THE REGULAR RING AND THE MAXIMAL RING OF QUOTIENTS OF A FINITE BAER *-RING

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In the first section of this paper we extend the construction of the regular ring **C**, which was defined in the work of S. K. Berberian [1], [2, Chapter 8]. The following theorem, the central result of this article, can be found in the second section. *If* A *is a finite Baer *-ring satisfying the condition* LP ~ RP *and containing sufficiently many projections, then the involution of* A *is extendible to the maximal ring of right quotients* Q *of* A. Next we show that the matrix ring $\mathbf{C}_n$ is also a Baer *-ring. In the last section we discuss the connection with Berberian's construction.

Because of the considerable overlap with the work of E. S. Pyle, Jr. (of which the author was informed after submission of the paper), some of the proofs that can be found in [6] are omitted.

1. CONSTRUCTION OF THE RING C

In this section we follow [2, Chapter 8], where the reader will find the missing definitions and proofs. We assume that A is a finite Baer *-ring satisfying the condition LP ~ RP; that is, the statement LP(x) ~ RP(x) is valid for all $x \in A$. Then A satisfies the parallelogram law (P) and generalized comparability (GC). The lattice of projections in A is a continuous geometry: if D is a directed index set and $\gamma < \delta$ ($\gamma, \delta \in D$) implies $e_\gamma \leq e_\delta$ (e_γ, e_δ are projections in A), and f is a projection in A, then

$$f \cap \sup_{\delta \in D} \{e_\delta\} = \sup_{\delta \in D} \{f \cap e_\delta\}.$$

LEMMA 1. *Let* $\{e_\delta\}$ *be a set of projections in* A, *let* D *be a directed index set, and let* $\gamma < \delta$ *imply* $e_\gamma \leq e_\delta$. *If* $e_\delta \lesssim f$ *for all* $\delta \in D$, *then* $\sup\{e_\delta\} \lesssim f$.

Proof. See [5, p. 115, Hilfssatz 1.5], [2, Section 33, Exercises 1 and 4; Section 34, Exercise 3].

In this section, D denotes a fixed directed index set.

A *strongly dense domain* (SDD) in A is a family of projections $\{e_\delta\}$ such that $\gamma < \delta$ ($\gamma, \delta \in D$) implies $e_\gamma \leq e_\delta$ and $\sup_{\delta \in D} \{e_\delta\} = 1$.

LEMMA 2 [2, p. 213, Lemma 1]. *If* $\{e_\delta\}$ *and* $\{f_\delta\}$ *are* SDD, *then* $\{e_\delta \cap f_\delta\}$ *is an* SDD.

Let $\{e_\delta\}$ be an SDD, and let $x \in A$. Then it can be shown that if $e_\delta x e_\delta = 0$ for all $\delta \in D$, then $x = 0$. Similarly, if $e_\delta x e_\delta$ is self-adjoint for all δ, then x is self-adjoint [2, p. 218, Exercise 2].

An *operator with closure* (OWC) is a pair of sequences (x_δ, e_δ), where $x_\delta \in A$ and $\{e_\delta\}$ is an SDD, such that $\gamma < \delta$ implies $x_\delta e_\gamma = x_\gamma e_\gamma$ and $x_\delta^* e_\gamma = x_\gamma^* e_\gamma$.

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If (x_δ, e_δ) and (y_δ, f_δ) are OWC, then so are (x_δ^*, e_δ) and $(x_\delta + y_\delta, e_\delta \cap f_\delta)$.

If $x \in A$ and e is a projection in A, we write $x^{-1}(e)$ for the largest projection g such that $(1 - e)xg = 0$, that is, $exg = xg$; thus $x^{-1}(e) = 1 - RP((1 - e)x)$. We can also show that $e \lesssim x^{-1}(e)$.

LEMMA 3. *Let* $\{e_\delta\}$, $\{x_\delta\}$, *and* $\{f_\delta\}$ *be families such that* $\gamma < \delta$ *implies* $e_\gamma \leq e_\delta$, $x_\delta e_\gamma = x_\gamma e_\gamma$, $f_\gamma \leq f_\delta$. *If* $g_\delta = e_\delta \cap x_\delta^{-1}(f_\delta)$, *then* $\gamma < \delta$ *implies* $g_\gamma \leq g_\delta$.

