ON THE SIGNATURE OF FERMAT SURFACES

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1. In this note, we study the signature of certain hypersurfaces of complex projective space. Specifically, we are interested in questions of divisibility of the signature of these hypersurfaces by certain primes; the questions are interesting both because of applications to cobordism theory [9] and because they are related to number-theoretic problems. That such considerations lead to difficult problems in elementary number theory is not surprising. However, it is perhaps unexpected that in certain cases these problems are related to the Fermat conjecture.

Let $\mathbb{C}P^n$ denote complex projective space of dimension n. We define the Fermat surface $Q_n(q)$ to be the hypersurface of complex dimension n and degree q in $\mathbb{C}P^{n+1}$ given by the formula

$$Q_n(q) = \{[z_0, z_1, \cdots, z_{n+1}] \in \mathbb{C}P^{n+1} \mid z_0^q + z_1^q + \cdots + z_{n+1}^q = 0\}.$$

These hypersurfaces admit, in a natural way, an action of an $(n+2)$-dimensional torus, and hence, actions of subgroups of that torus. Study of the equivariant signature under a variety of these actions has led to a number of interesting connections between number theory and topology ([5], [6], [12]). It is our thesis that on an even more elementary level, the study of divisibility of the signature of $Q_n(q)$ by powers of q, where q is an odd prime, leads to interesting questions. In this respect, the present paper is somewhat preliminary; it does not answer the questions, but rather explores the relations between number theory and topology in our special context.

That such relations exist seems to be a consequence of the fact that the signature of the Fermat surfaces can be computed (at least in principle) in divers ways. We point out at the start that although for $q = 2$ and 3 simple formulas for the signature can be obtained, for even modest values of q the numbers involved are quite large, and no simple closed formula, suitable for answering these questions of divisibility, seems to exist. (The signature of $Q_8(11)$ is 48 162 411, while the rank of the middle-dimensional cohomology is 909 090 911.) We have organized this paper from the point of view of obtaining three different formulas for the signature, each with its own purpose.

In Section 2, we briefly sketch the proofs of some well-known facts about the cohomology and Euler characteristic of $Q_n(q)$. Applying the Riemann-Roch theorem, we compute the χ^k-invariants of $Q_n(q)$. Using the Hodge theorem, we therefore establish our first formula for the signature of $Q_n(q)$ as a sum of products of pairs of binomial coefficients. The purpose of this calculation is not so much to investigate divisibility of the signature by powers of q, but to prove a conjecture about $\chi^k(Q_n(q))$. It is easy to compute the Todd genus of $Q_n(q)$ and to see that when $n \geq q - 1$ it is 1. This implies that $h^{0,n}(Q_n(q)) = 0$ when $n \geq q - 1$ (see [4] for notation.) Hence one might conjecture that for fixed k, $\chi^k(Q_n(q)) = (-1)^k$ and

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therefore $h^{k,n-k}(Q_n(q)) = 0$, for sufficiently large n. Observation of the formula for χ^k in this section readily yields this result. There is another interesting consequence. Subsequent to our work, results of F. Hirzebruch and D. B. Zagier [7] have appeared in which $\chi^k(Q_n(q))$ is computed in terms of certain combinatorial quantities involving restricted partitions of multiples of q. Equating the two results yields some interesting combinatorial identities. We mention this aspect in the concluding remarks.

In Section 3, we use the signature theorem of Hirzebruch to compute the signature of $Q_n(q)$ and to reduce it modulo q^2 and q^3 when q is an odd prime. This provides an algorithm for determining when q^2 or q^3 divides the signature. The special case of $Q_{q-3}(q)$ is related to some unsolved problems in elementary number theory.

In Section 4, we derive a result of Hirzebruch, giving the signature of $Q_n(q)$ as a certain coefficient in the power series of a rational function, and we use it to obtain our last expression for the signature in terms of tangent sums. We can therefore rephrase questions of divisibility of the signature inside certain algebraic number rings. This expression also gives an estimate of the absolute size of the signature.

