GENERIC SURFACES IN E^4

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1. INTRODUCTION

In this paper, we consider some structures existing on surfaces M that are immersed in Euclidean 4-space E^4 and whose normal curvature has a generic property. Let Q be the vector bundle, of fiber-dimension 2, consisting of bilinear forms orthogonal to the induced Riemannian metric. Then the second fundamental form projects into Q to give a bundle map $f: \perp \to Q$, where $\perp$ is the normal bundle. We show that the normal curvature κ can be considered to be a function on M and that it coincides with the determinant of f. We call an immersion *κ-regular* if $d\kappa$ is never zero on the singular set $S = \kappa^{-1}(0)$. For a κ-regular immersion we show that f has rank 1 everywhere on S. The results consist of various relations between the Euler numbers of $\perp$ and M, the integral absolute normal curvature, an integral weighted Gaussian curvature, the winding number of im f on S, the geodesic curvature of S, and the rotation of ker f on S relative to parallel translation. Some of the constructions are valid for any κ-regular immersion; but for the stronger results we assume M is compact and oriented.

2. THE VECTOR BUNDLES

We shall use the following notation for the various vector bundles arising from an immersion $\phi: M \to E^4$ of a 2-dimensional manifold in 4-space.

T is the tangent bundle.

T^* is the cotangent bundle.

$\perp$ is the normal bundle.

$B = T^* \circ T^*$ is the bundle of symmetric bilinear forms.

I is the subbundle of B spanned by the induced metric.

Q is the orthogonal complement of I in B. If we view a bilinear form as a symmetric linear mapping $T \to T$, using the identification $T^* \otimes T^* \approx T \otimes T^*$ given by the metric, then Q consists of those symmetric linear mappings whose two eigenvalues at each point are the same in magnitude and opposite in sign.

The second fundamental form of ϕ can be considered to be a bundle map $h: \perp \to B$. We compose h with orthogonal projection into Q to obtain the map $f: \perp \to Q$.

An orientation at any point of one of T, $\perp$, Q leads to an orientation of the other two. This depends only on the fact that the ambient space E^4 has a standard orientation. We shall fix a convention relating the orientations of T, $\perp$, Q, and our expression for the convention consists of isomorphisms between the bivector line

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bundles: $\bigwedge^2 T \approx \bigwedge^2 \perp \approx \bigwedge^2 Q$. Specifically, if e_1, e_2, e_3, e_4 is an adapted frame at $\phi(m)$ that gives the orientation of E^4, and ω^1, ω^2 is the dual coframe to e_1, e_2, we set

$$q_1 = (\omega^1)^2 - (\omega^2)^2 \quad \text{and} \quad q_2 = \omega^1 \omega^2 \quad \text{(symmetric products)}.$$

Then the isomorphisms map $e_1 \wedge e_2$, $e_3 \wedge e_4$, and $q_1 \wedge q_2$ to each other. Note that all these bivectors have unit-length in the induced metric, so that the isomorphisms are also isometries.

If we rotate the frame e_1, e_2 through an angle θ, then the (orthogonal but not normal) basis q_1, q_2 is rotated through an angle 2θ.

Because the map f extends to a homomorphism of Grassmann algebras, the isomorphisms make it possible to define the determinant. Specifically, for any conventionally related e_3, e_4, q_1, q_2, we define

$$f(e_3) \wedge f(e_4) = (\det f)\, q_1 \wedge q_2 .$$

3. CONNEXIONS

The bundles introduced at the beginning of Section 2 have induced connexions for which parallel translation preserves the metric. The curvature of a bundle V with a metric connexion is an so(V)-valued 2-form: $\bigwedge^2 T \to so(V)$. The metric gives a standard isomorphism ψ of so(V) with $\bigwedge^2 V$. Specifically,

$$\langle \psi(v_1 \wedge v_2) v_3, v_4 \rangle = \langle v_1 \wedge v_2, v_3 \wedge v_4 \rangle .$$

When V has fiber-dimension 2 and $\bigwedge^2 V \approx \bigwedge^2 T$, then the curvature is identified with a real-valued function on M: the map $\bigwedge^2 T \to so(V) \approx \bigwedge^2 V \approx \bigwedge^2 T$ is a bundle map of a line bundle, and therefore it is given by a 1-by-1 matrix at each point.

