ON INNER FUNCTIONS WITH H^p-DERIVATIVE

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In this paper, we consider the problem of determining the H^p-classes $(p > 0)$ to which the derivative ϕ' of an inner function ϕ in the unit disk U belongs. Various conditions sufficient for the relation $\phi' \in H^p$ have been established; see J. G. Caughran and A. L. Shields [5], M. R. Cullen [7], and H. A. Allen and C. L. Belna [2] for the case where ϕ is a singular inner function, and D. Protas [11] for the case where ϕ is a Blaschke product.

Part I is devoted to results about general inner functions. In Section 1, we give some relevant results on angular derivatives of bounded analytic functions. Applying these results to inner functions (Section 2), we show that the relation $\phi' \in H^{1/2}$ implies that ϕ is a Blaschke product. This answers a question raised in [5] and [7]; see also [2]. Another consequence is that $\phi' \in H^p$ if and only if $\phi'/s \in H^p$, where s is the singular part of ϕ. This has been proved before in a special case in [7]; the corresponding statement with H^p replaced by B^p is known to be false (see [2]). We conclude Part I with an application of our results to exceptional and omitted sets.

In the second part, we consider the case where ϕ is a Blaschke product with zeros $\{a_n\}$. Protas [11] (see also Caughran and Shields [6]) has shown that the condition

$$\sum (1 - |a_n|)^\delta < \infty \tag{1}$$

with $\delta = 1 - p$ is sufficient for $\phi' \in H^p$ if $1/2 < p < 1$. Using the results of Section 2, we show in Section 3 that condition (1), with $\delta = (1 - p)/p$, is necessary for $\phi' \in H^p$ $(1/2 < p < 1)$. This is apparently the first known necessary condition for the derivative of a Blaschke product to lie in H^p. We show by example that both these conditions represent the best possible values of δ. In fact, if the zeros a_n converge to a boundary point nontangentially, then $\delta = (1 - p)/p$ is precisely the right order of convergence of (1) (Section 4). Section 5 gives sufficient conditions for the relation $\phi' \in H^p$ in some other cases.

PART I. INNER FUNCTIONS

1. ANGULAR DERIVATIVES

We consider the class $\mathbb{B}$ of functions that are holomorphic and bounded (in modulus) by 1 in U. A function $f \in \mathbb{B}$ has a factorization of the form

$$f(z) = \left\{ \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \right\} \exp\left\{ -\int \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) \right\}, \tag{2}$$

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where $\sum (1 - |a_n|) < \infty$ and where μ is a positive Borel measure on the unit circle T. (The convention here is that $|a_n|/a_n$ is to be replaced by 1 of $a_n = 0$, and that the set $\{a_n\}$ may be infinite, finite or empty.) If $\mu \equiv 0$, the function f is called a *Blaschke product*, and if μ is singular with respect to Lebesgue measure on T, then f is called an *inner function;* a nonvanishing inner function is called a *singular inner function.* Lebesgue measure on T will be denoted dm, and the phrase "almost everywhere" will always refer to dm. If $f \in \mathbb{B}$, then $f(\zeta) = \lim_{r \to 1} f(r\zeta)$ exists for almost all $\zeta \in T$; details of these facts about $\mathbb{B}$ can be found, for instance, in [8, Chapter 2].

Following C. Carathéodory, we shall say that a function $f \in \mathbb{B}$ has an *angular derivative* at $\zeta \in T$ if $f(\zeta)$ exists and has modulus 1 and if $f'(\zeta) = \lim_{r \to 1} f'(r\zeta)$ exists. If f fails to have an angular derivative at ζ, we shall write $|f'(\zeta)| = \infty$. Note that this does not imply that $|f'(r\zeta)| \to \infty$ as $r \to 1$. The basic results on angular derivatives, some of which we assemble in the following theorem, are due to Carathéodory [3, Sections 298-299].

THEOREM 1. *If* $f \in \mathbb{B}$ *and* $\zeta \in T$, *then*

(i) $|f'(\zeta)| = \lim_{r \to 1} (1 - |f(r\zeta)|)/(1 - r)$;

(ii) *if* f *has an angular derivative at* ζ, *then* $f'(\zeta) = \bar{\zeta} f(\zeta) |f'(\zeta)|$;

(iii) *if* $f_n \in \mathbb{B}$ *and* $f_n \to f$ *uniformly on compact subsets of* U, *then*

$$|f'(\zeta)| \leq \liminf_{n \to \infty} |f_n'(\zeta)| .$$

COROLLARY 1. *If* f *and* g *belong to* $\mathbb{B}$, *and if* $\phi = fg$, *then*

$$|\phi'(\zeta)| = |f'(\zeta)| + |g'(\zeta)| \tag{3}$$

for all $\zeta \in T$.