This last expression for the signature in terms of tangent sums is closely related to similar expressions for the signature of certain Brieskorn varieties and for the Browder-Livesay invariant of certain lens spaces. We briefly mention some of these connections in our concluding remarks in Section 5. Naturally, one might hope that reduction of these other invariants modulo q^2 and q^3 might be more easily accomplished.

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2. First we determine the integral homology of the hypersurfaces $Q_n(q)$ and calculate the Euler characteristic.

THEOREM 2.1. *Let* $Q_n(q)$ *be the* q*-dric of complex dimension* n. *Then*

$$H_{2i}(Q_n(q);\ \mathbb{Z}) \simeq \mathbb{Z} \qquad (0 \le i \le 2n;\ 2i \ne n),$$

$$H_{2i+1}(Q_n(q);\ \mathbb{Z}) \simeq 0 \qquad (2i+1 \ne n).$$

Moreover, $H_n(Q_n(q);\ \mathbb{Z})$ *is torsion-free.*

Proof. From the Lefschetz hyperplane theorem [1], we know that the canonical embedding $j\colon Q_n(q) \to \mathbb{C}P^{n+1}$ induces an isomorphism

$$j_*\colon H_i(Q_n(q);\ \mathbb{Z}) \to H_i(\mathbb{C}P^{n+1};\ \mathbb{Z})$$

when $i < n-1$. The first two statements follow immediately. The last statement follows since by Poincaré duality the torsion subgroup of $H_n(Q_n(q);\ \mathbb{Z})$ is isomorphic to that of $H_{n-1}(Q_n(q);\ \mathbb{Z})$. ∎

All that remains then to compute the integral homology is to find the rank of $H_n(Q_n(q);\ \mathbb{Z})$. We do so by computing the Euler characteristic. Fix q and let $\chi_n = \chi(Q_n(q))$. Since $Q_0(q)$ is simply a set of q points, we see that $\chi_0 = q$. For $n > 0$, define a map $f\colon Q_n(q) \to \mathbb{C}P^n$ by

$$f([z_0, z_1, \cdots, z_{n+1}]) = [z_0, z_1, \cdots, z_n].$$

It is easy to see that $f: Q_n(q) \to \mathbb{C}P^n$ is a q-fold branched cover with branching locus $Q_{n-1}(q)$. Hence, for a suitable triangulation, we obtain (by counting simplices) the formula

$$\chi_n = q\chi(\mathbb{C}P^n) - (q-1)\chi_{n-1} \qquad (n > 0).$$

Solving the recursive relation, we obtain the following result.

THEOREM 2.2. $\chi(Q_n(q)) = n + 2 + \dfrac{(1-q)^{n+2} - 1}{q}$.

From Theorem 2.2 we see that

$$\operatorname{rank} H_n(Q_n(q)) = \begin{cases} \left| 2 + \dfrac{(1-q)^{n+2} - 1}{q} \right| & (n \text{ even}), \\ \left| 1 + \dfrac{(1-q)^{n+2} - 1}{q} \right| & (n \text{ odd}). \end{cases}$$

Finally, $Q_n(q)$ is a Kähler manifold. Let the fundamental class of the Kähler metric on $Q_n(q)$ be $\Omega \in H^2(Q_n(q), \mathbb{C})$.

THEOREM 2.3. *Ω^i generates $H^{2i}(Q_n(q); \mathbb{C})$, for $1 \leq i \leq 2n$ and $2i \neq n$.*

We now compute the other invariants of $Q_n(q)$. Let γ be the Hopf bundle over $\mathbb{C}P^{n+1}$. Clearly, the normal bundle of $Q_n(q) \hookrightarrow \mathbb{C}P^{n+1}$ is γ^q.