Proof. See [2, p. 214, Lemma 5].

LEMMA 4. *In the notation of Lemma* 3, *let* $\{e_\delta\}$ *and* $\{f_\delta\}$ *both be* SDD. *Then* $\{g_\delta\}$ *is an* SDD.

For a proof, see [2, p. 214, Lemma 5].

If (x_δ, e_δ) and (y_δ, f_δ) are OWC, and if

$$k_\delta = (f_\delta \cap y_\delta^{-1}(e_\delta)) \cap (e_\delta \cap (x_\delta^*)^{-1}(f_\delta)), \tag{1}$$

then we can show, by means of the preceding lemma, that $(x_\delta y_\delta, k_\delta)$ is an OWC.

We say that the OWC (x_δ, e_δ) and (y_δ, f_δ) are *equivalent* (and we write $(x_\delta, e_\delta) \equiv (y_\delta, f_\delta)$) if there exists an SDD $\{g_\delta\}$ such that $x_\delta g_\delta = y_\delta g_\delta$ and $x_\delta^* g_\delta = y_\delta^* g_\delta$ for all δ. The equivalence is said to be implemented via the SDD $\{g_\delta\}$. The relation $\equiv$ is an equivalence relation in the set of all OWC. It has also the following properties:

If (x_δ, e_δ) is an OWC and $\{g_\delta\}$ is an SDD, then $(x_\delta, e_\delta \cap g_\delta)$ is also an OWC and $(x_\delta, e_\delta) \equiv (x_\delta, e_\delta \cap g_\delta)$.

Suppose $(x_\delta, e_\delta) \equiv (y_\delta, f_\delta)$ via an SDD $\{g_\delta\}$. Set $h_\delta = e_\delta \cap f_\delta \cap g_\delta$. Then (x_δ, h_δ) and (y_δ, h_δ) are OWC, and $(x_\delta, h_\delta) \equiv (y_\delta, h_\delta)$ via $\{h_\delta\}$.

Definition. We write $[x_\delta, e_\delta]$ for the equivalence class of the OWC (x_δ, e_δ) with respect to the equivalence relation defined above. The set of all equivalence classes is denoted by $\mathbf{C}$, and its elements are called *closed operators* (CO). We denote the elements of $\mathbf{C}$ by letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \cdots$. If $x \in A$, we write $\bar{x} = [x, 1]$ for the CO determined by the pair $(x, 1)$ of constant sequences, and we write $\bar{A} = \{\bar{x}: x \in A\}$.

In the set $\mathbf{C}$ we can introduce operations that make $\mathbf{C}$ a *-ring. If $\mathbf{x} = [x_\delta, e_\delta]$ and $\mathbf{y} = [y_\delta, f_\delta]$, we define

$$\mathbf{x} + \mathbf{y} = [x_\delta + y_\delta, e_\delta \cap f_\delta], \qquad \mathbf{x}^* = [x_\delta^*, e_\delta], \qquad \mathbf{x}\mathbf{y} = [x_\delta y_\delta, k_\delta],$$

where $\{k_\delta\}$ is an SDD defined by (1).

With operations so defined, $\mathbf{C}$ is a *-ring with unity $\bar{1}$, and the mapping $x \to \bar{x}$ is a *-isomorphism of A onto a *-subring $\bar{A}$ of $\mathbf{C}$.

LEMMA 5. *If* $\mathbf{x} \in \mathbf{C}$ *and* $\mathbf{x} = [x_\delta, e_\delta]$, *then* $\mathbf{x}\bar{e}_\delta = \overline{x_\delta e_\delta}$ *and* $\bar{e}_\delta \mathbf{x} = \overline{e_\delta x_\delta}$ *for all* $\delta \in D$.

The proof is the same as in [2, p. 219, Proposition 1], except that an SDD $\{f_\delta\}$ is defined so that $f_\delta = 0$ for $\delta \not\geq \gamma$, and $f_\delta = 1$ for $\delta \geq \gamma$, where γ is a fixed index.