The corollary is an obvious consequence of Theorem 1 (ii) in case both f and g (and hence ϕ) have an angular derivative at ζ. The only other case in which not both sides of (3) are infinite is that in which ϕ has an angular derivative. In that case, since

$$(1 - |f(r\zeta)|)/(1 - r) \leq (1 - |\phi(r\zeta)|)/(1 - r),$$

it follows that f and hence g has an angular derivative at ζ, and therefore Theorem 1 (ii) again implies the desired conclusion.

If $f, g \in \mathbb{B}$, we say that g is a *divisor* of f if $f = gh$, for some $h \in \mathbb{B}$.

COROLLARY 2. *If* $\phi, \phi_n \in \mathbb{B}$ $(n = 1, 2, \cdots)$, *if* ϕ_n *is a divisor of* ϕ *for every* n, *and if* $\phi_n \to \phi$ *uniformly on compact sets, then* $|\phi_n'(\zeta)| \to |\phi'(\zeta)|$ *for every* $\zeta \in T$.

For the proof, note that Theorem 1 (iii) implies $|\phi'(\zeta)| \leq \liminf |\phi_n'(\zeta)|$, while Corollary 1 implies $|\phi_n'(\zeta)| \leq |\phi'(\zeta)|$.

The following theorem was proved by M. Riesz [12] for singular inner functions, and by O. Frostman [10] for Blaschke products. The general case was proved in [1]. Here we sketch a proof that is simpler than any of those cited above; it indicates how easily the theorem follows from the basic results of Carathéodory.

THEOREM 2. *Let* $f \in \mathbb{B}$ *have the representation* (2). *Then for all* $\zeta \in T$,

$$|f'(\zeta)| = \sum_n (1 - |a_n|^2)/|\zeta - a_n|^2 + 2 \int |\lambda - \zeta|^{-2} d\mu(\lambda). \tag{4}$$

Proof. We notice first that if μ has a point mass at ζ, then both sides of (4) are infinite. Another case that follows without difficulty is that in which ζ is in neither the closed support of μ nor the closure of the set $\{a_n\}$.

For the case of arbitrary $\{a_n\}$ and μ (with $\mu(\{\zeta\}) = 0$), define f_m to have a representation (2) with $\{a_n\}$ replaced by $\{a_1, a_2, \cdots, a_m\}$ and μ replaced by its restriction to the complement of the arc on T centered at ζ with length 1/n. Then f_m is a divisor of f, $f_m \to f$ on compact subsets of U, and therefore, by Corollary 2, $|f_m'(\zeta)| \to |f'(\zeta)|$. As we noted above, (4) holds for f_m, and the monotone-convergence theorem shows that the right side of the analogue of (4) for f_m tends to the right side of (4).

This proof may appear to depend largely on our convention that $|f'(\zeta)| = \infty$ whenever the angular derivative of f at ζ fails to exist. The following corollary shows that in the situation most important to us this is not the case.

COROLLARY 3. *Let* f *be an inner function such that* $f'(z)$ *has a radial limit* $f'(\zeta)$ *for almost all* $\zeta \in T$; *then the two sides of* (4) *are finite* (*and equal*) *almost everywhere.*

Of course, the hypotheses imply that f has an angular derivative almost everywhere.

2. INNER FUNCTIONS

We are now ready to apply Theorems 1 and 2 to the main problem of the paper. Recall that, for $0 < p < \infty$, H^p is the class of functions f, analytic in U, and such that

$$\|f\|_p = \sup_r \left(\int |f(r\zeta)|^p \, dm(\zeta) \right)^{1/p} < \infty.$$

THEOREM 3. *If* ϕ *is an inner function and* $\phi' \in H^{1/2}$, *then* ϕ *is a Blaschke product.*

Proof. By Corollary 3, we have almost everywhere the relation

$$|\phi'(\zeta)| = \sum (1 - |a_n|^2)/|\zeta - a_n|^2 + 2 \int |\lambda - \zeta|^{-2} d\mu(\lambda) \geq \int |\lambda - \zeta|^{-2} d\mu(\lambda).$$

Now, if μ is not the zero measure, it follows that

$$\begin{aligned}
\int |\phi'(\zeta)|^{1/2} dm(\zeta) &\geq \int \left(\int |\lambda - \zeta|^{-2} d\mu(\lambda) \right)^{1/2} dm(\zeta) \\
&\geq \|\mu\|^{-1/2} \int \left(\int |\zeta - \lambda|^{-1} d\mu(\lambda) \right) dm(\zeta) \\
&= \|\mu\|^{-1/2} \int \left(\int |\zeta - \lambda|^{-1} dm(\zeta) \right) d\mu(\lambda),
\end{aligned}$$

and the inside integral diverges for *every* λ.

THEOREM 4. *Let* $\phi = Bs$ *be an inner function* (*where* B *is a Blaschke product and* s *a singular inner function*). *Let* $0 < p < 1$. *Then the following are equivalent.*

(i) $\phi'/s \in H^p$,

(ii) $\phi' \in H^p$,

(iii) $|\phi'(\zeta)| \in L^p$.

