ON A DIFFERENTIAL-DIFFERENCE EQUATION

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In this paper we consider the entire and the meromorphic solutions $f(z)$ of the differential-difference equation

$$f(z+1) = \exp[P(z)]f'(z), \tag{1}$$

where

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \qquad (a_0 \neq 0) \tag{2}$$

is an arbitrary polynomial of degree $n \geq 1$.

We agree to say that

(a) a meromorphic function $f(z)$ is *properly meromorphic* if $f(z)$ has at least one pole,

(b) an infinite set E of entire functions is *linearly independent* (over the field of complex numbers) if each finite system of functions of the set E is linearly independent.

We prove that the equation (1) has no properly meromorphic solutions but has a linearly independent infinite set E (with cardinality of the continuum) of entire solutions. Furthermore, for each $\rho \geq n+1$ in the case $n > 1$ and for $\rho > 2$ in the case $n = 1$, the equation (1) has a linearly independent set E_ρ (with cardinality of the continuum) of entire solutions of order ρ.

The equation (1) has no nontrivial entire solutions $f(z)$ ($f(z) \not\equiv 0$), either of order $\rho < n+1$ or of minimal type with respect to the order $\rho = n+1$. On the other hand, this equation has entire solutions of normal type with respect to the order $\rho = n+1$.

Besides the equation (1), we shall also consider the inhomogeneous equation

$$f(z+1) = \exp[P(z)]f'(z) + g(z) \tag{3}$$

with an entire or meromorphic free term $g(z)$.

Remark 1. In the case $n = 0$, the equation (1) reduces to an equation with constant coefficients. Such equations have been at the center of wide research, and we do not include them in our study although the method used in this paper may be applied to some extent to these equations also.

Remark 2. The results announced above on the entire solutions of the equation (1) contain an answer to the following question of Hurwitz [2, p. 752]:

Is it possible for a power series

$$h(\xi) = \sum_{k=0}^{\infty} a_k(\xi - \xi_0)^k \qquad (h(\xi) \not\equiv c \exp \xi,\ c = \text{const})$$

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to take the form

$$h'(\xi) = \sum_{k=1}^{\infty} ka_k(\xi - \xi_0)^{k-1},$$

after analytic continuation along some closed path L? Assume $\xi_0 \neq 0$ and the path L is the circle $|\xi| = |\xi_0|$ described in the positive sense. Writing

$$\xi = \exp 2\pi iz, \qquad h(\exp 2\pi iz) = f(z),$$

we reduce the problem above to the following one:

Does the equation

$$f(z+1) = (-i/2\pi)\exp(-2\pi iz)f'(z) \tag{4}$$

have any solutions $f(z)$ ($f(z) \not\equiv c\exp(\exp 2\pi iz)$, $c = \text{const}$) analytic in some strip $\alpha < \Im z < \beta$, where α and β are real numbers?

The equation (4) is a particular case of the equation (1); therefore it has a linearly independent, infinite set of entire solutions. Consequently, the equation (4) has entire solutions other than $f(z) \equiv c\exp(\exp 2\pi iz)$, $c = \text{const}$, and the answer to the question of Hurwitz is positive.

The problem of Hurwitz was solved earlier by H. Lewy, who kindly permitted us to present here his elegant solution

$$h(z) = \int_0^{-\infty} \exp[zt + (\log t - \pi i)^2/4\pi i]\,dt.$$

I would like to thank A. Marden, who called my attention to the subject of this paper.

1. FORMAL SOLUTIONS

Using the notation

$$Df(z) \equiv f'(z), \qquad (\exp D)f(z) \equiv f(z+1),$$

we write the equation (1) in either of the two forms

$$Lf(z) = 0, \qquad L = L(D) = \exp D - \exp[P(z)]D \tag{1.1}$$

and

$$f(z) = Kf(z), \qquad K = K(D) = \exp(-D)\exp[P(z)]D. \tag{1.2}$$

We shall restrict ourselves to operating on entire functions $\phi(z)$ that tend to zero rapidly enough as $z \to +\infty$ along the positive real axis.

Define K^{-1} by the formula

$$K^{-1}\phi(z) \equiv D^{-1}\exp[-P(z)](\exp D)\phi(z) \equiv \int_{+\infty}^{z} \exp[-P(t)]\phi(t+1)\,dt. \tag{1.3}$$

Here the integration is along a path consisting first of an infinite segment of the positive real axis from $z = +\infty$ to an arbitrary point x_0 ($\Im x_0 = 0$), and then of any rectifiable path from x_0 to z. It is obvious that $K^{-1}K = KK^{-1} = I$, where I is the identity operator.

Consider now the formal series

$$(1.4) \qquad U\phi(z) = \phi(z) + K\phi(z) + K^2\phi(z) + \cdots,$$

$$(1.5) \qquad V\phi(z) = K^{-1}\phi(z) + K^{-2}\phi(z) + K^{-3}\phi(z) + \cdots,$$

and

$$(1.6) \qquad W\phi(z) = U\phi(z) + V\phi(z) = \sum_{n=-\infty}^{+\infty} K^n\phi(z),$$

where $\phi(z)$ is an arbitrary entire function tending rapidly enough to zero as $z \to +\infty$ along the positive real axis.

It is easy to verify that $W\phi(z)$ is a formal solution of both the equivalent homogeneous equations (1.2) and (1.1), and that $U\phi(z)$ and $-V\phi(z)$ are formal solutions of the inhomogeneous equation

$$(1.7) \qquad Lf(z) = \phi(z + 1).$$

To explain the main idea, which we shall use for the study of the series (1.4) and (1.5), we denote by A the angle

$$(1.8) \qquad A = \{z: \pi - |\arg z| < \delta,\ -\pi \le \arg z \le \pi,\ \delta > 0\}$$

and notice that $K\phi(z) = \exp[P(z - 1)]\phi'(z - 1)$ depends linearly on $\phi'(z - 1)$. It is easy to see by induction that $K^j\phi(z)$ depends linearly on $\phi'(z - j)$, $\phi''(z - j)$, $\cdots$, $\phi^{(j)}(z - j)$. Besides, for each z there exists a nonnegative number j_0 such that $z - j \in A$ if $j > j_0$. Therefore the moduli of $\phi'(z - j)$, $\phi''(z - j)$, $\cdots$, $\phi^{(j)}(z - j)$, and $K^j\phi(z)$ will be arbitrarily small if $j > j_0$ and $|\phi(z)|$ is small enough in the angle A.

It will be shown that the series (1.4) converges absolutely and uniformly in each circle $|z| \le R$, and $U\phi(z)$ is therefore an entire solution of the equation (1.7), if the entire function $\phi(z)$ tends to zero rapidly enough as $z \to \infty$ in the angle A. Similarly, the series (1.5) converges absolutely and uniformly in each circle $|z| \le R$, and $-V\phi(z)$ is an entire solution of (1.7) if $\phi(z)$ tends to zero rapidly enough as $z \to \infty$ in the angle

$$(1.9) \qquad B = \{z: |\arg z| < \delta\}.$$

The function

$$(1.10) \qquad f(z) = W\phi(z) = U\phi(z) + V\phi(z)$$

is then an entire solution of the homogeneous equation (1.1). We get a linearly independent infinite set of entire solutions of the equation (1.1), on taking in (1.10) $\phi(z) = \phi_\alpha(z)$, where $\{\phi_\alpha(z)\}$ is a suitably chosen set of entire functions.