PROPOSITION 2.4. *The Todd genus $\operatorname{Tg}(Q_n(q))$ of $Q_n(q)$ is $1 + (-1)^n \binom{q-1}{n+1}$.*

Proof. Since the Todd polynomial is multiplicative, we see that

$$\operatorname{Td}(Q_n(q)) = (\operatorname{Td}(\mathbb{C}P^{n+1}))(\operatorname{Td}(\gamma^q))^{-1} = \left(\frac{\omega}{1 - e^{-\omega}}\right)\left(\frac{1 - e^{-q\omega}}{q\omega}\right),$$

where ω is the first Chern class of γ. Therefore $\operatorname{Tg}(Q_n(q)) = \langle \operatorname{Td}(Q_n(q)); [Q_n(q)] \rangle$. Recalling that $Q_n(q)$ is a q-fold branched cover of $\mathbb{C}P^n$, we see that a simple residue computation yields the answer. ■

Proposition 2.4 implies, in particular, that

$$h^{n,0}(Q_n(q)) = \text{geometric genus } (Q_n(q)) = \binom{q-1}{n+1}.$$

Hence, when $n > q - 2$, geometric genus $(Q_n(q)) = 0$.

In the case $n = 2$, we have enough information to apply the theorem of Hodge, and we obtain the following result.

PROPOSITION 2.5. *$\tau(Q_2(q)) = \dfrac{q(4-q^2)}{3}$, where τ is the signature.*

From now on, $Q_n = Q_n(q)$. In this section, we use the Riemann-Roch theorem of Hirzebruch to compute $\chi^k(Q_n)$. Let M be an algebraic manifold, [M] the fundamental class of M, and ξ a holomorphic vector bundle over M. Then the Hirzebruch-Riemann-Roch theorem states that

$$\chi^0(M, \xi) = \chi(M, \xi) = \langle \mathrm{Td}(M) \cdot \mathrm{ch}\, \xi, [M] \rangle .$$

We compute

$$\langle \mathrm{Td}(Q_n) \cdot \mathrm{ch}(\Lambda^k T^*(Q_n)), [Q_n] \rangle = \chi^k(Q_n) .$$

Since $\chi^k(Q_n) = (-1)^k h^{k,k}(Q_n) + (-1)^{n-k} h^{k,n-k}(Q_n)$ and $h^{k,k}(Q_n) = 1$, this enables us to compute $h^{k,n-k}(Q_n)$. Henceforth, we let

$$T = T(Q_n), \qquad T^* = T^*(Q_n), \qquad T_{n+1} = T(\mathbb{C}P^{n+1}), \qquad T^*_{n+1} = T(\mathbb{C}P^{n+1}) .$$

Then $T_{n+1} = T \oplus \gamma^q$, where γ is the Hopf bundle over $\mathbb{C}P^{n+1}$, and all the bundles are restricted to Q_n. In all that follows, restrictions of bundles will be clear from the context.

We observe that

$$T^*_{n+1} = T^* \oplus \gamma^{-q} \Rightarrow \Lambda^k T^*_{n+1} = \Lambda^{k-1} T^* \otimes \gamma^{-q} \oplus \Lambda^k T^* .$$

Using the fact that ch is both additive and multiplicative, we obtain the formula

$$\mathrm{ch}(\Lambda^k T^*_{n+1}) = \sum_{\ell=0}^{k} (-1)^\ell \binom{n+2}{k-\ell} e^{-(k-\ell)\omega} .$$

Finally, using the fact that $\mathrm{ch}\, T^* = \mathrm{ch}\, T^*_{n+1} - e^{-q\omega}$, we see that

$$\mathrm{ch}\, \Lambda^k T^* = \sum_{\ell=0}^{k} \sum_{s=0}^{k-\ell} (-1)^{\ell+s} \binom{n+2}{k-\ell-s} e^{-(k-\ell-s+\ell q)\omega} .$$

Hence,

$$\chi^k(Q_n) = \left\langle \mathrm{Td}(Q_n) \sum_{\ell=0}^{k} \sum_{s=0}^{k-\ell} (-1)^{\ell+s} \binom{n+2}{k-\ell-s} e^{-(k-\ell-s+\ell q)\omega}, [Q_n] \right\rangle .$$

A simple residue computation gives us the following.