When $V = T$, we get the Gaussian curvature K.

When $V = Q$, the fact that bases in Q rotate twice as fast as bases in T leads to the fact that the curvature of Q is 2K.

When $V = \perp$, the curvature is the *normal-curvature function*, denoted κ.

THEOREM 1. $\kappa = \det f$.

Proof. We calculate both κ and $\det f$ in terms of local moving frames E_1, E_2, E_3, E_4, with our conventional orientations, the dual coframe ω^1, ω^2, and the corresponding moving basis Q_1, Q_2 of Q.

The connexion on $\perp$ is given by the connexion 1-form $\omega = \omega_4^3 = -\omega_3^4$, which determines covariant derivatives of the local frame of $\perp$ by the formulas $D_X E_3 = -\omega(X) E_4$ and $D_X E_4 = \omega(X) E_3$. Since the group O(2) of $\perp$ is commutative, the second structural equation says that the curvature 2-form has matrix $(-d\omega_s^r)$ $(r, s = 3, 4)$ relative to the frame E_3, E_4. The isomorphism ψ takes $E_3 \wedge E_4$ into the transformation with matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; therefore its inverse takes the curvature 2-form to $d\omega\, E_3 \wedge E_4$. In turn this is carried to $d\omega\, E_1 \wedge E_2$ via $\bigwedge^2 \perp \approx \bigwedge^2 T$. Thus, if we write $\Omega = \omega^1 \wedge \omega^2$, the normal curvature is given by $d\omega = \kappa\Omega$.

To involve det f and prove it is κ, we must consider the way the connexion on $\perp$ is induced by the immersion and how the second fundamental form is defined. Viewing ϕ and the E_i as E^4-valued functions on M, we can express their differentials in terms of the frame E_i, with coefficients that are 1-forms on M:

$$d\phi = \sum_{\alpha=1}^{2} \omega^\alpha E_\alpha, \qquad dE_i = \sum_{j=1}^{4} \omega_i^j E_j .$$

Because E_i is a frame, $\omega_i^j = -\omega_j^i$. We obtain the connexion on $\perp$ by projecting dE_3, dE_4 into $\perp$ to get their covariant differentials DE_3 and DE_4; therefore this ω_4^3 agrees with the one above. The second fundamental form corresponding to a normal field N is the inner product $\langle dN, d\phi \rangle$; hence the value of h on E_r is

$$h(E_r) = \langle dE_r, d\phi \rangle = \sum_{\alpha=1}^{2} \omega_r^\alpha \omega^\alpha = \sum_{\alpha,\beta=1}^{2} h_{\alpha\beta r} \omega^\alpha \omega^\beta \qquad (r = 3, 4) .$$

That the coefficients $h_{\alpha\beta r}$ of ω_r^α are symmetric in α and β follows from $d^2\phi = 0$.

We obtain the expression for $f(E_r)$ by subtracting $\frac{1}{2} \sum_\alpha h_{\alpha\alpha r} \langle d\phi, d\phi \rangle$ from $h(E_r)$ so as to make the coefficients of $\omega^\alpha \omega^\alpha$ opposite for $\alpha = 1$ and 2:

$$\begin{aligned} f(E_r) &= \frac{1}{2}(h_{11r} - h_{22r})(\omega^1 \omega^1 - \omega^2 \omega^2) + 2h_{12r} \omega^1 \omega^2 \\ &= \frac{1}{2}(h_{11r} - h_{22r}) Q_1 + 2h_{12r} Q_2 . \end{aligned}$$

Thus, $\det f = (h_{113} - h_{223}) h_{124} - (h_{114} - h_{224}) h_{123} = \sum_{\alpha=1}^{2} (h_{\alpha 13} h_{\alpha 24} - h_{\alpha 23} h_{\alpha 14})$.