Remark. We can also get the series (1.4) and (1.5) by a method of bilateral iteration. To get the series (1.4), we write the equation (1.7) in the form $f(z) = Kf(z) + \phi(z)$, and we define a sequence of successive approximations

$$f_0(z) = 0, \quad f_{i+1}(z) = Kf_i(z) + \phi(z) \quad (i = 0, 1, 2, \cdots).$$

To get the series (1.5), we restrict ourselves to the case when the free term $\phi(z)$ tends to zero rapidly enough as $z \to \infty$ in the angle B (see (1.9)) and the unknown function $f(z)$ also tends to zero as $z \to \infty$ in this angle. Using (1.3), we write the equation (1.7) in the form

$$f(z) = K^{-1}[f(z) - \phi(z)] = \int_{+\infty}^{z} \exp[-P(t)][f(t+1) - \phi(t+1)]\,dt,$$

and we define the sequence of successive approximations

$$f_0(z) = 0, \quad f_{i+1}(z) = K^{-1}[f_i(z) - \phi(z)] \quad (i = 0, 1, 2, \cdots).$$

This method has been used previously by us to solve a linear difference equation with entire coefficients [6], and by L. Navickaite [7] and A. Gilis [1] to solve some differential-difference equations.

2. CONVERGENCE AND ORDER OF THE SERIES (1.4) AND (1.5)

1. Before beginning the study of the series (1.4) and (1.5), we agree on the following notation:

(a) Let $\rho \geq 2$ and $0 < \delta < (\pi/2\rho)$. By $H_\rho(A)$ and $H_\rho(B)$ we denote the class of entire functions $\phi(z)$ that satisfy in the angles A and B, respectively (see (1.8) and (1.9)), the inequality

$$|\phi(z)| \leq C \exp(-\gamma |x|^\rho) \quad (z = x + iy), \tag{2.1}$$

where C and γ are positive numbers that do not depend on z but may depend on $\phi(z)$. Sometimes we shall also use for these classes the notation $H_\rho(A, \gamma)$ and $H_\rho(B, \gamma)$, with the purpose of indicating the factor γ in the exponent in (2.1).

(b) We use the customary notation $M(r, h) = \max_{|z| \leq r} |h(z)|$, for any entire function $h(z)$.

(c) For each $\delta > 0$, we define

$$\alpha_0 = 1 + \operatorname{ctg} \delta, \quad \alpha_1 = 1 + \alpha_0 = 2 + \operatorname{ctg} \delta. \tag{2.2}$$

Remark. For $\rho = 2\tau$, where $\tau \geq 1$ is a natural number, the function

$$\phi(z) = \exp[-az^{2\tau}] \quad (a > 0)$$

belongs to both the classes $H_\rho(A)$ and $H_\rho(B)$. In addition, this function is of order $\rho = 2\tau$ and of normal type.

To get such a function $\phi(z)$ for other values of ρ ($\rho \geq 2$), we denote by $\tau > 0$ any nonintegral number and by $K(z, \tau)$ the canonical product with zeros at the points $-n^{(1/\tau)}$ ($n = 1, 2, 3, \cdots$). The asymptotic equality

$$\log K(z, \tau) \sim (-1)^p \frac{\pi}{\sin \pi(\tau - p)} z^\tau \quad (p = [\tau])$$

holds (see [8, p. 232]) in the angle $|\arg z| < \pi - \varepsilon$, for each $\varepsilon > 0$. We define the function

$$G(z, \tau) = K(az^2, \tau) \qquad (a > 0) \tag{2.3}$$

if p is an odd number, and

$$G(z, \tau) = K(a \exp(i\pi/\tau)z^2, \tau) \tag{2.4}$$

if $p \geq 2$ is an even number. This function belongs to both classes $H_\rho(A)$ and $H_\rho(B)$ ($\rho = 2\tau$), its order equals ρ, and its type is normal.

2. THEOREM 2.1. *If the function* $\phi(z)$ *belongs to the class* $H_\rho(A)$ $(\rho > n + 1)$ *or to the class* $H_{n+1}(A, \gamma)$ *with a large enough* γ, *then the series* (1.4) *converges absolutely and uniformly in each circle* $|z| \leq R$, *and its sum*

$$G(z) = U\phi(z) \tag{2.5}$$

is an entire solution of the equation (1.7). *Moreover,* $G(z)$ *belongs to the same class* $H_\rho(A)$ *as* $\phi(z)$, *and there is a constant* $C_1 = C_1(\phi)$ *such that*

$$|G(z)| \leq C_1 \exp[\alpha_1 r \cdot p(\alpha_1 r)] M[\alpha_1(r + 1), \phi] \qquad (|z| \leq r), \tag{2.6}$$

where

$$p(r) = |a_0| r^n + |a_1| r^{n-1} + \cdots + |a_n|. \tag{2.7}$$

If the function $\phi(z)$ *belongs to the class* $H_\rho(B)$ $(\rho > n + 1)$ *or to the class* $H_{n+1}(B, \gamma)$ *with a large enough* γ, *then the series* (1.5) *converges absolutely and uniformly in each circle* $|z| \leq R$, *and its sum with the negative sign*

$$H(z) = -V\phi(z) \tag{2.8}$$

is an entire solution of the equation (1.7). *In addition,* $H(z)$ *belongs to the same class* $H_\rho(B)$ *as* $\phi(z)$, *and there is a constant* $C_2 = C_2(\phi)$ *such that*

$$|H(z)| \leq C_2 \exp[2\alpha_1 rp(\alpha_1(r + 1))] M(\alpha_1(r + 1), \phi) \qquad (|z| = r). \tag{2.9}$$

COROLLARY 2.1. *The functions* $G(z) = U\phi(z)$ *and* $H(z) = -V\phi(z)$ *have the same order and type as the function* $\phi(z)$ (speaking of type, we distinguish only three kinds of it: the minimal, normal, and maximal types).

Proof of the Corollary. Let μ be the order of the function $\phi(z)$. Since $\phi(z) \in H_\rho(A)$ or $H_\rho(B)$, it follows that $\mu \geq \rho$, and if $\mu = \rho$, the type of $\phi(z)$ is at least normal (see [3, p. 21]). Since $\rho \geq n + 1$, it is easy to conclude from (2.6) and (2.9) that the order and type of $G(z)$ and $H(z)$ don't exceed the order and type of the function $\phi(z)$. On the other hand, the order and type of $\phi(z)$ do not exceed the order and type of each of the functions $G(z)$ and $H(z)$, as follows from the identities

$$\phi(z + 1) \equiv G(z + 1) - \exp[P(z)]G'(z),$$

$$\phi(z + 1) \equiv H(z + 1) - \exp[P(z)]H'(z)$$

and the fact that the order ρ of $\phi(z)$ is higher than the order n of $\exp[P(z)]$.

3. Theorem 2.1 consists of two statements: the first about the series (1.4), the second about the series (1.5). To prove the first statement, we use the following result.

