WEAK COMPLETENESS AND INVARIANT SUBSPACES

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Throughout, X will denote a complex locally convex topological vector space. An *operator* on X is a continuous linear transformation on X. An operator on X is *intransitive* if it has a non-trivial closed invariant subspace. The space X is *operator-intransitive* if every operator on X is intransitive. A *hyperinvariant subspace* of an operator T is a subspace that is left invariant by every operator commuting with T.

A. L. Shields [5, Theorem 2] showed that the space (s) of all complex sequences (topologized by the coordinate seminorms) is operator-intransitive. Later, B. E. Johnson and A. L. Shields [1, Theorem 1] proved that every operator on (s) that is not a scalar has a non-trivial closed hyperinvariant subspace.

Since (s) is a separable, locally convex Fréchet space, it seems natural to ask whether the techniques applied to (s) might also be applied to some infinite-dimensional Banach space.

This paper isolates the property (property #) that makes Shields's proof [5, Theorem 2] work, and it shows (Theorem 1) that this property is equivalent to weak completeness. (Note: weak completeness means that every weakly fundamental net is convergent.)

The remainder of this paper shows (Theorem 2), for a weakly complete space X with dimension greater than 1, that every operator on X that is not a scalar has a non-trivial hyperinvariant subspace if and only if the (continuous) dual of X has linear dimension less than $2^{\aleph_0}$.

The notation and terminology of [3] will be used. The set of all linear functionals on X will be denoted by X^*, and the set of those functionals in X^* that are continuous will be denoted by X'. Also, dim X will denote the linear dimension of X. If M is a subspace of X', then $M^\perp$ denotes the set of all vectors in X that annihilate M. The space X has *property* # provided that, for every subspace M of X', $M^\perp = 0$ only if $M = X'$.

The proof of the following proposition is almost a word-for-word copy of Shields's proof that (s) is operator-intransitive.

PROPOSITION. *If* X *has property # and* dim X > 1, *then* X *is operator-intransitive.*

Proof. Let T be an operator on X. Then T' (the adjoint of T) is a linear transformation on X'. By a theorem of H. H. Schaefer [4], there is a subspace M of X' such that $0 \neq M \neq X'$ and $T'(M) \subseteq M$. Therefore $M^\perp$ is a closed subspace of X and $T(M^\perp) \subseteq M^\perp$. Since $M \neq 0$, we see that $M^\perp \neq X$. Since X has property # and $M \neq X'$, it follows that $M^\perp \neq 0$. Thus T is intransitive.

Let $\mathbb{C}$ denote the field of complex numbers. If B is a nonempty set, let

$$\mathbb{C}^B = \{\phi : \phi \text{ is a function from } B \text{ to } \mathbb{C}\}$$

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with the product topology. If B is countably infinite, then $\mathbb{C}^B$ is isomorphic to (s). The next lemma lists most of the important properties of this space. The proof is contained in [2, Section 20, and the material on ω_d].

LEMMA 1. *Let* B *be a Hamel basis for* X'. *Then the following statements are equivalent.*

(i) X *is isomorphic to* $\mathbb{C}^B$,

(ii) X *is weakly complete,*

(iii) $(X')^* = X$,

(iv) *if* L *is any linear transformation on* X', *then there is an operator* T *on* X *such that* T' = L.

It follows from part (iii) of the preceeding lemma that every weakly complete space has property #. The following theorem shows that the converse is also true.

THEOREM 1. *If* X *has property #, then* X *is weakly complete.*

Proof. By Lemma 1, we need only show that $(X')^* = X$. Suppose $\psi \in (X')^*$. We may assume $\psi \neq 0$. The space X has property #; therefore there is an x in X such that $x \neq 0$ and $x \in (\ker \psi)^{\perp}$. If ψ_1 is the linear functional defined on X' by $\psi_1(f) = f(x)$ for every f in X', then $\ker \psi_1 = \ker \psi$. Hence there is a scalar α such that $\psi(f) = \alpha\psi_1(f) = f(\alpha x)$ for every f in X'.

We now turn our attention to hyperinvariant subspaces.

LEMMA 2. *If* V *is a complex vector space with dimension greater than* 1, *then the following two statements are equivalent:*

(i) $\dim V < 2^{\aleph_0}$.

(ii) *every non-scalar linear transformation on* V *has a non-trivial hyperinvariant subspace.*

Proof. (i) $\Rightarrow$ (ii). This can be proved in the same way as Lemma 1 of [1].

(ii) $\Rightarrow$ (i). Suppose $\dim V \geq 2^{\aleph_0}$. Then there is a field F that is an extension of $\mathbb{C}$ and such that $\dim_{\mathbb{C}} F = \dim_{\mathbb{C}} V$. Thus F and V are isomorphic as vector spaces over $\mathbb{C}$. It follows from Lemma 2 of [1] that there is a linear transformation T on V such that p(T) is invertible whenever p is a non-zero polynomial. Let

$$K = \{r(T): r \text{ is a rational function}\}.$$

Then K is a field and V is a vector space over K with scalar multiplication defined by $r(T)x = r(T)(x)$ for each x in V and each rational function r. Also, every subspace of V over $\mathbb{C}$ that is left invariant by T is a subspace of V over K, and every linear transformation of V over $\mathbb{C}$ that commutes with T is a linear transformation of V over K. Therefore T has a non-trivial hyperinvariant subspace on V over $\mathbb{C}$ if and only if T has a non-trivial hyperinvariant subspace on V over K. However, T is a scalar on V over K and therefore cannot have a non-trivial hyperinvariant subspace. Hence T, considered as a linear transformation on V over $\mathbb{C}$, has no non-trivial hyperinvariant subspace.

THEOREM 2. *Let* X *be a weakly complete space with dimension greater than* 1. *Then the following two statements are equivalent:*

(i) $\dim X' < 2^{\aleph_0}$,

(ii) *every non-scalar operator on* X *has a non-trivial closed hyperinvariant subspace.*

Proof. It follows from Lemma 1 that (ii) holds if and only if every non-scalar linear transformation on X' has a hyperinvariant subspace. The latter condition holds if and only if $\dim X' < 2^{\aleph_0}$.

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