Proof. Clearly, (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii). If (iii) holds, then (4) holds, both sides being finite almost everywhere. Now, since $\phi' = B's + s'B$, we see that $\phi'/s = B' + Bs'/s$, so that for the functions $f = \phi'/s$ and $g = s'/s$,

$$|f(r\zeta)| \leq |B'(r\zeta)| + |g(r\zeta)| \leq \sum(1 - |a_n|^2)/|1 - \bar{a}_n r\zeta|^2 + 2\int |\lambda - r\zeta|^{-2} d\mu(\lambda)$$

$$\leq 4\left(\sum(1 - |a_n|^2)/|\zeta - a_n|^2 + 2\int |\lambda - \zeta|^{-2} d\mu(\lambda)\right) = 4|\phi'(\zeta)| .$$

It follows that

$$\int |f(r\zeta)|^p \, dm(\zeta) \leq 4^p \int |\phi'(\zeta)|^p \, dm(\zeta) = 4^p \|\phi'\|_p^p$$

for $r < 1$, and hence $f \in H^p$.

Cullen [7] proved Theorem 4 in the special case where $\phi = s$ and the support of μ is a Carleson set. He showed that in this case Theorem 3 follows, in other words, that ϕ' cannot belong to $H^{1/2}$. Cullen's proof, together with Theorem 4 above, gives another proof of Theorem 3.

From the proof of Theorem 4, we can obtain a version with H^p replaced by the Nevanlinna class N.

COROLLARY 4. *Let* ϕ *be as in Theorem* 4. *Then the following are equivalent.*

(i) $\phi'/s \in N^+$,

(ii) $\phi' \in N^+$,

(iii) $\phi' \in N$,

(iv) $\log^+|\phi'| \in L^1$.

Proof. We need only prove that (iv) implies (i). As in the proof of Theorem 4, the function $f = \phi/s$ satisfies the inequality $|f(r\zeta)| \leq 4|\phi'(\zeta)|$ almost everywhere. Thus

$$\log^+|f(r\zeta)| \leq \log^+|\phi'(\zeta)| + \log 4 .$$

It follows that $f \in N$ and that, by the dominated-convergence theorem, $\int \log^+|f(r\zeta)| \, dm(\zeta)$ converges to $\int \log^+|f(\zeta)| \, dm(\zeta)$ as $r \to 1$. The corollary follows from [8, p. 26, Theorem 2.10].

The following theorem is our major tool.

THEOREM 5. *Let* $\phi, \phi_1, \phi_2, \cdots$ *be inner functions with* $\phi = \prod_{n=1}^\infty \phi_n$, *the product converging uniformly on compact subsets of* U. *Then, for* $0 < p < 1$, *we have the inequalities*

(5) $$\sum_{n=1}^{\infty} \|\phi_n'\|_p \leq \|\phi'\|_p \leq \left(\sum \|\phi_n'\|_p^p\right)^{1/p},$$

where each norm is to be replaced by ∞ *if the function fails to exist on a set of positive measure.*

Proof. By Corollary 2,

$$|\phi'(\zeta)| = \sum_{n=1}^{\infty} |\phi_n'(\zeta)|,$$

and the inequality on the left in (5) follows from Minkowski's inequality for $0 < p < 1$. For the other inequality, note that

$$|\phi'(\zeta)|^p = \left(\sum_{n=1}^{\infty} |\phi_n'(\zeta)|\right)^p \leq \sum |\phi_n'(\zeta)|^p,$$

and integrate with respect to dm.

One very simple corollary of Theorem 5 is worth noting.

COROLLARY 5. *If* ϕ *is an inner function and* $\phi = \phi_1 \phi_2$, *then* $\phi' \in H^p$ *implies* $\phi_i' \in H^p$ $(i = 1, 2)$.

We conclude this section with an application of Theorems 3 and 4 to exceptional sets. If ϕ is an inner function, we define $E(\phi)$ as the set of $\alpha \in U$ such that the inner function

$$\psi = (\phi - \alpha)/(1 - \bar{\alpha}\phi)$$

has a nontrivial singular inner factor. Of course, $E(\phi)$ contains $O(\phi)$, the set of *omitted values* of ϕ. According to Frostman [9], $E(\phi)$ has logarithmic capacity 0.

THEOREM 6. *If* $\phi' \in H^{1/2}$, *then* $E(\phi) = \emptyset$.

Proof. For $\alpha \in U$, set $\phi_\alpha = (\phi - \alpha)(1 - \bar{\alpha}\phi)^{-1}$. Then ϕ_α is an inner function, and

$$\phi_\alpha = (1 - |\alpha|^2)\phi'(1 - \bar{\alpha}\phi)^{-2}.$$

It follows that $\phi' \in H^{1/2}$ implies $\phi_\alpha' \in H^{1/2}$, and hence, by Theorem 3, that ϕ_α is a Blaschke product for every α.