On the other hand,

$$0 = d^2 E_3 = \sum_{j=1}^{4} \left(d\omega_3^j - \sum_{i=1}^{4} \omega_3^i \wedge \omega_i^j \right) E_j .$$

The coefficient of E_4 is

$$\begin{aligned} 0 &= d\omega_3^4 - \sum_{\alpha=1}^{2} \omega_3^\alpha \wedge \omega_\alpha^4 = -\kappa\Omega + \sum_{\alpha=1}^{2} \left(\sum_{\beta=1}^{2} h_{\alpha\beta 3} \omega^\beta \right) \wedge \left(\sum_{\beta=1}^{2} h_{\alpha\beta 4} \omega^\beta \right) \\ &= \left[-\kappa + \sum_{\alpha=1}^{2} (h_{\alpha 13} h_{\alpha 24} - h_{\alpha 23} h_{\alpha 14}) \right] \Omega . \end{aligned}$$

Hence, $\kappa = \det f$.

COROLLARY. *If the normal curvature never vanishes, then* $\perp \approx Q$. *This isomorphism is given by* f. *When the manifold is oriented,* f *preserves orientation if* $\kappa > 0$, *and it reverses orientation if* $\kappa < 0$.

Remarks. S. Smale [3] has proved that in the case M is the 2-sphere, the homotopy classes of regular immersions correspond to the Euler numbers $\chi(\perp)$. In general, for a compact orientable surface, $\chi(\perp)$ is twice the Whitney self-intersection number of M [4], [1].

4. THE SINGULAR LOCUS OF A REGULAR IMMERSION

An immersion ϕ: M $\to$ E^4 is *κ-regular* if κ has no critical poins on its 0-level $\kappa^{-1}(0) = S$ (we call S the *singular locus* of ϕ). It follows that for a κ-regular immersion, S is a 1-dimensional submanifold of M.

PROPOSITION 1. *For a κ-regular immersion,* rank f ≥ 1 *everywhere.*

Proof. Using the rule for differentiating a determinant, we can represent the differential $d\kappa$ as a sum of two determinants that have differentials in one column and entries of a matrix representing f in the other. A point where f = 0 would be in S; but then each determinant in the sum for $d\kappa$ would have a column of zeros, contradicting $d\kappa \neq 0$.

THEOREM 2. *The κ-regular immersions are a residual set in* $C^\infty(M, E^4)$ *with respect to the Whitney* C^2*-topology.*

Proof. The theorem follows from Thom's transversality theorem, once we establish that a κ-regular immersion is a map $\phi \in C^\infty(M, E^4)$ such that the 2-jet $j_2\phi$: M $\to J_2(M, E^4) = J_2$ is transverse to a finite number of submanifolds of J_2.

The first condition is that ϕ be an immersion. The nonimmersion 2-jets form submanifolds Σ_2 and Σ_1 of J_2, where Σ_2 has codimension 8 and corresponds to points where $d\phi = 0$, and where Σ_1 has codimension 3 and corresponds to points where $d\phi$ has rank 1.

Let $'J_2 = J_2 - \Sigma_2 \cup \Sigma_1$, and let $'J_1$ be the projection of $'J_2$ into J_1. An element of $'J_1$ can be represented by (p, x, τ), where $p \in M$, $x \in E^4$, and $\tau: M_p \to E^4$ is a linear imbedding. Thus, $\tau(M_p)$ has an orthogonal complement $\perp_\tau$ in E^4, and the union of these forms a vector bundle $\perp$ over $'J_1$ having fiber-dimension 2. We get an analogue of the bundle B by pulling back B via the projection $J_1 \to M$, and we still call it B. Over $'J_1$, we get an induced metric on B; now we can imitate the process used to define Q so as to get a subbundle Q of B over $'J_1$, having fiber-dimension 2. Then the second fundamental form can be regarded as a map h: $'J_2 \to \perp^* \otimes B$. More explicitly, h is given as follows. For $X \in {'J_2}$, let (p, x, τ) be the projection into $'J_1$. Choose coordinates u^1 and u^2 on M such that τ maps the coordinate vectors at p into a 2-frame e_α (α = 1, 2) in E^4. Let ω^1, ω^2 be the corresponding coframe in M_p^*. Let $\phi \in C^\infty(M, E^4)$ be such that $j_2\phi = X$, and let $\phi_{\alpha\beta} = \dfrac{\partial^2\phi}{\partial u^\alpha \partial u^\beta}(p)$. Then we realize h(X) as an element of $\mathrm{Hom}(\perp_\tau, B_\tau)$ by

$$h(X)(n) = -\sum_{\alpha,\beta=1}^{2} \langle n, \phi_{\alpha\beta}\rangle \omega^\alpha \omega^\beta .$$