LEMMA 2.6.

$$\left\langle e^{-m\omega} \left(\frac{\omega}{1-e^{-\omega}}\right)^{n+2} \left(\frac{1-e^{-q\omega}}{q\omega}\right); [Q_n] \right\rangle = \begin{cases} (-1)^{n+1} \binom{m-1}{n+1} + (-1)^n \binom{q+m-1}{n+1} & \textit{if } m \geq 1, \\ \mathrm{Tg}(Q_n) & \textit{if } m = 0 . \end{cases}$$

Hence,

$$\chi^k = \sum_{\ell=1}^{k}\sum_{s=0}^{k-\ell}\Big[(-1)^{\ell+s+n+1}\binom{n+2}{k-\ell-s}\binom{k-\ell-s+\ell q-1}{n+1} + (-1)^{\ell+s+n}\binom{n+2}{k-\ell-s}\binom{q+k-\ell-s+\ell q-1}{n+1}\Big]$$
$$+\sum_{s=0}^{k-1}\Big[(-1)^{s+n+1}\binom{n+2}{k-s}\binom{k-s-1}{n+1} + (-1)^{s+n}\binom{n+2}{k-s}\binom{q+k-s-1}{n+1}\Big]$$
$$+ (-1)^k\, Tg(Q_n)\,.$$

Recalling that $Tg(Q_n) = 1 + (-1)^n\binom{q-1}{n+1}$, we obtain the formula

$$\chi^k = (-1)^k + \sum_{\ell=0}^{k}\sum_{s=0}^{k-\ell}(-1)^{\ell+s+n}\binom{n+2}{k-\ell-s}\Big[\binom{q+k-\ell-s+\ell q-1}{n+1} - \binom{k-\ell-s+\ell q-1}{n+1}\Big].$$

Finally, by telescoping, we obtain the following result.

THEOREM 2.7. $\chi^k = (-1)^k + \sum_{t=0}^{k}(-1)^{n+k-t}\binom{n+2}{t}\binom{q(k-t+1)+t-1}{n+1}$.

We see immediately that if $q + kq < n + 2$, then $\chi^k(Q_n) = (-1)^k$, and this implies the following.

COROLLARY 2.8. *If* $q + kq < n + 2$, *then* $h^{k,n-k}(Q_n) = 0$. *In particular, if* $q < n + 2$, *geometric genus* $(Q_n) = 0$.

By a theorem of Hodge [4, page 125], we can, in principle, compute the signature $\tau_n(q)$ of $Q_n(q)$:

$$\tau_n(q) = \sum_{k=0}^{n}\chi^k(Q_n(q))\,.$$

3. Although we have in the preceding section obtained an explicit formula for the signature of $Q_n(q)$, its usefulness for actual computation is, in general, rather limited. We are interested in reducing the signature of $Q_n(q)$ modulo powers of q, when q is an odd prime. In Section 4 it will be shown that q divides the signature of $Q_{2n}(q)$ if and only if $2n \leq q - 3$. In this section, we use the signature theorem of Hirzebruch to reduce some topological questions to difficult unsolved problems in elementary number theory.

Throughout this section, let p denote an odd prime, and let $\tau_n(p)$ denote the signature of $Q_n(p)$. Recall that γ is the Hopf bundle and that ω is the first Chern class of γ. Clearly, the normal bundle of $Q_n(p)$ in $\mathbb{C}P^{n+1}$ is γ^p. From all this, it is easy to compute $L(Q_n(p))$, the L-polynomial of $Q_n(p)$:

$$L(Q_n(p)) = \left(\frac{\omega}{\tanh\omega}\right)^{n+2}\frac{\tanh p\omega}{p\omega}\,.$$

Hence,

$$\tau_n(p) = \left\langle \left(\frac{\omega}{\tanh \omega}\right)^{n+2} \frac{\tanh p\omega}{p\omega}, [Q_n(p)] \right\rangle = \left\langle \left(\frac{\omega}{\tanh \omega}\right)^{n+2} \frac{\tanh p\omega}{p\omega}, p[\mathbb{C}P^n] \right\rangle.$$

It follows that $\tau_n(p)$ is the coefficient of x^n in

$$(1) \qquad \left(\frac{x}{\tanh x}\right)^{n+2} \frac{\tanh px}{x}.$$

To facilitate statements of results, we work in $\mathbb{Z}_{(p)}$, the integers localized at p, which is the subring of rationals with denominators prime to p.