LEMMA 6. *If* $\mathbf{x} = [x_\delta, e_\delta]$ *and* $\mathbf{y} = [y_\delta, f_\delta]$, *and if* $\{g_\delta\}$ *is an* SDD *such that* $x_\delta g_\delta = y_\delta g_\delta$ *for all* δ, *then* $\mathbf{x} = \mathbf{y}$. *In fact, it suffices to assume that* $h_\delta x_\delta g_\delta = h_\delta y_\delta g_\delta$ *for a pair of* SDD $\{g_\delta\}$, $\{h_\delta\}$.

LEMMA 7. *If* $\mathbf{x} \in \mathbf{C}$, *then there exists a projection* $f \in A$ *such that*

(a) $\bar{f}\mathbf{x} = \mathbf{x}$ *and*

(b) $\mathbf{y}\mathbf{x} = 0$ *if and only if* $\mathbf{y}\bar{f} = 0$.

If $\mathbf{x} = [x_\delta, e_\delta]$, *then* $f = \sup\{LP(x_\delta f_\delta)\}$.

As a consequence of this lemma, we see that $\mathbf{C}$ has no new projections; that is, if $\mathbf{e} \in \mathbf{C}$ is a projection, then $\mathbf{e} = \bar{e}$, where e is a projection in A. Therefore $\mathbf{C}$ is a Baer *-ring. See also [6, p. 61, Theorem 4.14].

We can also show that if $\mathbf{x} \in \mathbf{C}$, then $LP(\mathbf{x}) \sim RP(\mathbf{x})$, via a partial isometry of the form $\bar{w}$, where w is a partial isometry in A [2, p. 220, Theorem 2]. As a consequence of this we see that

(i) $\mathbf{C}$ is finite, and $\mathbf{y}\mathbf{x} = 1$ implies $\mathbf{x}\mathbf{y} = 1$, and

(ii) if $\mathbf{e}$ and $\mathbf{f}$ are projections in $\mathbf{C}$, say $\mathbf{e} = \bar{e}$ and $\mathbf{f} = \bar{f}$, then $\mathbf{e} \sim \mathbf{f}$ in $\mathbf{C}$ if and only if $e \sim f$ in A.

2. MAXIMAL RING OF QUOTIENTS OF THE RING A

In the first part of this section, A denotes a finite Baer *-ring satisfying the condition LP ~ RP; later we shall add another hypothesis.

Let $\aleph$ denote a cardinal number that is at least as great as the cardinal number of any family of pairwise orthogonal, nonzero projections in A. Let D be the set of all finite subsets of the set $\aleph$. If $\alpha \in \aleph$, we write $\{\alpha\}$ for the set having α as its only element; thus $\{\alpha\} \in D$. If $\gamma, \delta \in D$, then $\gamma \cup \delta \in D$. Therefore D, ordered by inclusion, is a directed set.

A *semioperator with closure* (SOWC) is a pair of sequences $(x_\delta, e_\delta\}$ with $\delta \in D$, where $x_\delta \in A$ and $\{e_\delta\}$ is an SDD, such that $\gamma < \delta$ implies $x_\delta e_\gamma = x_\gamma e_\gamma$ (Pyle calls it a *right operator*).

We say that the SOWC $(x_\delta, e_\delta\}$ and $(y_\delta, f_\delta\}$ are equivalent ($\equiv$) if there exists an SDD $\{g_\delta\}$ such that $x_\delta g_\delta = y_\delta g_\delta$ for all δ. This relation is an equivalence relation in the set of all SOWC. In the set $\mathbf{S}$ of equivalence classes of SOWC's we introduce operations in the same way as in the set $\mathbf{C}$ (multiplication is defined by projections $k_\delta = f_\delta \cap y^{-1}(e_\delta)$). The set $\mathbf{S}$ then becomes an associative ring with unit, containing a subring that is isomorphic to the ring A.

LEMMA 1. *If* $(x_\delta, e_\delta\}$ *is an* SOWC, *then there exists an* SOWC $(y_\delta, f_\delta\}$ *such that* $\sup_{\alpha \in \aleph}\{f_{\{\alpha\}}\} = 1$, *where the* $f_{\{\alpha\}}$ *are pairwise orthogonal, and* $(x_\delta, e_\delta\} \equiv (y_\delta, f_\delta\}$. *Furthermore,* $y_\delta f_\delta = y_\delta$.