(The minus sign is the same as in the formula $II = dN \cdot d\phi = -N \cdot d^2\phi$.) The fiber of $'J_2 \to {'J_1}$ is an affine space, and we can give it coordinates corresponding to the Cartesian coordinates of the $\phi_{\alpha\beta}$. Hence h: $'J_2 \to \perp^* \otimes B$ is a bundle map over $'J_1$ for which the fiber map h_τ: $'J_{2\tau} \to \perp_\tau^* \otimes B_\tau$ is an affine surjection.

Composing h with orthogonal projection of B into Q, we obtain an affine surjection bundle map f: $'J_2 \to \perp^* \otimes Q$ that is the 2-jet version of the previous map f. In each fiber, the rank-1-values of f form a submanifold S of codimension 1. The rank-0-values of f similarly correspond to a submanifold Z in $'J_2$ having codimension 4. We can extend the structure defined on any immersion so that $\kappa = \det f$ becomes a function on $'J_2$ for which $S \cup Z = \kappa^{-1}(0)$. Thus, if ϕ is an immersion, the normal curvature of ϕ is given by $\kappa \circ j_2\phi$. In terms of local frames, f is represented by a matrix $(f_{\alpha\beta})$ $(\alpha, \beta = 1, 2)$, and the fact that f is an affine surjection shows that these $f_{\alpha\beta}$ can be taken as part of a coordinate system on $'J_2$. Therefore $d\kappa = d(f_{11}f_{22} - f_{12}f_{21})$ is never 0 on S. It follows that if $j_2\phi$ is transverse to S, then the normal curvature of ϕ has no critical points on $(j_2\phi)^{-1}(S)$.

Now we claim that a κ-regular map ϕ is exactly one such that $j_2\phi$ is transverse to Σ_2, Σ_1, S, and Z. For the codimensions of Σ_2, Σ_1, and Z all exceed 2, so that transversality to these means simply that $j_2\phi(M)$ does not meet them. Hence it means that ϕ is an immersion, and the zero-level of its normal curvature is $(j_2\phi)^{-1}(S)$. Finally, transversality to S is the remaining condition in the definition of κ-regularity. This completes the proof of Theorem 2.

For the remainder of this paper we shall assume that $\phi: M \to E^4$ is a κ-regular immersion.

Remark. In the proof of Theorem 2, one of the key ideas is the introduction of the universal second fundamental form for immersions $M \to E^n$ as an affine bundle map $'J_2 \to \perp^* \otimes B$ over $'J_1$. We can describe other generic properties of immersions by specifying pointwise singular submanifolds of $\perp^* \otimes B$. For some significant cases, see [2].

We call attention to the restriction of f to the singular set S. The kernel of $f \mid S$ is a subbundle Z of $\perp \mid S$. We let Y denote $Z^{\perp}$, the orthogonal complement of Z in $\perp \mid S$. The image of $f \mid S$ is the subbundle $L = f(Y)$ of $Q \mid S$, so that Y and L are isomorphic as line bundles on S.

Let C be a component of S. We can choose an orientation of C, represented by a smooth unit tangent field E to C. We also have a smooth nonzero normal field grad $\kappa = F$ on C, so that $TM \mid C$ has an orientation corresponding to that of C, given by the ordered basis E, F. If we reverse the orientation of C, then we get the other orientation -E, F for $TM \mid C$. It follows that $\perp \mid C$ and $Q \mid C$ also have corresponding orientations, and that $Q \mid C$ has a distinguished section Q_1, independent of the choice of orientation on C, such that the eigenvalues of Q_1 are $\pm 1/\sqrt{2}$ and the eigenspace for the eigenvalue $1/\sqrt{2}$ is spanned by E. The orientation of $Q \mid C$ then determines another section Q_2 such that Q_1, Q_2 is a frame. The "angle" θ of L is thus determined up to a multiple of π, and locally smoothly, by the requirement that L is spanned by $(\cos\theta)Q_1 + (\sin\theta)Q_2$. The derivative θ' of θ with respect to arclength is uniquely determined; in fact, this derivative does not depend on the choice of orientation, because a reversal of orientation changes the sign of both θ and arclength s, leaving $d\theta/ds$ invariant. Therefore, the integral $\int_C \theta' ds$ is an invariant of L on C, provided it exists; of course, it does exist when C is a circle, and in that case it must be a multiple of π. We call $\frac{1}{2\pi}\int_C \theta' ds$ the *singular index of ϕ on* C. The sum of the singular indices of ϕ on C, taken over all components C, is called the *singular index of ϕ*, and it is denoted σ. It is a half-integer when S is compact.