COROLLARY 6. *If* ϕ *is an inner function such that* $\phi' \in H^{1/2}$, *then* $\phi(U) = U$.

This follows from the inclusion $O(\phi) \subset E(\phi)$.

Corollary 7 follows from Corollary 4.

COROLLARY 7. *If* ϕ *is an inner function with* $\phi' \in N$, *then* $E(\phi)$ *is countable.*

Proof. Let $\phi_\alpha = (\phi - \alpha)(1 - \bar{\alpha}\phi)^{-1}$, and let $\phi_\alpha = s_\alpha B_\alpha$ be its representation as the product of a singular inner function and a Blaschke product. Since $\phi' \in N$, we see that $\phi_\alpha' \in N$, and thus $\phi_\alpha'/s_\alpha \in N^+$. This and the formula

$$\phi_\alpha' = \phi'(1 - |\alpha|^2)/(1 - \bar{\alpha}\phi)^2$$

imply $\phi'/s_\alpha \in N^+$. Now, if μ_α is the measure μ appearing in the representation (2) for s_α, then $\phi(r\zeta)$ tends to α almost everywhere $(d\mu_\alpha)$, and therefore the measures μ_α are carried on disjoint sets. Since the function $\phi' \in N^+$ has a factorization $\phi' = BsG$ (where B is a Blaschke product, s is a singular inner function, and G is an outer function) and since each singular inner function s_α divides s, we conclude that the number of s_α's is countable.

This argument is an adaptation of a proof of Caughran and Shields [5].

PART II. BLASCHKE PRODUCTS

3. GENERAL BLASCHKE PRODUCTS

In this section we consider a Blaschke product $B(z)$ with zeros a_n $(n = 1, 2, \cdots)$. The quantity $1 - |a_n|$ will be denoted by d_n. We deal with two theorems. Their proofs are very simple; but we shall construct examples to show that the theorems are the best possible results of their type. Theorem 7 is due to Protas [11] (see also Caughran and Shields [6]).

THEOREM 7. (i) *If* $0 < \alpha < 1/2$ *and* $\sum d_n^\alpha < \infty$, *then* $B' \in H^{1-\alpha}$;

(ii) *if* $\sum d_n^{1/2} \log d_n^{-1} < \infty$, *then* $B' \in H^{1/2}$.

Proof. The proof given by Protas and Caughran is short, and it may be based upon Theorem 5, as follows. Let $B_n(z) = (z - a_n)/(1 - \bar{a}_n z)$. By Theorem 5,

$$\|B'\|_p \leq \left(\sum \|B_n'\|_p^p\right)^{1/p}.$$

Now it is easy to see that

$$\|B_n'\|_p^p \leq C d_n^\alpha \quad \text{if } p = 1 - \alpha \text{ and } 0 < \alpha < 1/2,$$

and that

$$\|B_n'\|_p^p \leq c\, d_n^p \log d_n^{-1} \quad \text{if } p = 1/2.$$

THEOREM 8. (i) *If* $1/2 < p < 1$ *and* $B' \in H^p$, *then* $\sum d_n^{(1-p)/p} < \infty$;

(ii) *if* $B' \in H^{1/2}$, *then* $\sum d_n (\log d_n^{-1})^2 < \infty$.

Proof. Using the other inequality in Theorem 5, we find that

$$\sum \|B_n'\|_p \leq \|B'\|_p,$$

where, as before, $B_n = (z - a_n)/(1 - \bar{a}_n z)$. It is not hard to see, this time, that

$$\|B_n'\|_p \geq \varepsilon\, d_n^{(1-p)/p} \quad \text{if } 1/2 < p < 1,$$

and that

$$\|B_n'\|_p \geq \varepsilon\, d_n (\log d_n^{-1})^2 \quad \text{if } p = 1/2.$$

This completes the proof.

Our goal now is to show that both Theorem 7(i) and Theorem 8(i) are the best possible results of the form

$$\sum d_n^\alpha < \infty \quad \text{implies} \quad B' \in H^p.$$

The rest of this section is devoted to a series of lemmas leading to an example showing that Theorem 7(i) cannot be improved. That Theorem 8(i) is best possible will be shown in the next section when we consider Blaschke products with real zeros.

The first of our lemmas will later be useful in other ways.

LEMMA 1. *Let* $a_n = r_n e^{i\theta_n}$. *Then* $B' \in H^p$ *if and only if* $f \in L^p$, *where*

$$f(\theta) = \sum_{n=1}^{\infty} d_n/[d_n^2 + (\theta - \theta_n)^2].$$

Proof. Note that

$$(6) \qquad |e^{i\theta} - a_n|^2 = 1 + r_n^2 - 2r_n\cos(\theta - \theta_n) = d_n^2 + 4r_n\sin^2[(\theta - \theta_n)/2].$$

Thus, if $B' \in H^p$, so that $B'(e^{i\theta}) \in L^p$, then

$$|e^{i\theta} - a_n|^2 \le d_n^2 + (\theta - \theta_n)^2;$$

hence $f(\theta) \le |B'(e^{i\theta})|$, and therefore $f \in L^p$.