LEMMA 2.1. *Suppose* j *is a natural number,* K *the operator* (1.2), *and* $g(z)$ *a function regular in the circle* $e(z_j, a) = \{z\colon |z - z_j| \le a\}$, *where* $a > 0$, $z_j = z_0 - j$, *and* z_0 *is a fixed point of the* z*-plane. Then the function* $g_j(z) = K^j g(z)$ *is regular in the circle* $e(z_0, a)$ *and*

$$(2.10)\qquad |g_j(z_0)| \le (j/a)^j \exp[j \cdot p(|z_0| + j + a)]\mathfrak{M},$$

where $p(r)$ *is the function* (2.7) *and*

$$\mathfrak{M} = \mathfrak{M}(z_j, a, g) = \max_{z \in e(z_j, a)} |g(z)|.$$

Proof. From the equation

$$g_1(z) = Kg(z) = \exp[P(z - 1)]g'(z - 1)$$

we easily conclude that $g_1(z)$ is regular in the circle $e(z_{j-1}, a)$, where $z_{j-1} = z_0 - j + 1$. Writing $a_1 = a - (a/j)$ and using the Cauchy inequalities, we get

$$(2.11)\qquad \mathfrak{M}(z_{j-1}, a_1, g_1) \le (j/a)\exp[p(|z_0| + j + a_1)]\mathfrak{M}(z_j, a, g).$$

By the same reasoning, we prove that for $i = 2, 3, \cdots, j$ the function $g_i(z) = Kg_{i-1}(z)$ is regular in the circle $e(z_{j-i}, a)$ $(z_{j-i} = z_0 - j + i)$ and

$$(2.12)\qquad \mathfrak{M}(z_{j-i}, a_i, g_i) \le (j/a)\exp[p(|z_0| + j - i + 1 + a_i)]\mathfrak{M}(z_{j-i+1}, a_{i-1}, g_{i-1}),$$

where $a_i = a_{i-1} - (a/j) = a - (ia/j)$ and $\mathfrak{M}(z_0, a_j, g_j) = |g_j(z_0)|$.

Multiplying the inequalities (2.11) and (2.12) $(i = 2, 3, \cdots, j)$, and using the fact that $p(r)$ is an increasing function, we get (2.10).

4. We now assume $\phi(z)$ to be an entire function satisfying in the angle A the inequality (2.1), with $\rho \ge n + 1$ and with γ large enough for $\rho = n + 1$, and we consider the general term

$$(2.13)\qquad T_j(z) = K^j \phi(z)$$

of the series (1.4). We recall that 2δ is the opening of the angle A, and we write (see (2.2))

$$(2.14)\qquad \alpha_2 = \cot\delta/(1 + \cot\delta) = 1 - 1/\alpha_0.$$

Let $z \in E_{R+1}$, where

$$(2.15)\qquad E_R = \{z\colon |z| \le R\},$$

and let $j > \alpha_0(R + 1)$. Then $z - j \in A$, $\mathfrak{R}(z - j) < -\alpha_2 j$, and by (2.1),

$$|\phi(z - j)| \le C\exp(-\gamma(\alpha_2 j)^\rho) \qquad (z \in E_{R+1}).$$

Using Lemma 2.1 (with $a = 1$), we get the inequality

(2.16) $$|T_j(z)| \leq C j^j \exp[j \cdot p(R + j + 1)] \exp[-\gamma(\alpha_2 j)^\rho]$$

whenever $z \in E_R$ and $j > \alpha_0(R + 1)$. We notice that $R + 1 < j/\alpha_0$ and $\rho \geq n + 1$, and we suppose γ to be large enough for $\rho = n + 1$. We easily conclude then from (2.16) and (2.7) that

(2.17) $$|T_j(z)| \leq \exp\left(-\frac{\gamma}{2}(\alpha_2 j)^\rho\right) \qquad (j > \alpha_0(R + 1),\ z \in E_R),$$

which implies rapid uniform convergence in E_R of the series (1.4). By the Weierstrass theorem we can therefore apply the operator K term-by-term to the series (1.4), so that the sum G(z) is an entire solution of the equation (1.7).

5. We consider now the general term $T_j(z)$ when $z \in E_R$ and

(2.18) $$j \leq \alpha_0(R + 1).$$

If $z \in E_{R+1}$, then (see (2.2))

(2.19) $$|z - j| \leq R + 1 + \alpha_0(R + 1) = \alpha_1(R + 1)$$

and

$$|\phi(z - j)| \leq M[\alpha_1(R + 1), \phi] = \max_{|z| \leq \alpha_1(R+1)} |\phi(z)|$$

whenever $z \in E_{R+1}$. Using Lemma 2.1, we conclude that

(2.20) $$|T_j(z)| \leq j^j \exp[j \cdot p(R + j + 1)] M[\alpha_1(R + 1), \phi] \qquad (z \in E_R,\ j \leq \alpha_0(R + 1)).$$

Since

$$|G(z)| \leq \sum_{j \leq \alpha_0(R+1)} |T_j(z)| + \sum_{j > \alpha_0(R+1)} |T_j(z)|$$

and the degree n of the polynomial p(r) is not less than 1, we easily deduce from (2.17) and (2.20) the estimate (2.6).

6. To complete our study of the function G(z), it remains to show that $G(z) \in H_\rho(A)$. For this purpose, we denote by d_1 the distance from the point $z = -1$ to the boundary of the angle A, and we write $d = \min(d_1, 1/2)$. Then the circle

$$e_j = \{z: |z - z_0 + j| \leq d\} \quad (j = 1, 2, 3, \cdots)$$

belongs to A whenever $z_0 = x_0 + iy_0 \in A$. Consequently (see (2.1)),

$$|\phi(z)| \leq C \exp[-\gamma |x_0 - j + 1/2|^\rho] \qquad (\rho \geq n + 1)$$

if $z \in e_j$ $(j = 1, 2, 3, \cdots)$ and $z_0 \in A$. Using the inequalities $a^\rho + b^\rho \leq (a + b)^\rho$ $(a \geq 0,\ b \geq 0)$ and $x_0 \leq 0$, we get the estimate

$$|\phi(z)| \leq C \exp[-\gamma |x_0|^\rho] \exp[-\gamma(j - 1/2)^\rho] \qquad (z \in e_j).$$

By Lemma 2.1, we therefore have the relation

(2.21) $$|K^j\phi(z_0)| = a_j \cdot C \exp[-\gamma |x_0|^\rho],$$

where

$$a_j \leq (j/d)^j \exp[j \cdot p(|z_0| + j + d)] \exp[-\gamma(j - 1/2)^\rho].$$

We notice that $|z_0| < |x_0|/\cos\delta$, since $z_0 \in A$. Hence, for $j < |z_0|$ we have the inequality

$$a_j \leq C_1 \exp[(2/\cos\delta)|x_0| \cdot p((2/\cos\delta)|x_0| + d)] \exp[-\gamma(j - 1/2)^\rho]$$
$$\leq C_2 \exp(\alpha |x_0|^{n+1}) \exp[-\gamma(j - 1/2)^\rho],$$

where C_1, C_2, and α do not depend on j, x_0, ρ, and γ. A similar reasoning shows that

$$a_j \leq C_2 \exp[\alpha(j - 1/2)^{n+1}] \exp[-\gamma(j - 1/2)^\rho]$$

for $j > |z_0|$. We use these estimates for a_j and recall that $\rho \geq n + 1$ and γ is sufficiently large for $\rho = n + 1$. We conclude then from (2.21) that

(2.22) $$|K^j\phi(z)| \leq C_3 C \exp[-(\gamma/2)(j - 1/2)^\rho] \exp[-(\gamma/2)x^\rho]$$

for $j = 0, 1, 2, \cdots$ and $z \in A$. Here C is the constant of (2.1), and $C_3 = C_3(\rho, \gamma)$ does not depend on j and z. Moreover, $C_3(\rho, \gamma) \leq C_3(\rho, \gamma_0)$ for $\gamma \geq \gamma_0$ $(\gamma_0 > 0)$. From (2.22) it follows that $G(z) \in H_\rho(A)$.

7. We shall now analyze the series (1.5) and its general term

(2.23) $$S_j(z) = K^{-j}\phi(z).$$

In all this analysis we suppose $\phi(z)$ to satisfy in the angle B (see (1.9)) the condition (2.1) with $\rho \geq n + 1$ and with γ large enough if $\rho = n + 1$. Consequently (see (1.3))

(2.24) $$S_1(z) = K^{-1}\phi(z) = \int_{L_{+\infty z}} \exp[-P(t)]\phi(t + 1)\,dt,$$

where the integration is along the horizontal half-line

$$L_{+\infty_z} = \{t: \Im t = \Im z, \ \Re z \leq \Re t < \infty\}$$

joining the points $t = \infty$ and $t = z$.