The eigenspaces of a nonzero element of L form an orthogonal pair of tangent lines at the base point in S. As we move along S, this "cross field" rotates relative to E, F at a rate half as great as that of L.

5. EULER NUMBERS

Let V be one of the bundles T, $\perp$, and Q, and let X be a section or a line subbundle that has an isolated zero or singularity at p. Let W_1, W_2 be a local moving frame for V in a neighborhood of p, so that the bivector $W_1 \wedge W_2$ determines an orientation of V on that neighborhood, hence also an orientation of the neighborhood. On a small, positively rotating circle around p, we may write

$$X = \lambda(\cos\theta\, W_1 + \sin\theta\, W_2),$$

where $\lambda > 0$ for a section and λ runs through $\mathbb{R}$ for a line bundle. The total change in θ divided by 2π is a half-integer, the *index of* X *at* p, denoted $i_p(X)$. It is independent of the choice of W_1, W_2 and the circle, because equally-oriented choices are homotopic and there are compensating sign changes when the orientation is reversed. At a nonzero or nonsingular point p, let $i_p(X) = 0$.

When M is compact, the *Hopf index theorem* says that if X has a finite number of zeros or singularities, then $\sum i_p(X) = \chi(V)$ is independent of the choice of X, and it is called the *Euler number of* V. When V = T, this is the Euler characteristic $\chi(M)$ of M. For V = Q, we have the relation $\chi(Q) = 2\chi(M)$. When $V = \perp$ and M is orientable, $\chi(\perp)$ is twice Whitney's algebraic self-intersection number [1]. It was shown by Whitney that every even integer can occur as the value of $\chi(\perp)$ [4]. When ϕ is an imbedding, the self-intersection number is 0, so that $\chi(\perp) = 0$. Hence $\chi(\perp) = 0$ when ϕ is regularly homotopic to an imbedding. For the 2-sphere, the converse is true; in fact, two immersions are regularly homotopic if and only if they have the same $\chi(\perp)$ [3].

The Gauss-Bonnet theorem for V says that $\chi(V) = \frac{1}{2\pi}\int_M k\,d\mu$, where k is the curvature of V. The integration is with respect to Riemannian measure $d\mu$, and k is a function, not a 2-form, and therefore this result is also valid for nonorientable M. It suffices to prove it in the orientable case, because in the nonorientable case we get an induced immersion of the two-fold orientable covering ϕ_2, for which we can easily obtain the relations

$$\chi(V_2) = 2\chi(V) \qquad \text{and} \qquad \int_{M_2} k_2\,d\mu_2 = 2\int_M k\,d\mu$$

by pulling back a section of V to get one for $V_2 = \phi_2 {}^* V$, and by observing that $k_2 = \phi_2 {}^* k$.

The Gauss-Bonnet theorem can be proved by applying Stokes's theorem to a connexion form. The proof gives a version for a compact surface with boundary $S = \partial M$. Normalizing X, we get a unit-length section E, and rotating E through $\pi/2$, we get a section F and hence an oriented frame E, F. The connexion form ω for this frame is determined by the formula $D_Y E = \omega(Y) F$ for the covariant derivative of E with respect to the tangent vector field Y. If we had a line bundle X instead of a section, then we would have two choices, E, F and -E, -F, where E and