Now suppose $f \in L^p$. We may write B as the product of four factors, each of which has its zeros lying in an angle of opening at most $\pi/2$, and it is enough to show that each of these factors has its derivative in H^p. Assume, therefore, that $-\pi/4 \le \theta_n \le \pi/4$; we need to show that

$$\int_{-\pi/4}^{\pi/4} |B'(e^{i\theta})|^p \, d\theta < \infty.$$

For $|\theta| \le \pi/4$, we see from (6) that

$$|e^{i\theta} - a_n|^2 \ge d_n^2 + 4(1/2)(4/\pi^2)(\theta - \theta_n)^2/2 \ge \varepsilon_0 [d_n^2 + (\theta - \theta_n)^2],$$

since $|\theta - \theta_n| \le \pi/2$. This shows that

$$|B'(e^{i\theta})| \le \varepsilon_0^{-1} f(\theta)$$

for $|\theta| \le \pi/4$, so that $f \in L^p$ implies $B' \in H^p$, and this proves the lemma.

LEMMA 2. *Let* $\gamma > 1$, *and let* $a_n = r_n e^{i\theta_n}$, *where* $r_n = 1 - n^{-\gamma}$ *and* $\theta_n = n^{1-\gamma}$. *Then* $B' \notin H^{1-1/\gamma}$.

Proof. In the notation of Lemma 1, we shall show that there is a positive number ε such that $f(\theta) \ge \varepsilon\theta^{-\gamma/(\gamma-1)}$ for all sufficiently small θ. In fact, in the present situation,

$$f(\theta) = \sum n^\gamma/[1 + n^2(n^{\gamma-1}\theta - 1)^2].$$

For a fixed θ, we estimate this sum from below by its largest term. This will occur when n_0 is chosen so that

(7) $$\theta^{-1/(\gamma-1)} - 1 \leq n_0 < \theta^{-1/(\gamma-1)}.$$

(Here we assume θ is small enough so that $\theta^{-1/(\gamma-1)} \geq 2$.) From (7) it follows that

$$1 - \theta^{1/(\gamma-1)} \leq n_0\,\theta^{1/(\gamma-1)} < 1,$$

and hence that

$$[1 - \theta^{1/(\gamma-1)}]^{\gamma-1} \leq n_0^{\gamma-1}\theta < 1,$$

so that

$$(n_0^{\gamma-1}\theta - 1)^2 \leq [1 - (1 - \theta^{1/(\gamma-1)})^{\gamma-1}]^2.$$

This and the inequalities (7) give the relations

$$f(\theta) \geq \frac{n_0^{\gamma}}{1 + n_0^2(n_0^{\gamma-1}\theta - 1)^2} \geq \frac{(\theta^{-1/(\gamma-1)} - 1)^{\gamma}}{1 + \left[\dfrac{1 - (1 - \theta^{1/(\gamma-1)})^{\gamma-1}}{\theta^{1/(\gamma-1)}}\right]^2}.$$

It follows from elementary calculus that the denominator approaches $1 + (\gamma - 1)^2$ as θ approaches 0, and from this it follows that there is a positive number ε such that $f(\theta) \geq \varepsilon\,\theta^{-\gamma/(\gamma-1)}$ for all sufficiently small θ. This completes the proof of Lemma 2.

Notice that if we set $\beta = \gamma^{-1}$, then the lemma provides an example where $\sum d_n^{\alpha} < \infty$ for all $\alpha < \beta$, but $B' \notin H^{1-\beta}$.

We are now in a position to show that the exponent $p = 1 - \alpha$ in Theorem 7(i) is the best possible.

Example. If $0 < \alpha < 1/2$, *there is a Blaschke product* $B(z)$ *whose zeros converge to* 1 *and such that* $\sum d_n^{\alpha} < \infty$, *but* $B' \notin H^p$ *for any* $p > 1 - \alpha$.

We construct the example as follows. For each n, we find a Blaschke product B_n whose zeros converge to 1 and such that $\sum (d_k^n)^{\alpha} < \infty$, but $B_n' \notin H^{1-\alpha+1/n}$. This can be done, by Lemma 2. By discarding finitely many of the zeros of B_n, we may assume that $\sum_{k=1}^{\infty} (d_k^n)^{\alpha} < 2^{-n}$. It follows that $B = \prod_{n=1}^{\infty} B_n$ is a convergent Blaschke product whose zeros satisfy the condition $\sum d_k^{\alpha} < \infty$. By Theorem 5,

$$\|B'\|_p \geq \sum \|B_n'\|_p = \infty$$

when $p > 1 - \alpha$, so that $B' \notin H^p$ whenever $p > 1 - \alpha$.