It will be useful to consider the somewhat more general integral

(2.25) $$I(z, T) = \int_{L_{+\infty z}} \exp[-P(t)]\phi(t + T)\,dt \qquad (T > 0).$$

First we suppose z and T to satisfy the conditions

(2.26) $z + T - 1/2 \in B$ and either $x = \Re z \geq 0$ or $T \geq 2|x| + 1/2$ if $x < 0$,

and we write $\tau = \Re t$. For each $t \in L_{+\infty_z}$ we therefore have the inequalities

$$\tau + T - 1/2 > 0 \quad \text{and} \quad |t| \leq |t + T - 1/2| \leq (\tau + T - 1/2)/\cos \delta .$$

Consequently,

$$|\exp[-P(t)]| \leq \exp[p((\tau + T - 1/2)\cos \delta)] \quad (t \in L_{+\infty_z}). \tag{2.27}$$

By the law of the mean,

$$(\tau + T)^\rho \geq (\tau + T - 1/2)^\rho + (\rho/2)(\tau + T - 1/2)^{\rho - 1},$$

and therefore

$$|\phi(t + T)| \leq C \exp[-\gamma(\tau + T - 1/2)^\rho]\exp[-\gamma(\tau + T - 1/2)^{\rho - 1}] \quad (t \in L_{+\infty_z}), \tag{2.28}$$

where C is the constant of (2.1). Since $\rho \geq n + 1$ and n is the degree of the polynominal $p(r)$, we have for $t \in L_{+\infty_z}$ the estimate

$$|\exp[-P(t)]\phi(t + T)| \leq C_1 C \exp[-\gamma(\tau + T - 1/2)^\rho], \tag{2.29}$$

where $C_1 = C_1(\rho, \gamma)$ does not depend on z and T, and where $C_1(\rho, \gamma) \leq C_1(\rho, \gamma_0)$ when $\gamma > \gamma_0$.

We now notice that for $a > 0$,

$$\int_a^{+\infty} \exp(-\gamma x^\rho)\, dx \leq (1/\gamma\rho a^{\rho - 1}) \int_a^{+\infty} \gamma\rho x^{\rho - 1} \exp(-\gamma x^\rho)\, dx \leq (1/\gamma\rho a^{\rho - 1}) \exp(-\gamma a^\rho),$$

and we deduce from (2.25) and (2.29) that

$$|I(z, T)| \leq C_2 C \exp[-\gamma(x + T - 1/2)^\rho] \quad (x = \Re z), \tag{2.30}$$

whenever z and T satisfy the condition (2.26). Here $C_2 = C_2(\rho, \gamma)$ does not depend on z and T; and $C_2(\rho, \gamma) \leq C_2(\rho, \gamma_0)$ when $\gamma > \gamma_0$.

We need an estimate for $I(z, T)$ also in the case when (2.26) does not hold. To obtain it, we write

$$I(z, T) = I(z_1, T) + I_1, \qquad I_1 = \int_{z_1}^{z} \exp[-P(t)]\phi(t + T)\, dt, \tag{2.31}$$

where z_1 is the point of intersection of the line $L_{+\infty_z}$ and the boundary of the angle B. By (2.30), we have the inequality

$$|I(z_1, T)| \leq a, \tag{2.32}$$

where $a = a(\rho, \gamma)$ does not depend on z_1 and T, and where $a(\rho, \gamma) \leq a(\rho, \gamma_0)$ when $\gamma \geq \gamma_0$.

To get an estimate for I_1, we notice (see (2.2)) that $|z_1 - z| \leq \alpha_0 r$ $(r = |z|,$ $|t| < \alpha_1(r + 1)$ when $t \in L_{+\infty_z}$, and $\Re z \leq \Re t \leq \Re z_1)$. Therefore $|\exp[-P(t)]| \leq \exp[p(\alpha_1(r + 1))]$, and since $\phi(t)$ is small enough in the angle B, $|\phi(t + T)| \leq M(\alpha_1(r + 1), \phi)$ for $r > r_0$. Consequently, for $r > r_0$ we have the estimates

$$|I_1| \leq \alpha_0 r \exp[p(\alpha_1(r+1))] M(\alpha_1(r+1), \phi)$$

and

$$|I(z, T)| \leq \alpha_0 r \exp[p(\alpha_1(r+1))] M(\alpha_1(r+1), \phi) + a. \tag{2.33}$$

8. We return to the general term $S_j(z)$ (see (2.23)) of the series (1.5), and we prove the following proposition.

LEMMA 2.2. *Suppose* $\phi(z)$ *satisfies in the angle* B *the condition* (2.1) *with* $\rho \geq n+1$ *and with* γ *large enough for* $\rho = n+1$. *If*

$$z + j/2 \in B \text{ and either } x = \Re z \geq 0 \text{ or } j \geq 2|x| + 1/2 \text{ when } x < 0, \tag{2.34}$$

then

$$|S_j(z)| \leq C_3^j C \exp[-\gamma(x + j/2)^\rho], \tag{2.35}$$

where C *is the constant of* (2.1), $C_3 = C_3(\rho, \gamma)$ *does not depend on* z *and* j, *and* $C_3(\rho, \gamma) \leq C_3(\rho, \gamma_0)$ *when* $\gamma \geq \gamma_0$.

For any z *and* j *we have the inequality*

$$|S_j(z)| \leq h^j(r) M(\alpha_1(r+1), \phi) + \sum_{k=1}^{j} a^k (h(r))^{j-k}, \tag{2.36}$$

where $h(r) = \alpha_0 r \exp[p(\alpha_1(r+1))]$ *and* a *is defined by* (2.32).

Proof. We prove the first assertion of the lemma by induction. Since $S_1(z) = I(z, 1)$ (see (2.24) and (2.25)), the statement is true for $j = 1$ since it follows from (2.30). For this case $j = 1$, we may take $C_3 = C_2$.

Since the point z and the number j satisfy the conditions (2.34), these conditions are satisfied also for the point $t + 1$ and the number $j - 1$, when $t \in L_{+\infty_z}$. We assume (using the method of induction) that

$$|S_{j-1}(t+1)| \leq C_3^{j-1} C \exp[-\gamma(\tau + (j+1)/2)^\rho] \quad (\tau = \Re t) \tag{2.37}$$

whenever $t \in L_{+\infty_z}$.

Using (2.23), we write

$$S_j(z) = K^{-1} S_{j-1}(z) = \int_{L_{+\infty_z}} \exp[-P(t)] \phi_1(t+T)\, dt, \tag{2.38}$$

where $T = (j+1)/2$ and ϕ_1 is defined by the equation

$$\phi_1(t + (j+1)/2) = S_{j-1}(t+1).$$

By (2.34) and (2.37), we can apply the result (2.30) to the integral in (2.38), and we easily get (2.35) with $C_3 = C_2$.

The second statement of the lemma (the inequality (2.36)) may be proved similarly by means of (2.33).

By this lemma we complete the study of the series (1.5) in the same way as for the series (1.4).

9. From Theorem 2.1 it follows that $W\phi(z)$ (see (1.6)) is an entire solution of the equation (1) if $\phi(z)$ belongs to both the classes $H_\rho(A)$ and $H_\rho(B)$. But it may happen that $W\phi(z) \equiv 0$ although $\phi(z) \not\equiv 0$. For example, $W\phi(z) \equiv 0$ if $\phi(z) = \psi(z) - K\psi(z)$ and $\psi(z) \in H_\rho(A) \cap H_\rho(B)$. The existence of entire nontrivial solutions of the equation (1) shows the following.