THEOREM 3.1. *For* $2n \leq p - 3$,

$$\tau_{2n}(p) \equiv p\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}\right) \pmod{p^3}.$$

Proof. We have the well-known power series expansions

$$\frac{x}{\tanh x} = 1 + \sum_{i=1}^{\infty} (-1)^{i+1} \frac{B_i}{(2i)!} 2^{2i} x^{2i}$$

and

$$\tanh x = \sum_{i=1}^{\infty} (-1)^{i+1} 2^{2i}(2^{2i} - 1) \frac{B_i}{(2i)!} x^{2i-1},$$

where B_i is the ith Bernoulli number. By a theorem of Von Staudt, the denominator of B_i is prime to p if $i < (p - 1)/2$. Hence, for $2n \leq p - 3$, we can truncate the power series above, reduce modulo p^3, and, using (1) from above, conclude that $\tau_{2n}(p)$ is congruent (mod p^3) to the coefficient of x^{2n} in

$$p\left(\frac{x}{\tanh x}\right)^{2n+2}.$$

Let c denote the coefficient of x^{2n} in $(x/\tanh x)^{2n+2}$. Then, as usual, we perform a residue computation to evaluate c. Integrating around a small circle about the origin, we obtain the formula

$$c = \frac{1}{2\pi i} \oint \frac{x}{(\tanh x)^{2n+2}} dx.$$

Substituting $u = \tanh x$, we see that

$$c = \frac{1}{2\pi i} \oint \frac{\tanh^{-1} u}{u^{2n+2}(1 - u^2)} du.$$

Hence c is the coefficient of u^{2n+1} in $(\tanh^{-1} u)/(1 - u^2)$. However,

$$\tanh^{-1} u = u + \frac{u^3}{3} + \frac{u^5}{5} + \cdots,$$

and therefore

$$c = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}. \quad \blacksquare$$

Theorem 3.1 gives a relatively simple condition for p^2 or p^3 to divide $\tau_{2n}(p)$. The sums involved can easily be expressed in terms of harmonic numbers. For fixed, small values of $2n$ it is a simple calculation to compute those primes for which p^2 divides $\tau_{2n}(p)$ (for example, p^2 divides $\tau_{12}(p)$ if and only if $p = 88\,069$). No primes for which p^3 divides $\tau_{2n}(p)$ are known.

The special case where $2n = p - 3$ is interesting. It is well known (and easy to show) that

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{p-2} \equiv \frac{2^{p-1} - 1}{p} \pmod{p}.$$

Hence we obtain the following result.

COROLLARY 3.2. $\tau_{p-3}(p) \equiv 2^{p-1} - 1 \pmod{p^2}$.

Remark. A prime p such that $2^{p-1} - 1 \equiv 0 \pmod{p^2}$ is called a *Wieferich square*. In 1909, A. Wieferich [11] showed that if p is not a Wieferich square, then the equation $x^p + y^p = z^p$ cannot be solved in integers that are not divisible by p. Because of the interesting connection with the Fermat conjecture, an extensive search has been made for such primes. For $p < 3 \times 10^9$, the only Wieferich squares are 1093 and 3511 [2]. We conclude that p^2 probably seldom divides $\tau_{p-3}(p)$. A heuristic argument suggests that

$$\#\{p < x \mid p^2 \text{ divides } \tau_{p-3}(p)\} \sim \log\log x.$$

In view of these remarks, the following result of H. S. Vandiver [10] is even more remarkable.