We point out that the same method may be used to show that there is a Blaschke product B whose zeros converge to 1 such that $B' \notin H^p$ for any $p > 0$.

4. BLASCHKE PRODUCTS WITH $a_n \to 1$ NONTANGENTIALLY

The main result of this section is that if $a_n \to 1$ nontangentially, that is, if

$$(1 - |a_n|)/|1 - a_n| \geq c, \tag{8}$$

then Theorem 8(i) gives the right order of convergence of $\sum d_n^\alpha$ for $B' \in H^p$.

First we reduce the case of $a_n \to 1$ nontangentially to the case where the a_n are real and positive.

LEMMA 3. *Let* $a_n \to 1$ *nontangentially. Then there are constants* $c_1, c_2 > 0$ *such that*

$$c_1 \leq |a_n - \lambda|/||a_n| - \lambda| \leq c_2 \tag{9}$$

for each λ *of modulus* 1. *In particular, if* B(z) *and* P(z) *are the Blaschke products with zeros* $\{a_n\}$ *and* $\{|a_n|\}$ *respectively, then*

$$c_1^2 |B'(\lambda)| \leq |P'(\lambda)| \leq c_2^2 |B'(\lambda)| \tag{10}$$

for $|\lambda| = 1$ $(\lambda \neq 1)$.

Proof. The right-hand inequality in (9) follows from the relations

$$|a_n - \lambda|/||a_n| - \lambda| \leq |a_n - 1|/(1 - |a_n|) + 1 + (1 - |a_n|)/|\lambda - |a_n|| \leq c + 2,$$

where c comes from (8).

To prove the left-hand inequality in (9), we note that

$$|\lambda - |a_n|| \leq |\lambda - a_n| + (1 - |a_n|) + |1 - a_n| \leq |\lambda - a_n| + (1 - |a_n|) + c^{-1}(1 - |a_n|),$$

where again c comes from (8). The result is

$$|\lambda - |a_n|| \leq (2 + c^{-1}) |\lambda - a_n| .$$

This proves (9). Now (10) follows from (9) and from Theorem 2.

THEOREM 9. *If the zeros of* B *tend to* 1 *nontangentially, if* $p \geq 1/2$, *and if* $\sum d_n^\alpha < \infty$ *for some* $\alpha < (1 - p)/p$, *then* $B' \in H^p$.

Proof. By Lemma 3, we may assume $a_n \geq 0$, so that the function $f(\theta)$ in Lemma 1 is $f(\theta) = \sum d_n/[d_n^2 + \theta^2]$. From the inequality $x^{1-\alpha} \leq 1 + x^2$, with $x = d_n/\theta$, we get the further inequality

$$(d_n/\theta)^{1-\alpha} \leq 1 + (d_n/\theta)^2 ,$$

from which it follows that

$$d_n/[\theta^2 + d_n^2] \leq \theta^{-(1+\alpha)} d_n^\alpha ,$$

so that $f(\theta) \leq \theta^{-(1+\alpha)} \sum d_n^\alpha$; therefore $f \in L^p$ for $p < (1+\alpha)^{-1}$. This, together with Lemma 1, implies the theorem.

On the basis of Theorems 8(i) and 9, it is natural to ask whether, under the assumption that $a_n \to 1$ nontangentially and $p \geq 1/2$, the condition $\sum d_n^{(1-p)/p} < \infty$ is both necessary and sufficient in order that $B' \in H^p$. Our next result shows that the answer is negative, and it gives some more precise information.

THEOREM 10. *Suppose that the zeros of* B *converge to* 1 *nontangentially. Let* k_n *denote the number of values* $|a_n|$ *in the interval* $(1 - 2^{-n}, 1 - 2^{-(n+1)}]$.

(i) *If* $0 < \alpha < 1$ *and* $k_n/2^{n\alpha} = O(n^{-t})$ *for some* $t > 1 + \alpha$, *then* $B' \in H^{1/(1+\alpha)}$.

(ii) *The zeros of* B *can be chosen in such a way that* $k_n/2^{n\alpha} = O(n^{-1-\alpha})$, *but* $B' \notin H^{1/(1+\alpha)}$.

Notice that the series $\sum d_n^\alpha$ converges if and only if $\sum k_n/2^{n\alpha} < \infty$.

Proof. Assume, as before, that the a_n are replaced by the $|a_n|$.