THEOREM 2.2. *The equation* (1) *has an entire nontrivial solution.*

Proof. For each $\gamma > 0$, there exists an entire function $\phi(z)$ with the two properties (i) $|\phi(0)| \geq 1$ and (ii) $\phi(z)$ satisfies in both the angles A and B the inequality (2.1) with $C \leq 1$ and $\rho \geq n + 1$. For example, the function $\exp(-az^{2j})$ (where j is a natural number and $j \geq (n + 1)/2$) and the function $G(z, \tau)$ $(2\tau \geq n + 1)$ defined in (2.3) and (2.4) have for large enough values of a the required properties.

The function $f(z) = W\phi(z)$ (see (1.6)) is by Theorem 2.1 an entire solution of the equation (1), and $|f(0)| \geq |\phi(0)| - \alpha \geq 1 - \alpha$, where

$$\alpha = \sum_{m=1}^{\infty} |K^m \phi(0)| + \sum_{m=1}^{\infty} |K^{-m} \phi(0)| .$$

From the estimates (2.22) and (2.35) it follows that $\alpha < 1$ if γ is chosen large enough. Consequently, $f(0) \neq 0$ and $f(z) \not\equiv 0$.

Remark. The function $\phi(z)$ may be chosen so that in addition it is of normal type with respect to the order $\rho = n + 1$. Then, by Corollary 2.1, the order of the solution $f(z) = W\phi(z)$ is not greater than $n + 1$, and its type not more than normal. It will follow from the next theorem that the order of this solution is exactly $n + 1$ and its type is normal.

10. Suppose $f(z)$ $(f(z) \not\equiv 0)$ is an entire solution of the equation (1). Then it is easy to prove that the order of $f(z)$ is not less than n. Indeed, if the order of $f(z)$ were $\tau < n$, then the left side $f(z + 1)$ of the equation (1) would have the order τ, and the right side $\exp[P(z)]f'(z)$ the order n.

This result may be improved. For this purpose, we denote by $E[\mu, 0]$ the class of all entire functions $f(z)$ of order ρ not more than μ, and of minimal type if $\rho = \mu$.

THEOREM 2.3. *There does not exist an entire function* $f(z)$ $(f(z) \not\equiv 0)$ *belonging to the class* $E[n + 1, 0]$ *and satisfying the equation* (1).

Proof. Let $f(z)$ be an entire solution of the equation (1). Then

$$|f'(z_0 - 1)| = |\exp[-P(z_0 - 1)]| \, |f(z_0)|$$

for each point z_0. Denote by C the circle $|z - (z_0 - 1)| \leq 1/2$, and let $M = \max |f(z)|$ $(z \in C)$. By the Cauchy inequalities, $|f'(z_0 - 1)| \leq 2M$. Hence there exists a point $z_1 \in C$ such that $|f(z_1)| \geq (1/2)|\exp[-P(z_0 - 1)]| \, |f(z_0)|$. We agree to say that such a point z_1 is *associated* with z_0, and we note that $|z_0 - z_1| \leq 3/2$.

Consider the sequence of points $z_0, z_1, z_2, \cdots$, where z_{i+1} is associated with z_i. Clearly,

$$|z_i - z_{i+1}| \leq 3/2, \qquad |f(z_{i+1})| \geq (1/2)\,|\exp[-P(z_i - 1)]| \, |f(z_i)| \tag{2.39}$$

for $i = 0, 1, 2, \cdots$.

There exists an angle H with vertex at the point $z = 0$ such that (see (2))

(2.40) $$(1/2)\,|\exp[-P(z-1)]| \geq \exp[c\,|z|^n] \qquad (c = |a_0|/2),$$

whenever $z \in H$ and $|z| \geq R_0$ is large enough. Suppose the point z_0 belongs to the bisector L of the angle H, and $|z_0| \geq 2R_0$. From (2.39) it follows that $|z_i - z_0| \leq 3i/2$, and therefore there exists a positive number c_1, depending only on the angle H, such that $z_i - 1 \in H$ for $0 \leq i \leq j$ and $c_1\,|z_0| \leq j < c_1\,|z_0| + 1$. Moreover, the number $c_1 > 0$ may be chosen so small that $|z_i| \geq |z_0|/2$ for $i = 0, 1, 2, \cdots, j$.

We use again (2.39) and (2.40), and we get the inequalities

$$|f(z_{i+1})| \geq |f(z_i)|\,|\exp[c\,|z_i|^n] \qquad (i = 0, 1, 2, \cdots, j-1).$$

Let us multiply these inequalities, use $j \geq c_1\,|z_0|$, and write $\zeta_0 = z_j$. Then we get the inequality

(2.41) $$|f(\zeta_0)| \geq |f(z_0)|\exp[c_2\,|z_0|^{n+1}] \qquad (|\zeta_0| \leq c_3\,|z_0|),$$

where $c_2 = 2^n c_1 c$, $c_3 = 1 + 3c_1$.

We agree to say that the point $\zeta_0 = z_j$ is related to the point z_0, and by way of a contradiction we assume the solution f(z) to belong to the class $E[n+1, 0]$. Then (see [3, p. 21]) $\limsup[\log|f(z)|/|z|^{n+1}] \geq 0$ when $z \to \infty$ along the bisector L. Therefore there exists a sequence of points $z^{(i)}$ ($z^{(i)} \in L$, $i = 1, 2, 3, \cdots$) such that $z^{(i)} \to \infty$ and

(2.42) $$\lim_{i \to \infty}\,[\log|f(z^{(i)})|/|z^{(i)}|^{n+1}] \geq 0.$$

For the points $\zeta^{(i)}$ that are related to $z^{(i)}$, it follows from (2.41) that

$$|f(\zeta^{(i)})| \geq |f(z^{(i)})|\exp[c_2\,|z^{(i)}|^{n+1}] \qquad (|\zeta^{(i)}| \leq c_3\,|z^{(i)}|).$$

From this and (2.42) it follows that $\liminf[\log|f(\zeta^{(i)})|/|\zeta^{(i)}|^{n+1}] > 0$ when $i \to \infty$. But this inequality contradicts our assumption that f(z) belongs to the class $E[n+1, 0]$.

11. Let us consider a properly meromorphic function f(z), its poles, and the set of the orders of these poles. The minimum k of this set we call the *index* of the function f(z). Since f(z) has at least one pole, its index k is positive. Obviously, the function $f(z+1)$ has the same index k, but the index of the function $f'(z)\exp[P(z)]$ is $k+1$. Therefore the identity $f(z+1) \equiv f'(z)\exp[P(z)]$ does not occur. In other words, we have proved the following result.

THEOREM 2.4. *The equation* (1) *does not have properly meromorphic solutions.*

3. CONSTRUCTION OF LINEARLY INDEPENDENT SOLUTIONS

1. LEMMA 3.1. *Let* $a_1(z), a_2(z), \cdots, a_m(z)$, *and* $\phi(z)$ *be entire functions, and let* $D\phi(z) = \phi'(z)$. *Then*

$$\prod_{i=1}^{m} (\exp(-D)\, a_i(z)\, D)\phi(z) = \prod_{i=1}^{m} a_i(z - i)\phi^{(m)}(z - m) + \sum_{i=1}^{m-1} b_i(z)\phi^{(i)}(z - m),$$

where the $b_i(z)$ *are entire functions.*

This lemma may be proved easily by induction. The following is an immediate corollary.