Proof. Let $\{f_\beta\}$ be a maximal family of pairwise orthogonal nonzero projections such that for each β there exists $\delta(\beta) \in D$ such that $f_\beta \leq e_{\delta(\beta)}$. By maximality, we see that $\sup\{f_\beta\} = 1$; indeed, if $f = \sup\{f_\beta\}$, then $(1 - f) \cap e_\delta = 0$ for all $\delta \in D$ implies $1 - f = 0$, by continuity. The indices β are from $\aleph$. Since $|\{f_\beta\}| \leq \aleph$ ($|\ |$ denotes cardinality), we assign the projection 0 to the remaining indices (if there are any). Set $f_{\{\alpha\}} = f_\alpha$, $y_{\{\alpha\}} = x_{\delta(\alpha)} f_{\{\alpha\}}$, or $y_{\{\alpha\}} = 0$ when $f_{\{\alpha\}} = 0$ ($\alpha \in \aleph$, $\{\alpha\} \in D$). Here $\delta(\alpha) \in D$ denotes an index for which $f_\alpha \leq e_{\delta(\alpha)}$. If $y_{\{\alpha\}} = x_{\delta(\alpha)} f_{\{\alpha\}}$ and $\gamma > \delta(\alpha)$, then

$$(2) \qquad y_{\{\alpha\}} = x_{\delta(\alpha)} f_{\{\alpha\}} = x_{\delta(\alpha)} e_{\delta(\alpha)} f_{\{\alpha\}} = x_\gamma e_{\delta(\alpha)} f_{\{\alpha\}} = x_\gamma f_{\{\alpha\}}.$$

The function $\delta(\beta)$ is not uniquely determined. Let $\delta_1(\beta)$ and $\delta_2(\beta)$ be functions such that the relations $f_\beta \leq e_{\delta_1(\beta)}$ and $f_\beta \leq e_{\delta_2(\beta)}$ hold for all β. If we set $\gamma(\beta) = \delta_1(\beta) \cup \delta_2(\beta)$, then it follows from (2) that

$$y_{\{\alpha\}} = x_{\delta_1(\alpha)} f_{\{\alpha\}} = x_{\gamma(\alpha)} f_{\{\alpha\}} = x_{\delta_2(\alpha)} f_{\{\alpha\}};$$

thus $\{y_{\{\alpha\}}\}$ is uniquely determined.

Define

$$f_{\{\alpha_1,\cdots,\alpha_n\}} = f_{\{\alpha_1\}} + \cdots + f_{\{\alpha_n\}}, \qquad y_{\{\alpha_1,\cdots,\alpha_n\}} = y_{\{\alpha_1\}} + \cdots + y_{\{\alpha_n\}}.$$

Let $\gamma = \{\alpha_1, \cdots, \alpha_n\} \subset \{\alpha_1, \cdots, \alpha_n, \cdots, \alpha_m\} = \delta$. Since $\alpha_i \neq \alpha_j$ implies $y_{\{\alpha_i\}} f_{\{\alpha_j\}} = x_{\delta(\alpha_i)} f_{\{\alpha_i\}} f_{\{\alpha_j\}} = 0$, we see that

$$y_\delta f_\gamma = (y_{\{\alpha_1\}} + \cdots + y_{\{\alpha_n\}} + \cdots + y_{\{\alpha_m\}})(f_{\{\alpha_1\}} + \cdots + f_{\{\alpha_n\}})$$

$$= (y_{\{\alpha_1\}} + \cdots + y_{\{\alpha_n\}})(f_{\{\alpha_1\}} + \cdots + f_{\{\alpha_n\}}) = y_\gamma f_\gamma .$$

Therefore $(y_\delta, f_\delta\}$ is an SOWC. From the relation $y_{\{\alpha\}} f_{\{\alpha\}} = y_{\{\alpha\}}$ we deduce that $y_\delta f_\delta = y_\delta$ for all $\delta \in D$.