$-E$ are the unit-length sections spanning X. These two frames give the same connexion form ω. The *curvature* k of V is determined by the structural equation $d\omega = -k\,\Omega$, where Ω is the oriented volume element (2-form) of M. To apply Stokes's theorem, we excise a small disk $D_p(\varepsilon)$ of radius ε around each singular point p. By considering the relation of E, F to a frame W_1, W_2 defined in a neighborhood of p, one shows that

$$\lim_{\varepsilon \to 0} \int_{\partial D_p(\varepsilon)} \omega = 2\pi\, i_p(X),$$

where the orientation of $\partial D_p(\varepsilon)$ is the positive one, hence opposite to that induced from $M - D_p(\varepsilon)$. The resulting Gauss-Bonnet theorem is

$$\int_M k\,\Omega = 2\pi \sum i_p(X) + \int_S \omega,$$

where S has the orientation for which its distinguished tangent, paired with the outward normal, is an oriented frame for M along S. The interpretation of $\int_S \omega$ is this:

$\int_S \omega$ = the total rotation of X along S with respect to a parallel oriented frame along S.

For a nonorientable M, we get the same result except that we replace $\int_M k\,\Omega$ by $\int_M k\,d\mu$ and use an orientation of a collar for S to determine $\int_S \omega$. Since a reversal of orientation of the collar (or a component of it) changes the sign of ω and reverses the direction of S, the boundary piece is independent of the choice of orientation.

In either case, we denote $\int_S \omega$ by $\mathrm{rot}_S X$, since it represents the amount X rotates on S relative to a parallel frame having the same orientation as E, F, where E is a (chosen) unit tangent field to S and F is an outward normal.

6. SPECIAL SECTIONS OF THE BUNDLES

Returning to the structure of a regular immersion of a compact M, we extend the subbundle Y to a subbundle of $\perp$ having a finite number of singularities, and we still call it Y. Then $f(Y) = L$ is an extension of L to a subbundle of Q having singularities at the singular points of Y. The indices of the singularities of L are the same as those of Y, at points where $\kappa > 0$, and the opposite where $\kappa < 0$. Thus the Hopf theorem applied to L gives us the following.

LEMMA 1. $2\chi(M) = \chi(Q) = \sum_p (\text{sign}\,\kappa(p))\, i_p(Y)$.

Now let $M_- = \kappa^{-1}((-\infty, 0])$. By the Gauss-Bonnet theorem for Y on M_-, we have the formula

$$\sum_{p \in M_-} i_p(Y) = \frac{1}{2\pi}\left(\int_{M_-} \kappa - \mathrm{rot}_S\, Y\right),$$

where $\mathrm{rot}_S\, Y$ depends on the fact that grad κ is the outward normal. Similarly, on $M_+ = \kappa^{-1}([0, \infty))$,

$$\sum_{p \in M_+} i_p(Y) = \frac{1}{2\pi}\left(\int_{M_+} \kappa + \mathrm{rot}_S\, Y\right),$$

where we have changed the sign on $\mathrm{rot}_S\, Y$ because here the outward normal is -grad κ. The following theorem is proved by taking the difference of these two formulas, applying Lemma 1, and observing that $\mathrm{rot}_S\, Z = \mathrm{rot}_S\, Y$.

THEOREM 3. *Let* M *be compact and* κ*-regularly immersed in* E^4; *let* $Z = \ker(f \mid S)$; *and let* $\mathrm{rot}_S\, Z$ *be the rotation of* Z *on* S *relative to parallel translation. Then*

$$4\pi\chi(M) = \int_M |\kappa| + 2\,\mathrm{rot}_S\, Z\,.$$

We can obtain corresponding sections of $\perp$ and Q by starting with a section of Q and modifying it in a neighborhood of S so that it will be the image of a section of $\perp$ under f. The initial section Q_1 of Q to be modified is defined in terms of a Morse function k that is an extension of κ restricted to a neighborhood of S. We force Q_1 to have an eigenvalue $-1/\sqrt{2}$ with its eigenspace spanned by grad k. This determines Q_1 uniquely, and Q_1 has singularities only at the critical points of k. The index of each singularity of Q_1 is twice the index of dk at the corresponding zero of dk.

The modification of Q_1 in a neighborhood of S is a weighted sum of Q_1 and a section that is close to being the projection of Q_1 onto a smooth extension of L. However, to ensure that the projection has only a finite number of zeros on S, we must rotate Q_1 by a small angle ψ that is a function on that neighborhood of S.