PROPOSITION 3.3. *Let* p *be an odd prime greater than* 3. *Then* $2^{p-1} - 1 \equiv 0 \pmod{p^3}$ *if and only if*

$$p\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{p-2}\right) \equiv 0 \pmod{p^3}.$$

COROLLARY 3.4. *If* p *is an odd prime, then* p^3 *divides* $\tau_{p-3}(p)$ *if and only if* $2^{p-1} \equiv 1 \pmod{p^3}$.

Remarks. No primes less than 3×10^9 satisfy the condition of Corollary 3.4 ([2]). A heuristic argument suggests that the number of such primes is finite. It has been shown by Ž. B. Linkovskiĭ [8] that if the equation $x^p + y^p = z^p$ has a solution in integers not divisible by p, then $2^{p-1} - 1 \equiv 0 \pmod{p^3}$. It is interesting to consider the possibility of answering by topological methods the question when p^3 divides $\tau_{p-3}(p)$.

4. In this section we use the results of the preceding section to develop an expression for the signature of $Q_n(q)$ in terms of "tangent sums." Our result has two consequences. First, the formula obtained "explains" the alternation of sign of the signature, and it gives a crude estimate of the size. Second, it relates some of the divisibility questions of Section 3 to questions of divisibility in the Kummer ring $\mathbb{Z}[e^{2\pi i/p}]$. Although these questions are no more amenable to solution, the possible connection with work of Kummer on the Fermat conjecture cannot go unnoticed.

As before, we denote by p an odd prime and by $\tau_{2n}(p)$ the signature of $Q_{2n}(p)$. From Section 3 we see that $\tau_{2n}(p)$ is the coefficient of x^{2n} in

$$\left(\frac{x}{\tanh x}\right)^{2n+2} \frac{\tanh px}{x}.$$

As usual, using residues, and integrating around a small circle about the origin we obtain the formula

$$\tau_{2n}(p) = \frac{1}{2\pi i} \oint \frac{\tanh px}{(\tanh x)^{2n+2}} dx.$$

Making the substitution $u = \tanh x$, we see that

$$\tau_{2n}(p) = \frac{1}{2\pi i} \oint \frac{1}{u^{2n+2}} \cdot \frac{1}{(1-u^2)} \cdot \frac{(1+u)^p - (1-u)^p}{(1+u)^p + (1-u)^p} du.$$

Hence we deduce the result of Hirzebruch [3]:

THEOREM 4.1. *$\tau_{2n}(p)$ is the coefficient of u^{2n+1} in*

$$\frac{1}{1-u^2}\left[\frac{(1+u)^p - (1-u)^p}{(1+u)^p + (1-u)^p}\right].$$

COROLLARY 4.2.

$$\tau_{2n}(p) \equiv \begin{cases} 0 \pmod p & (2n \leq p-3), \\ 1 \pmod p & (2n > p-3). \end{cases}$$

Proof. Reducing the rational function in Theorem 4.1 modulo p, we see that $\tau_{2n}(p)$ is congruent (mod p) to the coefficient of u^{2n+1} in $u^p/(1-u^2)$. ■

For $p = 2$ or 3, it is possible to obtain closed-form expressions for the signature of $Q_n(p)$ from Theorem 4.1. One could attempt to generalize this to larger primes, by first using Theorem 4.1 to obtain an inductive formula for $\tau_{2n}(p)$ and then using finite-difference methods to obtain a closed form expression. A more systematic approach is the following.

Consider the rational function

$$f(u) = \frac{1}{(1-u^2)}\left[\frac{(1+u)^p - (1-u)^p}{(1+u)^p + (1-u)^p}\right] - \frac{u}{1-u^2}.$$