Let $\delta = \{\alpha_1, \cdots, \alpha_n\}$, and let $\delta(\alpha_1), \cdots, \delta(\alpha_n)$ denote indices for which $y_{\{\alpha_i\}} = x_{\delta(\alpha_i)} f_{\{\alpha_i\}}$ or $\delta(\alpha_i) = \{\alpha_i\}$ when $f_{\{\alpha_i\}} = 0$. Let γ denote an index that is larger than any of $\delta, \delta(\alpha_1), \delta(\alpha_2), \cdots, \delta(\alpha_n)$. From (2) we obtain the relations

$$x_\delta(e_\delta \cap f_\delta) = x_\delta e_\delta(e_\delta \cap f_\delta) = x_\gamma(e_\delta \cap f_\delta),$$

$$y_\delta(e_\delta \cap f_\delta) = (y_{\{\alpha_1\}} + \cdots + y_{\{\alpha_n\}})(e_\delta \cap f_\delta)$$

$$= (x_\gamma f_{\{\alpha_1\}} + \cdots + x_\gamma f_{\{\alpha_n\}})(e_\delta \cap f_\delta) = x_\gamma f_\delta(e_\delta \cap f_\delta);$$

Therefore $(x_\delta, e_\delta\} \equiv (y_\delta, f_\delta\}$ via the SDD $\{e_\delta \cap f_\delta\}$.

LEMMA 2. *If* $g_\alpha \lesssim e_\alpha$ $(\alpha \in I)$ *and the* e_α *are pairwise orthogonal, then* $\sup\{g_\alpha\} \lesssim \sup\{e_\alpha\}$.

Proof. The statement clearly holds if I is a finite index set and the projections g_α are also pairwise orthogonal. Let us take $I = \{1, 2\}$. Then, by the parallelogram law, we have the relations

$$(g_1 \cup g_2) - g_2 \sim g_1 - (g_1 \cap g_2) \lesssim e_1 .$$

When we add this to $g_2 \lesssim e_2$, we see that $g_1 \cup g_2 \lesssim e_1 + e_2$.

If I is an infinite set, then

$$g_{\alpha_1} \cup \cdots \cup g_{\alpha_n} \lesssim e_{\alpha_1} + e_{\alpha_2} + \cdots + e_{\alpha_n} \lesssim \sup\{e_\alpha\}$$

for each finite subset $\{\alpha_1, \cdots, \alpha_n\}$ of the set I. The finite unions $g_{\alpha_1} \cup \cdots \cup g_{\alpha_n}$ form an increasing family. Therefore, by Lemma 1 of the first section, we see that $\sup\{g_\alpha\} \lesssim \sup\{e_\alpha\}$.

LEMMA 3. *In each equivalence class of* SOWC, *there exists an* OWC.

Proof. Let $(x_\delta, e_\delta\}$ be an SOWC with the following properties: $\sup\{e_{\{\alpha\}}\} = 1$, where the $e_{\{\alpha\}}$ are pairwise orthogonal, and $x_\delta e_\delta = x_\delta$. By Lemma 1, there exists such an SOWC in each equivalence class. Set $f'_\alpha = LP(x_{\{\alpha\}}) = LP(x_{\{\alpha\}} e_{\{\alpha\}})$. The equality $x_{\{\alpha\}} e_{\{\alpha\}} = x_{\{\alpha\}}$ tells us that

$$f'_\alpha = LP(x_{\{\alpha\}}) \sim RP(x_{\{\alpha\}}) \leq e_{\{\alpha\}}.$$

Therefore $f'_\alpha \lesssim e_{\{\alpha\}}$. Set

$$f_\delta = 1 - \sup_{\alpha \in \aleph\setminus\delta} \{f'_\alpha\} \qquad (\delta = \{\alpha_1, \cdots, \alpha_n\} \in D). \tag{3}$$