If $k_n/2^{n\alpha} = O(n^{-t})$ for some $t > 1+\alpha$, we write

$$f(\theta) = \sum_{d_n<|\theta|} d_n/(d_n^2+\theta^2) + \sum_{d_n\geq|\theta|} d_n/(d_n^2+\theta^2) \leq \sum_{d_n<|\theta|} d_n/\theta^2 + \sum_{d_n\geq|\theta|} d_n^{-1}.$$

Concerning the first sum we invoke the inequality

$$\sum_{d_n<|\theta|} d_n = \sum_{n=N}^{\infty} \sum_{\frac{1}{2^{n+1}}<d_k\leq\frac{1}{2^n}} d_k \leq \sum_{n=N}^{\infty} k_n/2^n = O\left(\sum_{n=N}^{\infty} n^{-t}\, 2^{-n(1-\alpha)}\right),$$

where $2^{-(N+1)} \leq |\theta| \leq 2^{-N}$. If we notice that

$$\sum_{n=N}^{\infty} n^{-t}\, 2^{-n(1-\alpha)} \leq N^{-t} \sum_{n=N}^{\infty} 2^{-n(1-\alpha)} = O(N^{-t} 2^{-N(1-\alpha)}) = O(|\theta|^{1-\alpha}/[\log|\theta|^{-1}]^t),$$

we see that

$$\theta^{-2} \sum_{d_n<|\theta|} d_n = O(|\theta|^{-1-\alpha}/[\log|\theta|^{-1}]^t).$$

For the second sum, we get the estimate

$$\sum_{d_n\geq|\theta|} d_n^{-1} = \sum_{n=1}^{N}\left(\sum_{2^{-n-1}<d_k\leq 2^{-n}} d_k^{-1}\right) \leq \sum_{n=1}^{N} k_n 2^{n+1} = O\left(\sum_{n=1}^{N} 2^{n(1+\alpha)}/n^t\right),$$

where $2^{-N-1} \leq |\theta| \leq 2^{-N}$. Now it is easy to see that for each $t' < t$,

$$\sum_{n=1}^{N} 2^{n(1+\alpha)} n^{-t} = O(2^{N(1+\alpha)} N^{-t'}).$$

Since $2^{-(N+1)} \leq |\theta| \leq 2^{-N}$, we see that

$$f(\theta) = O(\theta^{-(1+\alpha)}/[\log|\theta|^{-1}]^{t'}) \quad \text{for each } t' < t.$$

Choosing t' so that $1 + \alpha < t' < t$, we see that $f \in L^{1/(1+\alpha)}$.

In order to prove (ii), we choose for k_n the unique integer satisfying the condition

$$2^{n\alpha} n^{-(1+\alpha)} \leq k_n < 2^{n\alpha} n^{-(1+\alpha)} + 1.$$

Clearly, $k_n 2^{-n\alpha} = O(n^{-(1+\alpha)})$. On the other hand,

$$f(\theta) \geq \frac{1}{2} \sum_{d_n > |\theta|} d_n^{-1} \geq \frac{1}{2} \sum_{n=1}^{N} \sum_{2^{-(n+1)} < d_k < 2^{-n}} d_k^{-1} \geq \frac{1}{2} \sum_{n=1}^{N} 2^n k_n,$$

where $2^{-N} \leq |\theta| \leq 2^{-N+1}$, so that

$$f(\theta) \geq \frac{1}{2} 2^N k_N \geq \frac{1}{2} 2^N 2^{N\alpha} N^{-(1+\alpha)}$$

$$= \frac{1}{2} 2^{N(1+\alpha)}/N^{1+\alpha} \geq \varepsilon |\theta|^{-(1+\alpha)}/[\log|\theta|^{-1}]^{1+\alpha}.$$

It follows that $f \notin L^{1/(1+\alpha)}$ and hence that $B' \notin H^{1/(1+\alpha)}$.

The following corollary shows that Theorem 8(i) is best possible in a stronger sense than Theorem 9.

COROLLARY 8. *For each* p $(1 > p > 1/2)$, *there is a Blaschke product* B *such that* $B' \in H^p$ *and* $\sum d_n^\beta = \infty$ *for every* $\beta < (1 - p)/p$.

Proof. Let $\alpha = (1 - p)/p$, and choose k_n so that $k_n 2^{-n\alpha} = O(n^{-t})$ for some $t > 1 + \alpha$, but $\sum k_n 2^{-n\beta} = \infty$ for every $\beta < \alpha$.

5. TWO MORE SUFFICIENT CONDITIONS

In this section, we give two sufficient conditions for the relation $B' \in H^p$, under assumptions different from those of Section 4. The first condition generalizes the estimate applied to a $B(z)$ having real zeros in Theorem 9, and the second is based on a lemma that generalizes part of Lemma 3.

The first of our two theorems generalizes Theorem 7 when q is sufficiently small. Note also that it holds for all $\alpha \leq 1$.

THEOREM 11. *Let* B *be a Blaschke product with zeros* $a_n = r_n e^{i\theta_n}$. *Let* E *be the closure of the set* $\{\theta_n, n = 1, 2, \cdots\}$. *Suppose* E *has measure* 0, *and let* $\{\varepsilon_n\}$ *be the sequence of lengths of the intervals on the circle* T *complementary to* E. *Further, suppose that* $\sum \varepsilon_n^q < \infty$ *for some* q $(0 < q < 1)$, *and that* $\sum d_n^\alpha < \infty$ *for some* α $(0 < \alpha \leq 1)$; *then* $B' \in H^p$ *for* $p \leq (1 - q)/(1 + \alpha)$.

