathscr{B}]$.

The proof of [4, Theorem 2.2] establishes the following result.

THEOREM D. *Let X be a compact Hausdorff space with an infinite number of points. Then there is a bounded map Φ from $C(X)$ into the compact operators on a separable Hilbert space such that Φ is* not *completely bounded.*

Using Theorem D, we can establish the converse to Lemma 5.

THEOREM E. $B_1[\mathscr{A}, \mathscr{B}] = B_\infty[\mathscr{A}, \mathscr{B}]$ *for all $\mathscr{B}$ if and only if $\mathscr{A}$ is finite-dimensional.*

Proof. We need only establish the theorem in one direction; assume therefore that $B_1[\mathscr{A}, \mathscr{B}] = B_\infty[\mathscr{A}, \mathscr{B}]$ for all $\mathscr{B}$. Let $S = S^* \in \mathscr{A}$, and consider the commutative C^*-subalgebra generated by 1 and S. This subalgebra is $C(X)$, for some compact Hausdorff space X. If X is infinite, we argue as follows. The proof of Theorem D, like that of Theorem C, involves choosing positive linear functionals of disjoint support on $C(X) \subseteq \mathscr{A}$. By Krein's theorem [5, p. 227] these functionals can be extended to be positive on $\mathscr{A}$. But this extends the map Φ of Theorem D to all of $\mathscr{A}$ in such a way that the norm of the extension equals $\|\Phi\|$. Theorem D then implies that the extended map is not completely bounded.

Hence, X must be finite; that is, every self-adjoint $S \in \mathscr{A}$ has finite spectrum. Further, every abelian *-subalgebra must also be finite-dimensional. This forces $\mathscr{A}$ to be finite-dimensional (see [3]). For the sake of completeness, we sketch a proof.

Notice that self-adjoint elements having finite spectrum implies the existence of spectral projections. Therefore, let $E = \{e_\alpha\}$ be a maximal family of commuting nonzero projections in $\mathscr{A}$. This family is finite, since otherwise the abelian algebra they generate would be infinite-dimensional. Order E by the usual partial ordering ($e \leq f$ if and only if $ef = fe = e$); since we have a finite set, there exist minimal elements with respect to this ordering; call them $e_1, \cdots, e_n$. Then $1 = e_1 + \cdots + e_n$, by maximality of E and by the definition of the e_n.

Now $\mathscr{A} = \sum_{ij} e_i \mathscr{A} e_j$ is a decomposition of $\mathscr{A}$. Note that $e_i \mathscr{A} e_i$ is a C*-algebra with identity. By [5, Theorem 1.6.15], for $a \in \mathscr{A}$,

$$\mathrm{sp}_{\mathscr{A}}(a) = \mathrm{sp}_{e_i \mathscr{A} e_i}(a) \cup \{0\};$$

in particular, every self-adjoint element in $e_i \mathscr{A} e_i$ has finite spectrum. Therefore the spectral projections of such an element are in $e_i \mathscr{A} e_i$, and they are subprojections of e_i. This contradicts the choice of e_i. Hence, every self-adjoint element in $e_i \mathscr{A} e_i$ is a scalar multiple of e_i; therefore $e_i \mathscr{A} e_i = \mathbb{C} e_i$, and thus $e_i \mathscr{A} e_i$ is one-dimensional.

Let $i \neq j$, and suppose $e_i \mathscr{A} e_j \neq \{0\}$. If $a \neq 0 \in e_i \mathscr{A} e_j$, then $0 \neq a^*a \in e_j \mathscr{A} e_j = \mathbb{C} e_j$. By scaling, we can assume $a^*a = e_j$. Define the map $T: e_i \mathscr{A} e_j \to e_i \mathscr{A} e_i$ by $Tx = xa^*$. Then T is one-to-one, for

$$xa^* = ya^* \Rightarrow xa^*a = ya^*a,$$

or $xe_j = ye_j$; but e_j is a right identity for $e_i \mathscr{A} e_j$. Since $e_i \mathscr{A} e_i$ is one-dimensional, $e_i \mathscr{A} e_j$ is also one-dimensional. Hence, $\dim \mathscr{A} \leq n^2$.

We now ask about the ranges.

LEMMA 6. *Let* $\mathscr{A}$ *be a* C*-*algebra. Then* $B_1[\mathscr{A}, M_n] = B_\infty[\mathscr{A}, M_n]$.