It follows immediately from (3) that $\gamma < \delta$ implies $f_\gamma \leq f_\delta$. By Lemma 2, we see that

$$\sup_{\alpha \in \aleph\setminus\delta} \{e_{\{\alpha\}}\} \gtrsim \sup_{\alpha \in \aleph\setminus\delta} \{f'_\alpha\},$$

and from this that

$$f_\delta = 1 - \sup_{\alpha \in \aleph\setminus\delta} \{f'_\alpha\} \gtrsim 1 - \sup_{\alpha \in \aleph\setminus\delta} \{e_{\{\alpha\}}\} = e_\delta = e_{\{\alpha_1\}} + \cdots + e_{\{\alpha_n\}}.$$

Therefore $\sup_{\gamma \in D}\{f_\gamma\} \gtrsim e_\delta$, and $\sup_{\delta \in D}\{f_\delta\} \gtrsim \sup_{\delta \in D}\{e_\delta\} = 1$.

Define the OWC (x_δ, g_δ) so that $g_\delta = e_\delta \cap f_\delta$. Let

$$\gamma = \{\alpha_1, \cdots, \alpha_n\} \subset \{\alpha_1, \cdots, \alpha_n, \cdots, \alpha_m\} = \delta.$$

Then

$$x_\delta g_\gamma = x_\delta e_\gamma g_\gamma = x_\gamma e_\gamma g_\gamma = x_\gamma g_\gamma,$$

$$x^*_\delta g_\gamma = x^*_\delta f_\gamma g_\gamma = (x^*_{\{\alpha_1\}} + \cdots + x^*_{\{\alpha_n\}} + \cdots + x^*_{\{\alpha_m\}}) f_\gamma g_\gamma.$$

But $\alpha_i \notin \gamma$ implies

$$x^*_{\{\alpha_i\}} f_\gamma = x^*_{\{\alpha_i\}}(1 - \sup_{\alpha \in \aleph\setminus\gamma} \{f'_\alpha\}) = 0,$$

since $RP(x^*_{\{\alpha_i\}}) = LP(x_{\{\alpha_i\}}) = f'_{\alpha_i} \leq \sup_{\alpha \in \aleph\setminus\gamma}\{f'_\alpha\}$. Therefore $x^*_\delta g_\gamma = x^*_\gamma g_\gamma$.

THEOREM 1. *There exists a natural ring isomorphism of* **C** *onto* **S**; *in particular,* **S** *has an involution extending that of* A.

Proof. Since every OWC is also SOWC, and the equivalence between OWC implies the equivalence of SOWC, there exists a natural mapping of the set **C** into the set **S**, defined by

$$[(x_\delta, e_\delta)] \to [(x_\delta, e_\delta\}].$$

The brackets denote the corresponding equivalence classes. By Lemma 6 of the preceding section, we see that the equivalence between OWC as SOWC implies equivalence as OWC; thus the mapping is injective, and by the preceding lemma it is also surjective. It is obvious that the mapping preserves the ring operations and leaves fixed the elements of A. Since $\mathbf{C}$ possesses a natural involution (extending that of A), the isomorphism induces an involution of $\mathbf{S}$ (extending that of A).

LEMMA 4. *Each* SOWC $(x_\delta, e_\delta\}$ *defines a homomorphism of the right ideal* N (*as right* A-*module*) *generated by projections* $\{e_\delta\}$ *into the ring* A.

For a proof, see [6, p. 45, proof of Theorem 3.35].

Till the end of this section, let the ring A satisfy also the condition that each nonzero right ideal contains a nonzero projection. In such a case we say that A has *sufficiently many projections* (see [7], [6, p. 24]).

A right ideal N is called *essential* if $N \cap N' \neq \{0\}$ for each nonzero right ideal N'.

LEMMA 5. *A right ideal* N *of* A *is essential if and only if there exists in* N *a set of pairwise orthogonal projections with the supremum* 1.

For a proof, see [6, p. 44, Proposition 3.34, and p. 49, proof of Corollary 3.37].