Since $u/(1-u^2) = u + u^3 + u^5 + \cdots$, we see that $\tau_{2n} - 1$ is the coefficient of u^{2n+1} in the power series expansion of $f(u)$. Moreover, a simple calculation shows that

$$f(u) = \frac{(1+u)^{p-1} - (1-u)^{p-1}}{(1+u)^p + (1-u)^p}.$$

Let $g(u) = (1+u)^p + (1-u)^p$. Clearly,

$$f(u) = \frac{1}{p}\frac{g'(u)}{g(u)}.$$

Now g(u) has p - 1 zeros $s_k = \frac{\zeta^k + 1}{\zeta^k - 1}$ (k = 1, 2, ⋯, p - 1), where $\zeta = e^{2\pi i/p}$. Hence

$$f(u) = \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{(u - s_k)} .$$

Therefore the coefficient of u^{2n+1} in f(u) is

$$-\frac{1}{p} \sum_{k=1}^{p-1} \left(\frac{1}{s_k}\right)^{2n+2} .$$

Finally, using the fact that $(\zeta^k - 1)/(\zeta^k + 1) = i \tan(k\pi/p)$, we deduce the following result.

THEOREM 4.3. *If* p *is an odd prime and* $\zeta = e^{2\pi i/p}$, *then*

$$\tau_{2n}(p) = 1 - \frac{1}{p} \sum_{k=1}^{p-1} \left(\frac{\zeta^k - 1}{\zeta^k + 1}\right)^{2n+2} = 1 + \frac{(-1)^n}{p} \sum_{k=1}^{p-1} \left(\tan \frac{k\pi}{p}\right)^{2n+2} .$$

Since $|\tan(k\pi/p)|$ is maximum when $k = (p - 1)/2$ or $(p + 1)/2$, we can obtain a crude estimate of the size of $\tau_{2n}(p)$.

COROLLARY 4.4. *If* p *is an odd prime, then*

$$\frac{1}{p}\left(\tan \frac{(p - 1)\pi}{2p}\right)^{2n+2} < |\tau_{2n}(p) - 1| < \frac{(p - 1)}{p}\left(\tan \frac{(p - 1)\pi}{2p}\right)^{2n+2} .$$

We can also obtain an asymptotic estimate for $\tau_{2n}(p)$ by using the approximation $\tan x \sim 2/(\pi - 2x)$ for x sufficiently close to $\pi/2$. Rewriting the formula of Theorem 4.3, we see that

$$\tau_{2n}(p) = 1 + \frac{2(-1)^n}{p} \sum_{k=0}^{(p-3)/2} \left(\tan\left(\frac{p - 1}{2} - k\right)\pi/p\right)^{2n+2}$$

$$\sim \frac{2(-1)^n}{p} \sum_{k=0}^{(p-3)/2} \left(\frac{2p}{\pi}\right)^{2n+2} \left(\frac{1}{2k + 1}\right)^{2n+2}$$

$$\sim \frac{2(-1)^n}{p} \sum_{k \geq 0} \left(\frac{2p}{\pi}\right)^{2n+2} \left(\frac{1}{2k + 1}\right)^{2n+2}$$

$$= 2(-1)^n p^{2n+1} (2^{2n+2} - 1) \zeta(2n + 2)/\pi^{2n+2} .$$

(Here ζ denotes the zeta function.) Using the formula $\zeta(2i) = 2^{2i-1} B_i \pi^{2i}/(2i)!$, we obtain the following result.

COROLLARY 4.5. *For* n *fixed and* $p \to \infty$,

$$\tau_{2n}(p) \sim (-1)^n 2^{2n+2} (2^{2n+2} - 1) \frac{B_{n+1}}{(2n + 2)!} p^{2n+1} .$$

Remark 1. It is amusing to note that for the special case of interest in Section 3, namely $Q_{1090}(1093)$, the signature has approximately 3100 digits.

Remark 2. We can also prove Corollary 4.4 from the relation

$$\tau_{2n}(p) = \operatorname{res}_{x=0}\left(\frac{\tanh px}{(\tanh x)^{2n+2}}\right)$$

by substituting the expression

$$\tanh px = \sum_{i=1}^{\infty} (-1)^{i+1}\, 2^{2i}\,(2^{2i} - 1)\frac{B_i}{(2i)!}\, p^{2i-1}\, x^{2i-1}.$$