Proof. If $\phi: \mathscr{A} \to M_n$, then $\phi(a) = [\phi_{ij}(a)]$, where $\phi_{ij}: \mathscr{A} \to \mathbb{C}$, and $\|\phi_{ij}\| \leq \|\phi\|$. Therefore $\phi \otimes \mathrm{id}_k = [\phi_{ij} \otimes \mathrm{id}_k]$. It follows that

$$\|\phi \otimes \mathrm{id}_k\| \leq \lambda_n \sup_{ij} \|\phi_{ij} \otimes \mathrm{id}_k\|,$$

where λ_n is a constant depending on n but not on ϕ. By Lemma 1,

$$\|\phi_{ij} \otimes \mathrm{id}_k\| = \|\phi_{ij}\|$$

for all k; therefore $\|\phi \otimes \mathrm{id}_k\| \leq \lambda_n \|\phi\|$.

LEMMA 7. *Let* $\mathscr{A}$ *be a* C*-*algebra, and let* $\mathscr{B} \subseteq \mathscr{C} \otimes M_n$, *where* $\mathscr{C}$ *is commutative. Then* $B_1[\mathscr{A}, \mathscr{B}] = B_\infty[\mathscr{A}, \mathscr{B}]$.

Proof. The result follows from an easy modification of the proof of Lemma 6, which also establishes the following (possibly trivial) result.

LEMMA 7'. *Suppose that* $B_1[\mathscr{A}, \mathscr{B}] = B_\infty[\mathscr{A}, \mathscr{B}]$ *for all* $\mathscr{A}$. *Then* $B_1[\mathscr{A}, \mathscr{B} \otimes M_n] = B_\infty[\mathscr{A}, \mathscr{B} \otimes M_n]$ *for all* n.

We feel that the converse to Lemma 7 is also true; but we have been able to show it only in such special cases as Type I von Neumann algebras.

CONJECTURE 1. *If* $B_1[\mathscr{A}, \mathscr{B}] = B_\infty[\mathscr{A}, \mathscr{B}]$ *for all* $\mathscr{A}$, *then* $\mathscr{B} \subseteq \mathscr{C} \otimes M_n$, *for some commutative* C*-*algebra* $\mathscr{C}$ *and some integer* n.

The major difficulty encountered in working on this conjecture is a shortage of maps that fail to be completely bounded. An affirmative answer to Conjecture 1 would make Lemma 7' trivial.

A proof of Conjecture 1 would establish the following analogue of Theorem B.

CONJECTURE 2. *If* $B_1[\mathscr{A}, \mathscr{B}] = B_\infty[\mathscr{A}, \mathscr{B}]$, *then either* $\mathscr{A}$ *is finite-dimensional or* $\mathscr{B} \subseteq \mathscr{C} \otimes M_n$.

It would be desirable to have a quantitative method for deciding whether a given map is completely contractive or completely bounded. We have found such methods in special cases, and the following conjecture has held in every example we have tried. It is an analogue of Theorems 5, 6, 7, and 8 of [2].

CONJECTURE 3. *If* $\mathscr{A}$ *or* $\mathscr{B}$ *is a subalgebra of* M_n (*where* n *is minimal*) *and* $\phi: \mathscr{A} \to \mathscr{B}$, *then* $\sup_k \|\phi_k\| = \|\phi_n\|$.

The reader should notice how Conjecture 3 has been used to prove the results of this paper.

Added March 8, 1976. We have established the validity of Conjectures 1 and 2 in case $\mathscr{B}$ is a von Neumann algebra.

REFERENCES

1. W. B. Arveson, *Subalgebras of* C*-*algebras*. Acta Math. 123 (1969), 141-224.

2. M.-D. Choi, *Positive linear maps on* C*-*algebras*. Canad. J. Math. 24 (1972), 520-529.

3. R. A. Hirschfeld and B. E. Johnson, *Spectral characterization of finite-dimensional algebras*. Indag. Math. 34 (1972), 19-23.

4. R. I. Loebl, *A Hahn decomposition for linear maps*. Pacific J. Math. (to appear).

5. C. E. Rickart, *General theory of Banach algebras*. Von Nostrand, New York, 1960.

6. B. Russo and H. A. Dye, *A note on unitary operators in* C*-*algebras*. Duke Math. J. 33 (1966), 413-416.

Wayne State University
Detroit, Michigan 48202