The singular ideal of a Baer *-ring is zero [7], [3]. When the singular ideal of a ring R is zero, we may characterize the maximal ring of (right) quotients in the following way [4, Section 4.3]: The maximal ring of quotients Q of a ring R is

$$\bigcup_N \mathrm{Hom}_R(N, R)/\theta,$$

where N runs over the set of all essential right ideals, and where θ denotes the relation defined by the rule that two homomorphisms are equivalent provided they coincide on the intersection of their domains. The operations are defined by the formulas

$$(\phi_1 + \phi_2)\,d = \phi_1(d) + \phi_2(d) \qquad \text{for } d \in \mathscr{D}_{\phi_1} \cap \mathscr{D}_{\phi_2},$$

$$(\phi_1\phi_2)\,d = \phi_1(\phi_2(d)) \qquad \text{for } d \in \mathscr{D}_{\phi_2} \text{ such that } \phi_2(d) \in \mathscr{D}_{\phi_1}.$$

THEOREM 2. *If* A *is a finite Baer *-ring satisfying the condition* LP ~ RP *and having sufficiently many projections, and if* D *is a directed set, defined as at the beginning of this section, then* $\mathbf{C}$ *is isomorphic to the maximal ring of quotients of* A. *Since the singular ideal of the ring* A *is zero,* $\mathbf{C}$ *is a regular ring; thus* $\mathbf{C}$ *is a regular Baer *-ring with the same projection lattice as* A.

Proof. Using Lemmas 4 and 5, we can show that the ring $\mathbf{S}$ is isomorphic to the ring Q [6, p. 49, Corollary 3.37]. Then we apply Theorem 1 [6, p. 62, Theorem 4.17]. See also [7].

Let A satisfy also the condition

(4) $$x_1x_1^* + \cdots + x_nx_n^* = 0 \quad \text{implies} \quad x_1 = \cdots = x_n = 0.$$

LEMMA 6. $\mathbf{C}$ *also satisfies condition* (4) [2, Section 50, Proposition 1], *and the matrix ring* $\mathbf{C}_n$ *is *-regular* [2, Section 56, Proposition 2].

COROLLARY. *Under the assumption* (4), $\mathbf{C}_n$ *is a regular Baer *-ring.*

Proof. By Theorem 2, the ring $\mathbf{C}$ is isomorphic to Q. Since Q as a right Q-module is injective [4, Section 4.3, Proposition 3], the right Q-module Q^n of n-dimensional rows is also injective [4, Section 4.2, Proposition 2]. Let M be an injective submodule of Q^n. Since M is direct summand of Q^n [4, Section 4.2, Proposition 6], we can write $M + N = Q^n$, where $M \cap N = \{0\}$. Then there exist elements $a_i \in M$ and $b_i \in N$ such that $a_i + b_i = (0, \cdots, 1, \cdots, 0)$, where $i = 1, \cdots, n$. Let M' and N' denote the submodules of M and N generated by $\{a_i\}$ and $\{b_i\}$. Then we can write

$$x = (x_1, \cdots, x_n) = \sum_i (0, \cdots, 1, \cdots, 0) x_i = \sum_i (a_i + b_i) x_i$$

$$= \sum_i a_i x_i + \sum_i b_i x_i \in M' + N'.$$

Therefore $M' + N' = Q^n$ and $M' = M$, $N' = N$. Consequently, M and N are finitely generated.

Let M be a finitely generated submodule of Q^n. Then it has a complement in Q^n [8, p. 15, Lemma]. By [4, Section 4.2, Proposition 2], M is injective. Therefore the injective submodules of Q^n form a lattice [8, p. 16, Theorem 4] that is isomorphic to the lattice of projections in Q_n, by *-regularity.

Let $\{M_\alpha\}$ be a family of injective submodules of Q^n, and let M denote a minimal injective extension of the submodule that is generated by the elements of M_α [4, Section 4.2, Proposition 10]. Since the injective submodules form a lattice, M is unique, and $M = \sup \{M_\alpha\}$. Thus the projection lattice of $\mathbf{C}_n = Q_n$ is complete; therefore $\mathbf{C}_n$ is a Baer *-ring. See also [9].

3. THE CONNECTION WITH THE CONSTRUCTION OF BERBERIAN

The construction of Berberian [1], [2, Chapter 8] is a special case of our construction if for D we take the set N of natural numbers or the set of all finite subsets of the set N.

Let $\aleph_1$ and $\aleph_2$ be the cardinal numbers, and let $D_{\aleph_1}$ and $D_{\aleph_2}$ be the corresponding sets of all finite subsets. With these directed sets we construct rings $\mathbf{C}_{\aleph_1}$ and $\mathbf{C}_{\aleph_2}$.