Choose an orientation of $T \mid S$. We consider only a sufficiently small tubular neighborhood U of S onto which this orientation extends. In U, it makes sense to rotate Q_1 by $\pi/2$; this gives Q_2, which together with Q_1 makes an oriented frame for $Q \mid U$. Consider the restriction f_1 of f to the unit circle bundle in $\perp \mid U$. Above S, the function $\langle f_1, f_1 \rangle$ has a unique pair of maximum points $\pm Y_1$ in each circle, namely, the unit vectors in the subbundle Y. Hence, if U is small enough, $\langle f_1, f_1 \rangle$ still has a unique pair of maximum points $\pm Y_1$ in each circle. Some easy calculations (involving the matrix for f with respect to a smooth local frame) show that $\langle f_1, f_1 \rangle$ has a unique pair of minimum points $\pm Z_1$ in each circle, for which $\langle Z_1, Y_1 \rangle = 0$ and $\langle f(Z_1), f(Y_1) \rangle = 0$, and Y_1, Z_1 can be chosen locally to be a smooth oriented frame. For such a frame we can define smooth functions λ, μ, and θ such that

$$f(Y_1) = \lambda(\cos\theta\, Q_1 + \sin\theta\, Q_2), \quad \text{where } \lambda > 0,$$

$$f(Z_1) = \mu(-\sin\theta\, Q_1 + \cos\theta\, Q_2).$$

Hence $\det f = \lambda\mu = \kappa$, and on S the subbundle L is spanned by $f(Y_1)$. The requirement $\lambda > 0$ makes λ, hence μ, independent of the choice of Y_1, Z_1. Moreover, $\mu \mid S = 0$ and $d\mu \mid S = \frac{1}{\lambda} d\kappa \mid S$ is never 0.

Let h be a function defined on all of M so that $0 < h \leq 1$ on M - S and $h = 1$ outside U, and so that $h = \mu^2$ in some smaller tubular neighborhood V. Let ψ be a function that is 0 outside U, with $|\psi| < 1$ everywhere, and such that $\cos(\theta + \psi)$ has isolated nondegenerate zeros on S. Because the change in θ when we take $-Y_1$ instead of Y_1 is a translation by π, the nondegeneracy condition is independent of the choice of Y_1. Now define a section P of $\perp \mid U$ by

$$P = \frac{1}{\lambda}\cos(\theta + \psi)\, Y_1 - \frac{h}{\mu}\sin(\theta + \psi)\, Z_1 .$$

Since $\cos(\theta + \psi)$, Y_1, $\sin(\theta + \psi)$, and Z_1 all change sign for the other possible local expression, P is defined independently of such choices. Therefore

$$\begin{aligned} f(P) &= \frac{1}{\lambda}\cos(\theta + \psi) \cdot \lambda(\cos\theta\, Q_1 + \sin\theta\, Q_2) \\ &\quad - \frac{h}{\mu}\sin(\theta + \psi) \cdot \mu(-\sin\theta\, Q_1 + \cos\theta\, Q_2) \\ &= [\cos(\theta + \psi)\cos\theta + h\sin(\theta + \psi)\sin\theta]Q_1 \\ &\quad + [\cos(\theta + \psi)\sin\theta - h\sin(\theta + \psi)\cos\theta]Q_2 . \end{aligned}$$

When $h = 1$ and $\psi = 0$, this reduces to Q_1. Hence P can be extended as $f^{-1}(Q_1)$ outside U, except for singularities at the zeros of dk.

Now we determine the zeros of P and f(P) in U. Clearly, if $P = 0$, then $\frac{1}{\lambda}\cos(\theta + \psi) = 0$ and $\frac{h}{\mu}\sin(\theta + \psi) = 0$, hence $h/\mu = 0$. The only such points are those on S at which $\cos(\theta + \psi) = 0$. Thus the zeros of P in U are isolated, and they coincide with the points on S where $\theta + \psi$ is an odd multiple of $\pi/2$. Moreover, there are no other zeros of f(P) on U, because they would have to be on S; hence $\cos(\theta + \psi)\cos\theta = 0$ and $\cos(\theta + \psi)\sin\theta = 0$.