Proof. We consider the relations

$$
\begin{aligned}
f(\theta) &= \sum_{d_n<|\theta-\theta_n|} d_n/[d_n^2+(\theta-\theta_n)^2] + \sum_{d_n\geq|\theta-\theta_n|} d_n/[d_n^2+(\theta-\theta_n)^2] \\
&\leq \sum_{d_n<|\theta-\theta_n|} d_n(\theta-\theta_n)^{-2} + \sum_{d_n\geq|\theta-\theta_n|} d_n^{-1} \\
&= \sum_{d_n<|\theta-\theta_n|} d_n^\alpha d_n^{1-\alpha}(\theta-\theta_n)^{-2} + \sum_{d_n\geq|\theta-\theta_n|} d_n^\alpha d_n^{-(1+\alpha)} \\
&\leq \sum_{d_n<|\theta-\theta_n|} d_n^\alpha |\theta-\theta_n|^{1-\alpha}/(\theta-\theta_n)^2 + \sum_{d_n\geq|\theta-\theta_n|} d_n^\alpha/|\theta-\theta_n|^{1+\alpha} \\
&= 2\sum_{n=1}^\infty d_n^\alpha/|\theta-\theta_n|^{1+\alpha} \leq \left(2\sum_{n=1}^\infty d_n^\alpha\right) d(\theta)^{-1-\alpha},
\end{aligned}
$$

where $d(\theta)$ denotes the distance from θ to E. Since

$$\int_0^{2\pi} d(\theta)^{-(1+\alpha)p}\, d\theta = O\left(\sum_{n=1}^\infty \varepsilon_n^{1-(1+\alpha)p}\right),$$

we see that $f \in L^p$ as long as $1-(1+\alpha)p \geq q$. This completes the proof.

In the next lemma, we have a generalization of (part of) inequality (9) to the case of a Blaschke product whose zeros approach 1 tangentially. Following G. T. Cargo [4], we define, for $\gamma \geq 1$ and $\delta > 0$,

$$R(\delta, \gamma) = \{z\colon 1-|z| \geq \delta|1-z|^\gamma\}.$$

When $\gamma = 1$, the region $R(\delta, \gamma)$ is essentially a Stoltz angle. As γ increases, $R(\delta, \gamma)$ touches the unit circle T with a greater degree of tangency.

LEMMA 4. *If* $a \in R(\delta, \gamma)$ *and* $|\zeta| = 1$, *then*

$$|\zeta - |a||^\gamma \leq (8+\delta^{-\gamma})^\gamma |\zeta - a|.$$

Proof. Let $\varepsilon = 1/\gamma$. Then, since $a \in R(\delta, \gamma)$, we see that

$$|1-a| \leq \delta^{-\gamma}(1-|a|)^\varepsilon.$$

Thus

$$
\begin{aligned}
|\zeta - |a|| &\leq |\zeta - a| + (1-|a|) + |1-a| \leq 2|\zeta-a| + \delta^{-\gamma}|\zeta-a|^\varepsilon \\
&= [2|\zeta-a|^{1-\varepsilon} + \delta^{-\gamma}]|\zeta-a|^\varepsilon \leq (8+\delta^{-\gamma})|\zeta-a|^\varepsilon.
\end{aligned}
$$

Of course, Lemma 4 is used in the same way as Lemma 3.

THEOREM 12. *Suppose* $a_n \in R(\delta, \gamma)$ $(n = 1, 2, \cdots)$ *and* $\sum d_n^\alpha < \infty$. *Then* $B' \in H^p$ *for* $p < (\alpha + 2\gamma - 1)^{-1}$.

Proof. By Lemma 4,

$$|e^{i\theta} - a_n|^{-2} = O(|e^{i\theta} - |a_n||^{-2\gamma}),$$

so that it suffices to show that

$$g(\theta) = \sum d_n |e^{i\theta} - |a_n||^{2\gamma} \in L^p.$$

But clearly $g(\theta)$ is dominated by

$$G(\theta) = \sum d_n (d_n^2 + \theta^2)^{-\gamma} \leq \sum_{d_n < |\theta|} d_n |\theta|^{-2\gamma} + \sum_{d_n \geq |\theta|} d_n^{1-2\gamma}$$

$$\leq \sum_{d_n < |\theta|} d_n^{\alpha} |\theta|^{1-\alpha-2\gamma} + \sum_{d_n \geq |\theta|} d_n^{\alpha} |\theta|^{1-\alpha-2\gamma} = O(\theta^{1-\alpha-2\gamma}),$$

and this proves Theorem 12.

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