THEOREM 1. *If* $\aleph_1 < \aleph_2$, *then* $\mathbf{C}_{\aleph_1}$ *is *-isomorphically imbedded in* $\mathbf{C}_{\aleph_2}$.

Proof. Corresponding to each OWC (x_δ, e_δ) in the construction of $\mathbf{C}_{\aleph_1}$ we assign an OWC in the construction of $\mathbf{C}_{\aleph_2}$ in the following way: Write $\delta \cup \gamma$ as an index in $D_{\aleph_2}$, where $\delta \subset \aleph_1$ and $\gamma \subset \aleph_2 \setminus \aleph_1$, and set

$$e_{\delta\cup\gamma} = e_\delta, \qquad x_{\delta\cup\gamma} = x_\delta.$$

If $(x_\delta, e_\delta) \equiv_{\aleph_1} (y_\delta, f_\delta)$ and $\{g_\delta\}$ implies the equivalence for $D_{\aleph_1}$, then $\{g_{\delta\cup\gamma} = g_\delta\}$ implies the equivalence for $D_{\aleph_2}$. Conversely, let

$(x_{\delta\cup\gamma}, e_{\delta\cup\gamma}) \equiv_{\aleph_2} (y_{\delta\cup\gamma}, f_{\delta\cup\gamma})$ via $\{g_{\delta\cup\gamma}\}$, where $[x_\delta, e_\delta]$ and $[y_\delta, f_\delta]$ are in $\mathbf{C}_{\aleph_1}$. Thus $x_{\delta\cup\gamma}g_{\delta\cup\gamma} = y_{\delta\cup\gamma}g_{\delta\cup\gamma}$. Since $x_{\delta\cup\gamma} = x_\delta$ and $y_{\delta\cup\gamma} = y_\delta$ for all $\gamma \subset \aleph_2 \setminus \aleph_1$ we see that

$$(x_\delta - y_\delta)g_{\delta\cup\gamma} = 0.$$

Therefore $(x_\delta - y_\delta)g_\delta = 0$ if $g_\delta = \sup_\gamma\{g_{\delta\cup\gamma}\}$. Since $\{g_\delta\}$ is an SDD, we conclude that $(x_\delta, e_\delta) \equiv_{\aleph_1} (y_\delta, f_\delta)$ via $\{g_\delta\}$. This means that the function from $\mathbf{C}_{\aleph_1}$ to $\mathbf{C}_{\aleph_2}$ thus defined is well-defined. We can also prove that it is a *-monomorphism.

We can extend the study of $\mathbf{C}_\aleph$ by taking $\aleph$ to be the cardinal number we fixed in the preceding section, with the additional hypotheses (1^0), (2^0), (3^0), (4^0) from [2, Section 51]. The hypothesis (5^0) [2, Section 52] is not necessary.

THEOREM 2. *If* A *satisfies the* (US)*-axiom, then* $\mathbf{C}_\aleph$ *is *-isomorphic to* $\mathbf{C}_N$ (*that is, to the ring constructed by Berberian*).

Indeed, each $\mathbf{x} \in \mathbf{C}_\aleph$ with $\mathbf{x} = \mathbf{x}^*$ can be written in the form

$$\mathbf{x} = i(1 + \mathbf{u})(1 - \mathbf{u})^{-1}.$$

Since $(1 - \mathbf{u})^{-1} \in \mathbf{C}_N$ with $1 - \mathbf{u} = 1 - \bar{u}$, it follows that $\mathbf{x} \in \mathbf{C}_N$ [2, Section 52, Proposition 2].

To extend [2, Section 54], we must instead of (6^0) in [2, Section 54] take the following axiom. *If* $\{f_\alpha\}$ *is a set of pairwise orthogonal projections with* $\sup\{f_\alpha\} = 1$, *and if* $a_\alpha \in f_\alpha A f_\alpha$ $(0 \leq a_\alpha \leq 1)$, *then there exists an* $a \in A$ *such that* $af_\alpha = a_\alpha$ *for all* α.

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