Now we show that the indices of f(P) at the zeros on S are 0. Let $Q^* = \cos\psi\, Q_1 - \sin\psi\, Q_2$. Then

$$\langle f(P), Q^* \rangle = 2[\cos^2(\theta + \psi) + h\sin^2(\theta + \psi)] \geq 0 .$$

In a neighborhood of a zero of f(P), this inner product is positive except at the zero. Since Q^* is never zero in such a neighborhood, the index of f(P) is 0.

Let $p \in S$ be a zero of P. Then $d(\theta + \psi)$ and $d\mu$ are linearly independent at p, so that the components of P,

$$u = \frac{1}{\lambda}\cos(\theta + \psi) \qquad \text{and} \qquad v = -\frac{h}{\mu}\sin(\theta + \psi) = -\mu\sin(\theta + \psi),$$

can be taken as coordinates in a neighborhood of p. Thus, relative to the orientations on T_p and $\perp_p$ given by the bases $\partial/\partial u$, $\partial/\partial v$ and Y_1, Z_1, we see that

$P = uY_1 + vZ_1$ has index 1. The actual index of P relative to the conventional orientations is either 1 or -1, depending on whether $\partial/\partial u$, $\partial/\partial v$ agrees or disagrees with our chosen orientation on T_p. Calculating $du \wedge dv$ at p (where $\cos(\theta + \psi) = 0$), we get $\frac{1}{\lambda} d(\theta + \psi) \wedge d\mu$, so that the coordinates (u, v) give the same orientation as $(\theta + \psi, \mu)$. This agrees or disagrees with the orientation on T_p according to whether $\theta + \psi$ is increasing or decreasing at p in the positive direction on S. It follows that the total change of $\theta + \psi$ on S is π times the sum of the indices of P on S. But the restriction $|\psi| < 1$ makes the total change of $\theta + \psi$ the same as that of θ, which is $2\pi\sigma$. Combining the indices of P on the rest of M with those on S, we obtain the proof of the following proposition.

LEMMA 2. $\chi(\perp) = \sum_p (\text{sign } \kappa(p))\, i_p(f(P)) + 2\sigma$.

COROLLARY. *If* M *is compact and oriented, then* σ *is an integer (not half an odd integer).*

Proof. By [1], $\chi(\perp)$ is an even integer, and by construction all $i_p(f(P))$ are even.

We can evaluate the sum in Lemma 2 in terms of integrals of the Gaussian curvature on M_- and M_+, by using the Gauss-Bonnet theorem for Q on these manifolds with boundary. The connexion form ω for the frame Q_1, Q_2 of Q is twice the connexion form for the corresponding frame of T. On S, this corresponding frame has for its first member the the unit tangent field E of S. This means that $\int_S \omega = 2 \int_S k_g$, where k_g is the geodesic curvature of S. But

$$2 \int_S k_g = 2 \operatorname{rot}_S E = \operatorname{rot}_S Q_1,$$

and $2\pi\sigma$ is the total rotation of L relative to Q_1. Since the rotation of L relative to a parallel field is the sum of the rotation of L relative to Q_1 and the rotation of Q_1 relative to a parallel field, we get the equation $2 \int_S k_g + 2\pi\sigma = \operatorname{rot}_S L$. The results are the formulas in the following theorem, which is also valid in the nonorientable case, by virtue of the double-covering trick.

THEOREM 4. *Let* M *be compact and* κ*-regularly immersed in* E^4, *let* $L = \operatorname{im}(f \mid S)$, *let* k_g *be the geodesic curvature of* S, *and let* σ *be the singular index. Then*

$$\pi\chi(\perp) = \int_M (\text{sign } \kappa) K + 2 \int_S k_g + 2\pi\sigma = \int_M (\text{sign } \kappa) K + \operatorname{rot}_S L .$$

Remark. By imposing another transversality condition on $j_3\phi$, we could ensure that $\cos\theta$ in the proof of Lemma 2 has isolated nondegenerate zeros on S. Then we could take $\psi = 0$. Moreover, the singular index σ would then be simply half of the net number of times θ goes through $\cos\theta = 0$ in the algebraic sense.

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