This gives $\tau_{2n}(p) = \sum_{j=0}^{n} a_{nj}\, p^{2j+1}$, where

$$a_{nj} = (-1)^j\, 2^{2j+2}\,(2^{2j+2} - 1)\frac{B_{j+1}}{(2j+2)!}\operatorname{res}_{x=0}\left(\frac{x^{2j+1}}{(\tanh x)^{2n+2}}\right).$$

In particular,

$$a_{nn} = (-1)^n\, 2^{2n+2}\,(2^{2n+2} - 1)\frac{B_{n+1}}{(2n+2)!}.$$

Remark 3. Although we have considered $\tau_{2n}(p)$ only in the case where p is an odd prime, it is clear that the techniques used in this section can be used to obtain estimates for $\tau_{2n}(q)$, where q is any positive integer. For even values of q the results are slightly different.

5. In this last section, we make some remarks concerning the preceding computations.

Hirzebruch and Zagier [7] have recently published work in which they have computed the invariants $\chi^k(Q_n(q))$ in terms of certain combinatorial quantities. Specifically, let $\tilde{N}_{k+1}$ denote the number of partitions of $(k+1)q$ into $n+2$ parts, not exceeding $q-1$. Then

$$\tilde{N}_{k+1} = \#\{0 < a_1, a_2, \cdots, a_{n+2} < q \mid a_1 + a_2 + \cdots + a_{n+2} = (k+1)q\}.$$

Hirzebruch and Zagier have shown that

$$\chi^k(Q_n(q)) = (-1)^k + (-1)^{n-k}\tilde{N}_{k+1}.$$

By virtue of the results in Section 2 about the cohomology of $Q_n(q)$, this shows immediately that when n is sufficiently large (k fixed), $h^{k,n-k}(Q_n(q)) = 0$. Moreover, comparing their result with the formula obtained in Theorem 2.7 we see that

$$\tilde{N}_{k+1} = \sum_{t=0}^{k} (-1)^k \binom{n+2}{t}\binom{q(k-t+1)+t-1}{n+1}.$$

The signature of $Q_{2n}(p)$ is also related to two other topological invariants. The first is the signature of the Brieskorn variety V given by

$$V = \{(z_0, z_1, \cdots, z_{2n}) \in \mathbb{C}^{2n+1} \mid z_0^p + \cdots + z_{2n}^p = 1\}.$$

We easily see that

$$\operatorname{sign}(Q_{2n}(p)) = 1 + \operatorname{sign} V,$$

by decomposing $Q_{2n}(p)$ into a tubular neighborhood of $Q_{2n-1}(p) \subset Q_{2n}(p)$, which has signature 1, and the complement, which is essentially V. (See [7, page 203].)

The second related invariant is the Browder-Livesay invariant of a certain lens space. Consider S^{4n+3} as the unit sphere in $\mathbb{C}^{2n+2}$. Let the cyclic group of order p act on S^{4n+3} by

$$(z_0, z_1, \cdots, z_{2n+1}) \mapsto (\zeta z_0, \zeta z_1, \cdots, \zeta z_{2n+1}),$$

and let $L(p; 4n+3)$ denote the orbit space. The antipodal map induces a free involution T on $L(p; 4n+3)$. If $\alpha(T; L(p, 4n+3))$ denotes the Browder-Livesay invariant of this involution, then

$$\operatorname{sign} Q_{2n}(p) = 1 + \alpha(T; L(p, 4n+3))$$

We can see this either by comparing the formula for $Q_{2n}(p)$ developed in Section 4 with the results of [5], or more directly, by noting that $L(p, 4n+3)$ is the sphere bundle of γ^p, which implies the relationship after calculation. Naturally, one might hope to reduce these two invariants modulo p^2 and p^3.

Are these curious and perhaps unexpected results merely accidents, or do they follow from some deep relationship between the topology of Fermat surfaces and their numerical invariants? We do not know. Certainly, other mathematicians have obtained similar results. The G-signature theorem, in particular, has led to the discovery of a number of such relationships [7]. Perhaps a better understanding of the topology of Fermat surfaces will shed new light on long-standing problems in number theory.

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