LEMMA 3.2. *Let* $\phi(z)$ *be an entire function, and let* K *be the operator* (1.2). *The function* $K^m\phi(z)$ $(m = 1, 2, 3, \cdots)$ *equals zero at the point* $z = \lambda$ *if* $\lambda - m$ *is a zero of multiplicity at least* $m + 1$ *of the function* $\phi(z)$. *If* $\lambda - m$ *is a zero of multiplicity* m *of the function* $\phi(z)$, *then* $K^m\phi(\lambda) = \prod_{i=1}^{m} \exp[-P(\lambda - i)]\phi^{(m)}(\lambda - m)$.

2. Let us fix a positive number σ $(0 < \sigma \leq \pi/4)$ and a point λ $(|\arg\lambda| \geq \sigma)$ of the z-plane. Denote by $B(\lambda)$ the set of all points $\lambda - j$ $(j = 0, 1, 2, \cdots)$, where each point $\lambda - j$ is counted $j + 1$ times, and denote by $n(B(\lambda), r)$ the number of points of the set $B(\lambda)$ belonging to the circle $T = \{z: |z| \leq r\}$. Then there exists a constant C, independent of λ and r, such that

$$n(B(\lambda), r) \leq Cr^2 \qquad (r \geq 1)\,. \tag{3.1}$$

Of course, it is impossible for a point $\lambda - j$ of the set $B(\lambda)$ to belong to the circle T, unless $|\Im\lambda| \leq r$ and $\Re\lambda - j \geq -r$. Since $|\arg\lambda| \geq \sigma$, the inequalities

$$j \leq r + \Re\lambda \leq C_1 r \quad (C_1 = 1 + \cot\sigma) \qquad \text{and} \qquad n(B(\lambda), r) \leq j(j+1) \leq Cr^2 \quad (C = 2C_1^2)$$

will hold.

Now take a sequence $S = \{z_m\}$ $(z_m \to \infty$ and $|\arg z_m| \geq \sigma)$, and write $B(S) = \bigcup_{m=1}^{\infty} B(z_m)$. If for some j the point $z_m - j$ is in T, then $|\Im z_m| \leq r$ and $|z_m| \leq r/\sin\sigma$. From this and (3.1) it follows that $n(B(S), r) \leq C r^2 n(S, r/\sin\sigma)$, where $n(Q, r)$ denotes the number of points of an arbitrary set Q in the circle $|z| \leq r$. From this we easily obtain the following result.

LEMMA 3.3. *Let* S *and* B(S) *be the sequences defined above. If the exponent of convergence of the set* S *is* μ, *then the exponent of convergence of the set* B(S) *is not greater than* $\mu + 2$.

3. THEOREM 3.1. *For each* $\mu \geq n + 1$ *if* $n > 1$ *and for each* $\mu > 2$ *if* $n = 1$, *the equation* (1) *has a linearly independent infinite set* F_μ (*with cardinality of the continuum*) *of entire solutions of order* μ.

Proof. For each μ described in the theorem, we fix a number ρ such that $n + 1 \leq \rho \leq \mu$ (if $n > 1$) or $2 < \rho \leq \mu$ (if $n = 1$). Denote by $E_\mu = E_{\mu\rho}$ the set of all entire functions $\phi(z)$ with the following properties:

(i) $\phi(z)$ is of order μ,

(ii) $\phi(z) \in H_\rho(A) \cap H_\rho(B)$ if $\rho > n + 1$, and

(iii) $\phi(z) \in H_\rho(A, \gamma) \cap H_\rho(B, \gamma)$, with $\gamma > \gamma_0$ large enough, if $\rho = n + 1$.

There exists a sequence S_1 of points ζ_m $(m = 1, 2, 3, \cdots;\ \zeta_m \to \infty)$ such that

$$\text{(3.2)}\qquad \begin{aligned} &\lim\,[\log(\log|\phi(\zeta_m)|)/\log|\zeta_m|] = \mu \ \ (m \to \infty) && \text{if } \mu > n+1, \\ &\lim\,[\log|\phi(\zeta_m)|/\zeta_m^{\mu}] > \gamma_0 \ \ (m \to \infty) && \text{if } \mu = n+1. \end{aligned}$$

For $m > m_0$, the points ζ_m belong to neither of the angles A and B, since $\phi(z)$ is small in these angles. Taking a subsequence if necessary, we may assume that $\zeta_m \notin A \cup B$ for $m = 1, 2, 3, \cdots$ and

$$\text{(3.3)}\qquad |\Im\,\zeta_1| \geq 1, \qquad |\Im\,\zeta_{m+1}| \geq 2\,|\Im\,\zeta_m|.$$

Denote by j_m a natural number such that

$$\text{(3.4)}\qquad z_m = \zeta_m + j_m \in B \text{ but } z_m - 1 \notin B.$$

Since $0 < \delta \leq |\arg \zeta_m| \leq \pi - \delta$, there exists a constant c, independent of m, such that

$$\text{(3.5)}\qquad j_m \leq c\,|z_m| \quad \text{and} \quad |z_m| \leq c\,|\zeta_m|.$$

As in Lemma 3.3, we denote by S the set $\{z_m\}$ and by B(S) the set $\{z_m - j\}$ $(m = 1, 2, 3, \cdots;\ j = 0, 1, 2, \cdots)$, where $z_m - j$ is counted $j + 1$ times. Notice that by (3.4) the point ζ_m is contained $j_m + 1$ times in the set B(S). For every m, we exclude from B(S) a single point ζ_m and denote the remaining set by $B^1(S)$. Thus $B^1(S)$ contains $j + 1$ times each point $z_m - j \neq \zeta_m$, but only j_m times the point $z_m - j_m = \zeta_m$.

From (3.3) and (3.4) it follows that the exponent of convergence of the set S is zero. By Lemma 3.3, the exponent of convergence of the set $B^1(S)$ is therefore not greater than 2. Not greater than 2 is also the order of the canonical product M(z) with zeros in all the points of the set $B^1(S)$ (and with no other zeros). Since $\mu \geq \rho > 2$ and $\phi(z) \in E_{\mu\rho}$, it is easy to see that $\psi(z) = M(z)\phi(z)$ also belongs to the set $E_{\mu\rho}$.

Now consider the function $f(z) = W\psi(z) = U\psi(z) + V\psi(z)$ (see (1.6)). By Theorem 2.1, this function is a solution of the equation (1), and its order (see Corollary 2.1) is not greater than μ. We shall show that its order is exactly μ. For this purpose, we agree to say that an entire or meromorphic function g(z) has the order ν on the sequence $\{\eta_m\}$ $(\eta_m \to \infty)$ if $\lim\,[\log(\log|g(\eta_m)|)/\log|\eta_m|] = \nu$ $(m \to \infty)$, and we shall prove that the solution f(z) has order not less than μ on the sequence $S = \{z_m\}$.

By Theorem 2.1, the function $V\psi(z) \in H_\rho(B)$, and therefore (see (3.4)) $V\psi(z_m) \to 0$ as $m \to \infty$. Therefore the function f(z) has on the sequence S the same order as $U\psi(z)$. To find this order, we notice that $\psi(z) = \phi(z)M(z)$ has zeros at all points of the sequence $B^1(S)$, and we apply Lemma 3.2. From (1.4) and (3.4) we see that

$$\text{(3.6)}\qquad U\psi(z_m) = \phi(\zeta_m)\,a_m \cdot b_m,$$

where $a_m = \prod_{i=1}^{j_m} \exp[-P(z_m - i)]$ and $b_m = M^{(j_m)}(\zeta_m)$.

From (2) and (3.5) it easily follows that there exist two positive constants C and c_1 such that

$$\text{(3.7)}\qquad |a_m| \geq C \exp[-c_1 |z_m|^{n+1}].$$

It remains to estimate the magnitude of

$$b_m = M^{(j_m)}(\zeta_m) = j_m!\ \lim [M(z)/(z - \zeta_m)^{j_m}] \qquad (z \to \zeta_m). \tag{3.8}$$

Since the order of $M(z)$ is not greater than 2 and the least distance between the zeros of this function is 1, there exists for each $\varepsilon > 0$ a constant $C = C(\varepsilon) > 0$ such that

$$|M(z)| \geq C \exp(-|\zeta_m|^{2+\varepsilon}) \quad \text{whenever } z \in T_m \qquad (T_m = \{z\colon |z - \zeta_m| = 1/2\}).$$

Therefore $|(z - \zeta_m)^{j_m}/M(z)| \leq \exp(|\zeta_m|^{2+\varepsilon})/(2^{j_m} \cdot C)$ on the circle T_m, and therefore, by the maximum principle (see (3.8)), $|b_m| \geq C\, j_m!\, 2^{j_m} \exp(-|\zeta_m|^{2+\varepsilon})$. From (3.2), (3.5), (3.6), (3.7), and the last inequality, it follows that on the sequence S the order of $U\psi(z)$ is not less than μ. Thus we have proved that for each function $\phi(z) \in E_\mu$ satisfying (3.2), the order of the corresponding solution

$$f(z) = W\psi(z) = W[\phi(z)M(z)]$$

equals μ.

To complete the proof of Theorem 3.1, we choose from the set E_μ, for every $\gamma > 0$ if $\mu > n+1$ and for every $\gamma > \gamma_0$ if $\mu = n+1$, a function $\phi(z) = \phi_\gamma(z)$ satisfying the two requirements

(i) the type of $\phi_\gamma(z)$ equals γ and

(ii) there exists a set of points $\{\zeta_m\}$ such that (3.2) holds for each $\phi(z) = \phi_\gamma(z)$.

The set generated by these functions $\phi_\gamma(z)$ we denote by E_μ^1. If μ is an even natural number, we let the set E_μ^1 consist of the functions $\exp[-az^\mu]$, where $a > 0$ for $\mu > n+1$ and $a > a_0$ is large enough for $\mu = n+1$. For other values of μ, we let E_μ^1 consist of the functions $G(z, \tau)$ $(\mu = 2\tau)$ given in (2.3) and (2.4).

As it has already been proved for each $\phi_\gamma(z) \in E_\mu^1$, the function

$$f_\gamma(z) = W\psi_\gamma(z) = W[\phi_\gamma(z)M(z)]$$

is a solution of order μ of the equation (1). It remains to prove that the set F_μ consisting of these solutions $f_\gamma(z)$ is linearly independent.

Take a linear combination

$$f(z) = \sum_{i=1}^{m} c_i f_{\gamma_i}(z),$$

where the c_i are arbitrary nonzero complex numbers and the γ_i are arbitrary distinct positive numbers. The function $f(z)$ may be written in the form $f(z) = W[\phi(z)M(z)]$, where

$$\phi(z) = \sum_{i=1}^{m} c_i \phi_{\gamma_i}(z).$$

Since $\gamma_i \neq \gamma_j$ when $i \neq j$, the function $\phi(z)$ belongs to E_μ and satisfies (3.2). Consequently, $f(z)$ is a solution of order μ. Thus $f(z) \neq 0$, and the set F_μ is linearly independent.

4. THE INHOMOGENEOUS EQUATION

1. In this section we consider the inhomogeneous equation

$$Lf(z) \equiv f(z+1) - \exp[P(z)]f'(z) = h(z), \tag{3}$$

where $P(z)$ is the polynomial (2) and $h(z)$ is an entire or meromorphic function.

LEMMA 4.1. *Let* $h(z)$ *be an entire function. For each* $\rho \geq 2$ *there exist two entire functions* $h_1(z)$ *and* $h_2(z)$ *such that*

$$h_1(z) \in H_\rho(A), \qquad h_2(z) \in H_\rho(B), \qquad h(z) \equiv h_1(z) + h_2(z).$$

If $h(z)$ *is of finite order* μ, *and* ρ *is an arbitrary number satisfying the inequalities* $\rho \geq 2$ *and* $\rho > \mu$, *the functions* $h_1(z)$ *and* $h_2(z)$ *may be chosen so that not only* $h_1(z) \in H_\rho(A)$ *and* $h_2(z) \in H_\rho(B)$, *but also the order of both* $h_1(z)$ *and* $h_2(z)$ *does not exceed* ρ.

This lemma was used and partly proved in [6]. A full proof may be found in [7].

2. THEOREM 4.1. *For each entire function* $h(z)$, *the equation* (3) *has an entire solution* $f(z)$. *If* $h(z)$ *is of order* μ, *then for each* ρ $(\rho \geq n+1,\ \rho > \mu)$ *there exists an entire solution* $f(z)$ *of the equation* (3) *of order not exceeding* ρ.

To prove this theorem, we use Lemma 4.1, write $h(z) = h_1(z) + h_2(z)$, and notice that by Theorem 2.1 the function $f(z) = Uh_1(z-1) - Vh_2(z-1)$ is an entire solution of the equation (3). Using Corollary 2.1, we get the required estimate for the order of the solution $f(z)$.

Remark. Let $f(z)$ be an entire solution of the equation (3). Then $f(z+1) \equiv \exp[P(z)]f'(z) + h(z)$. If the order μ of $h(z)$ is less than n, then it follows from this identity that the order of $f(z)$ is not less than n. If $\mu > n$, the order of $f(z)$ is not less than μ.

3. Consider now the equation (3) with a meromorphic free term $h(z)$. Denote by M_δ $(\delta > 0)$ the class of all meromorphic functions $h(z)$ that have in the angle A (see (1.8)) only a finite number of poles (note that $h(z)$ may have infinitely many poles outside the angle A).

THEOREM 4.2. *For each* $h(z) \in M_\delta$, *there exists a meromorphic solution* $f(z) \in M_\delta$ *of the equation* (3). *If the function* $h(z)$ $(h(z) \in M_\delta)$ *is of order* μ, *and if* ν *is the exponent of convergence of the sequence of its poles, then for each* ρ $(\rho > \chi = \max(\mu, \nu+2, n+1))$ *there exists a solution* $f(z) \in M_\delta$ *of the equation* (3) *of order not exceeding* ρ.

Proof. Since $h(z) \in M_\delta$, there is on the real axis a point x_0 such that $h(z)$ is regular in the angle $A_1 = \{z\colon \pi\ |\arg(z - x_0)| \leq \alpha,\ \alpha \leq \delta\}$. Therefore, for each $\sigma \geq n+1$ and each $\gamma > 0$, there exist by a theorem of Mergelyan (see [4, p. 86]) an entire function $g(z)$ and a constant C such that

$$|h(z) - g(z)| \leq C \exp[-\gamma |x|^\sigma] \qquad (z = x+iy \in A_1,\ \sigma \geq n+1), \tag{4.1}$$

if the width of the angle A_1 is small enough $(\alpha < \pi/2\sigma)$.

Write $\psi(z) = h(z) - g(z)$. Obviously $\psi(z) \in M_\delta$, and therefore, for each $r > 0$, there exists a number j_0 such that $\psi(z-j)$ is regular in the circle $|z| \leq r$ for $j > j_0$. By virtue of this property and (4.1), the series (see (1.4)) $G(z) = U\psi(z-1)$

converges absolutely and uniformly in each circle $|z| \leq r$. The function $G(z)$ is therefore a meromorphic solution of the equation

$$\text{(4.2)} \qquad Lf(z) = \psi(z) \equiv h(z) - g(z).$$

This assertion may be proved in the same way as in Theorem 2.1, even though $\psi(z)$ is not entire.

Now we use in (3) the function $f(z) = F(z) + G(z)$. We get the equation

$$\text{(4.3)} \qquad LF(z) = h_1(z),$$

where $h_1(z) = h(z) - LG(z)$. Since $G(z)$ is a solution of (4.2), the function $h_1(z) \equiv h(z) - (h(z) - g(z)) \equiv g(z)$ is entire. By Theorem 4.1, Equation 4.3 has an entire solution $F(z)$. The function $f(z) = F(z) + G(z)$ is then a meromorphic solution of the equation (3). Obviously $G(z) \in M_\delta$ and $f(z) \in M_\delta$.

To get the estimate for the order of the solution $f(z)$ required in the theorem, we take in (4.1) $\sigma = n + 1$ and notice that by the theorem of Mergelyan mentioned above the function $g(z)$ in (4.1) may be chosen so that the order of $g(z)$ is not greater than $\chi_1 = \max(n + 1, \mu)$ (μ is the order of $h(z)$). Such will also be the order of $\psi(z) = h(z) - g(z)$.

Denote the poles of the function $h(z)$ by λ_i ($i = 1, 2, 3, \cdots$; $|\lambda_i| \leq |\lambda_{i+1}|$), and surround each λ_i ($|\lambda_i| \neq 0$) by the two circles

$$e_i = \{z: |z - \lambda_i| \leq c\,|\lambda_i|^{-h}\} \quad \text{and} \quad e_i^1 = \{z: |z - \lambda_i| \leq 3c\,|\lambda_i|^{-h}\},$$

where h is chosen so that $h > \mu$, and $c > 0$ so that

$$\text{(4.4)} \qquad 6c\left(1 + \sum_{i=1}^{\infty} |\lambda_i|^{-h}\right) < 1 \qquad (\lambda_i \neq 0).$$

If $\lambda_1 = 0$, we take $e_1 = \{z: |z| < c\}$ and $e_1^1 = \{z: |z| \leq 6c\}$. Outside the set $E = \bigcup_{i=1}^{\infty} e_i$ the function $\psi(z)$ satisfies for each $\varepsilon > 0$ the condition

$$\text{(4.5)} \qquad |\psi(z)| = O(|z|^{\mu+\varepsilon}).$$

From (4.4) it follows (see [5]) that there exists a point z_0 such that the net N consisting of the lines $\Re z = \Re z_0 + j$ and $\Im z = \Im z_0 + j$ ($j = 0, \pm 1, \pm 2, \cdots$) intersects none of the circles e_i^1. Consequently, (4.5) holds on the net N and its neighborhood N_1 described as follows: $z \in N_1$ if there exists a point $\zeta \in N$ such that $|z - \zeta| \leq c\,|\zeta|^{-h}$ when $|\zeta| \geq 1$ and $|z - \zeta| \leq c$ if $|\zeta| < 1$ (here c is the same as in (4.4)). We also notice that $z \in N$ implies $z - j \in N$ for $j = 0, 1, 2, \cdots$, and it is then easy to see that on the net N we may treat the series $G(z) = U\psi(z)$ in exactly the same manner as the series $U\phi(z)$ in the proof of Theorem 2.1. We conclude that for each $\varepsilon > 0$ we have the estimate

$$\text{(4.6)} \qquad |G(z)| = O(|z|^{\chi_1+\varepsilon}) \qquad (z \in N,\ \chi_1 = \max(\mu, n + 1)).$$

If λ is a pole of $\psi(z)$ of multiplicity k, then $\lambda + 1$ is a pole of $K\psi(z)$ (see (1.2)) of multiplicity $k + 1$, and $\lambda + 2$ is a pole of $K^2\psi(z)$ of multiplicity $k + 2$, and so forth. Consequently, all the poles of the function $G(z)$ belong to the set

$$T = \{\lambda_i + j: i = 1, 2, 3, \cdots; j = 0, 1, 2, \cdots\},$$

in which the point $\lambda_i + j$ is contained $j + 1$ times. We recall that the exponent of convergence of the set $\{\lambda_i\}$ equals ν. Therefore the exponent of convergence of the sequence T is by Lemma 3.3 not greater than $\nu + 2$. Not greater than $\nu + 2$ is also the order of the canonical product M(z) with zeros at all the points of the set T. From (4.6) and the maximum principle, it follows now that the order of the function M(z) G(z) is not greater than $\chi = \max(\chi_1, \nu + 2) = \max(n + 1, \mu, \nu + 2)$. The same is true of the order of both the functions G(z) and $h_1(z) = h(z) - LG(z)$. By Theorem 4.1, for each $\rho > \chi$ there exists an entire solution F(z) of the equation (4.3) of order not exceeding ρ. The order of the solution f(z) = F(z) + G(z) of the equation (4) likewise does not exceed ρ.

4. We complete the study of the meromorphic solutions of the equation (3) by showing that the condition $h(z) \in M_\delta$ is to some extent essential for the existence of such solutions.

First we prove the following statement: *If the free term* h(z) *of the equation* (3) *is regular at the points* $\lambda - j$ $(j = 1, 2, 3, \cdots)$, *then each meromorphic solution* f(z) *of this equation* (3) *is regular at the point* λ.

Suppose λ is a pole of order k of the solution f(z), and consider the identity

$$f(z + 1) \equiv \exp[P(z)] f'(z) + h(z) \tag{4.7}$$

at the point $\lambda - 1$. The term f(z + 1) has at this point a pole of order k, the term h(z) is regular at the point, and $\exp[P(z)] \neq 0$. Therefore the function f'(z) has at $\lambda - 1$ a pole of order k, and consequently f(z) has at the same point a pole of order k - 1. By the same reasoning, we deduce that f'(z) has at the point $\lambda - 2$ a pole of order k - 1, that f(z) has a pole of order k - 2, and so forth. Thus f'(z) has at the point $\lambda - k$ a simple pole, which is impossible.

Now suppose that h(z) is regular at the points $\lambda \pm j$ $(j = 1, 2, 3, \cdots)$, but has a pole of order k at the point λ. From the statement above and the identity (4.7) it follows that the solution f(z) has a pole of order k at the point $\lambda + 1$. We consider again the identity (4.7) at the points $\lambda + 1, \lambda + 2, \cdots$, and we conclude that f(z) has a pole of order k + 1 at $\lambda + 2$, of order k + 2 at $\lambda + 3$, and so forth.

Let us agree to say that two points λ and μ are *congruent* if $\lambda - \mu$ is an integer, and take a free term h(z) having no pairs of congruent poles. Also, suppose the function h(z) has infinitely many poles λ_n in the half-strip

$$S = \{z: -a \leq \Im z \leq a, \; a > 0, \; \Re z < 0\}.$$

If for such a free term h(z) a meromorphic function f(z) were a solution of the equation (4), then f(z) would have poles at all the points $\lambda_i + j$ $(i, j = 1, 2, 3, \cdots)$. But this is impossible. Of course, for each $i = 1, 2, 3, \cdots$, there is an integer j_i such that $\lambda_i + j_i$ belongs to the rectangle $R = \{z: -a \leq \Im z \leq a, \; 0 \leq \Re z < 1\}$. Moreover, $\lambda_i + j_i \neq \lambda_k + j_k$ if $i \neq k$. Thus the function f(z) would have infinitely many poles in R.

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